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ON PARALLELISM OF ASCREEN HALF LIGHTLIKE SUBMANIFOLDS OF INDEFINITE COSYMPLECTIC MANIFOLDS

WENJIE WANG AND XIMIN LIU

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Abstract. In this paper, we mainly study the parallelism of the second fundamental forms of ascreen half lightlike submanifolds of indefinite cosymplectic manifolds. It is proved that a half lightlike submanifold M of an indefinite cosymplectic space form $(\overline{M}(c), \overline{g})$ with semi-parallel second fundamental form h either satisfies $c = 0$ or is $(\overline{J}(Rad(TM)), TM)$-mixed geodesic. Moreover, some properties of ascreen half lightlike submanifolds with parallel second fundamental forms are also obtained.

1. Introduction

Since the intersection of the normal bundle and the tangent bundle of a submanifold of a semi-Riemannian manifold may be not trivial, it is more difficult and interesting to study the geometry of lightlike submanifolds than non-degenerate submanifolds. The two standard methods to deal with the above difficulties were developed by Kupeli [11] and Duggal-Bejancu [3], Duggal-Jin [5] and Duggal-Sahin [6] respectively. Let M be a lightlike submanifold immersed in a semi-Riemannian manifold, it is obvious to see that there are two cases of codimension 2 lightlike submanifolds, since for this type the dimension of their radical distributions is either 1 or 2. A codimension 2 lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ is called a half lightlike submanifold [2, 4] if $\dim(Rad(TM)) = 1$, where $Rad(TM)$ denotes the degenerate radical distribution of M. For more results about half lightlike submanifolds, we refer the reader to [4, 6, 8, 9].

In the theory of submanifolds of Riemannian manifolds, the parallel and semi-parallel immersions were studied by Ferus [7] and Deprez [1] respectively. Recently, F. Massamba [12, 13, 14] and Upadhyay-Gupta [15] studied the parallel and semi-parallel lightlike hypersurfaces of an indefinite Sasakian, Kenmotsu and cosymplectic manifolds, respectively. However, the parallel and semi-parallel ascreen half

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lightlike submanifolds of indefinite cosymplectic manifolds have not yet been considered. The object of this paper is to study the parallelism of ascreen half lightlike submanifolds of indefinite cosymplectic manifolds.

2. Preliminaries

First of all, we follow Duggal-Sahin [6] and Jin [9] for the notations and fundamental equations for half lightlike submanifolds of indefinite cosymplectic manifolds. A $(2n+1)$-dimensional semi-Riemannian $(\overline{M}, \overline{g})$ is said to be an indefinite cosymplectic manifold if it admits a normal almost contact metric structure $(\overline{J}, \zeta, \theta, \overline{g})$, where $\overline{J}$ is a tensor field of type (1,1), ζ is a vector field which is called characteristic vector field, θ is a 1-form and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ such that

$$\overline{J}^2 X = -X + \theta(X)\zeta,\ \overline{J}\zeta = 0,\ \theta \circ \overline{J} = 0,\ \theta(\zeta) = 1, \tag{2.1}$$

$$\theta(X) = \overline{g}(\zeta, X),\ \overline{g}(\overline{J}X, \overline{J}Y) = \overline{g}(X, Y) - \theta(X)\theta(Y), \tag{2.2}$$

$$d\theta = 0,\ (\overline{\nabla}_X \overline{J})Y = 0,\ \overline{\nabla}_X \zeta = 0,\ \forall X, Y \in \Gamma(T\overline{M}), \tag{2.3}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection of a semi-Riemannian metric $\overline{g}$.

A plane section of an indefinite cosymplectic manifold $(\overline{M}, \overline{J}, \zeta, \theta, \overline{g})$ is called a $\overline{J}$-section if it is spanned by a unit vector filed X orthogonal to ζ and $\overline{J}X$, where X is a non-null vector field on $\overline{M}$. The sectional curvature $K(X, \overline{J}X)$ of a $\overline{J}$-section is called a $\overline{J}$-sectional curvature. If $\overline{M}$ has a constant $\overline{J}$-sectional curvature c which is not depend on the $\overline{J}$-section at each point, then c is a constant and $\overline{M}$ is called a cosymplectic space form, denoted by $(\overline{M}(c), \overline{g})$. The curvature tensor $\overline{R}$ of an indefinite cosymplectic space form $(\overline{M}(c), \overline{g})$ is given in [15] as follows:

$$\begin{aligned} \overline{R}(X,Y)Z = & \frac{c}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \theta(X)\theta(Z)Y \\ & - \theta(Y)\theta(Z)X + \overline{g}(X,Z)\theta(Y)\zeta - \overline{g}(Y,Z)\theta(X)\zeta \\ & + \overline{g}(\overline{J}Y,Z)\overline{J}X - \overline{g}(\overline{J}X,Z)\overline{J}Y - 2\overline{g}(\overline{J}X,Y)\overline{J}Z\}. \end{aligned} \tag{2.4}$$

A submanifold (M, g) of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ of codimension 2 is called a half lightlike submanifold if the radical distribution $Rad(TM) = TM \cap TM^\perp$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^\perp$ is of rank 1, where the metric g induced from ambient space $\overline{M}$ is degenerate. Thus there exist non-degenerate complementary distribution $S(TM)$ and $S(TM^\perp)$ of $Rad(TM)$ in TM and $TM^\perp$ respectively, which are called the screen and screen transversal distribution on M respectively. Thus we have

$$TM = Rad(TM) \oplus_{orth} S(TM), \tag{2.5}$$

$$TM^\perp = Rad(TM) \oplus_{orth} S(TM^\perp), \tag{2.6}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Consider the orthogonal complementary distribution $S(TM)^\perp$ to $S(TM)$ in $T\overline{M}$, it is easy to see that $TM^\perp$ is a subbundle of $S(TM)^\perp$. As $S(TM^\perp)$ is a non-degenerate subbundle of $S(TM)^\perp$, the orthogonal complementary distribution $S(TM^\perp)^\perp$ to $S(TM^\perp)$ in $S(TM)^\perp$ is also a non-degenerate distribution. Clearly $Rad(TM)$ is a subbundle of $S(TM^\perp)^\perp$. Choose $L \in \Gamma(S(TM^\perp))$ as a unit vector field with $\overline{g}(L, L) = \pm 1$. In this paper, we may assume that $\overline{g}(L, L) = 1$ without lose the generality. For any null section

$\xi \in Rad(TM)$, there exists [3] a uniquely defined null vector field $N \in \Gamma(S(TM^{\perp})^{\perp})$ satisfying

$$\overline{g}(\xi, N) = 1, \quad \overline{g}(N, N) = \overline{g}(N, X) = \overline{g}(N, L) = 0, \quad \forall X \in \Gamma(S(TM)). \tag{2.7}$$

Denote by $ltr(TM)$ the vector subbundle of $S(TM^{\perp})^{\perp}$ locally spanned by N. Then we show that $S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM)$. We put $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$. We call N, $ltr(TM)$ and $tr(TM)$ the lightlike transversal vector field, lightlike transversal vector bundle and transversal vector bundle of M with respect to the chosen screen distribution $S(TM)$ respectively. Then $T\overline{M}$ is decomposed as follows:

$$\begin{aligned} T\overline{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned} \tag{2.8}$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.8). For any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$, $\xi \in \Gamma(Rad(TM))$ and $L \in \Gamma(S(TM)^{\perp})$, the Gauss and Weingarten formulas of M and $S(TM)$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{2.9}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.10}$$

$$\overline{\nabla}_X L = -A_L X + \phi(X)N, \tag{2.11}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.12}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.13}$$

where ∇ and ∇^* are induced connection on TM and $S(TM)$ respectively, B and D are called local second fundamental forms of M and C is called the local second fundamental form on $S(TM)$. A_N, A^*_ξ and A_L are linear operators on TM and τ, ρ and ϕ are 1-forms on TM. We put $h(X, Y) = B(X, Y)N + D(X, Y)L$ for any $X, Y \in \Gamma(TM)$, where h is called the second fundamental form of M. Note that the connection ∇ is torsion free but is not metric tensor and the connection ∇^* is metric. We also know that both B and D are symmetric tensors on $\Gamma(TM)$ and independent of the choice of a screen distribution. Using (2.9)-(2.13) we obtain the following equations.

$$B(X, \xi) = 0, \ D(X, \xi) = -\phi(X), \ \overline{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi - \phi(X)L, \tag{2.14}$$

$$g(A_N X, PY) = \overline{g}(N, h(X, PY)), \quad \overline{g}(A_N X, N) = 0, \tag{2.15}$$

$$g(A^*_\xi X, PY) = \overline{g}(\xi, h(X, PY)), \quad \overline{g}(A^*_\xi X, N) = 0, \tag{2.16}$$

$$D(X, Y) = g(A_L X, Y) - \phi(X)\eta(Y), \quad \overline{g}(A_L X, N) = \rho(X), \tag{2.17}$$

for any $X, Y, Z \in \Gamma(TM)$, where $\eta(X) = \overline{g}(X, N)$. Denote by $\overline{R}$ and R the curvature tensor of semi-Riemannian connection $\overline{\nabla}$ of $\overline{M}$, then we have

$$
\begin{aligned}
&\overline{R}(X,Y)Z \\
=&R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\
&+ D(X,Z)A_L Y - D(Y,Z)A_L X \\
&+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\
&+ \phi(X)D(Y,Z) - \phi(Y)D(X,Z)\}N \\
&+ \{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) - \rho(Y)B(X,Z)\}L.
\end{aligned} \tag{2.18}
$$

3. Semi-parallel of ascreen half lightlike submanifolds

In this section, we mainly investigate semi-parallelism of ascreen half lightlike submanifolds of indefinite cosymplectic manifolds.

Lemma 3.1 (see [10]). *Let M be a half lightlike submanifold of an indefinite almost contact metric manifolds $\overline{M}$. Then there exists a screen distribution $S(TM)$ such that*

$$\overline{J}(S(TM)^{\perp}) \subset S(TM). \tag{3.1}$$

Moreover, the structure vector field ζ does not belong to $Rad(TM)$ and $ltr(TM)$.

Definition 3.2 (see [8]). *A half lightlike submanifold M of an indefinite cosymplectic manifold $\overline{M}$ is said to be an ascreen half lightlike submanifold if the structure vector field ζ of $\overline{M}$ belongs to the distribution $Rad(TM) \oplus ltr(TM)$.*

For any ascreen half lightlike submanifold M, the vector field ζ is decomposed as

$$\zeta = a\xi + bN \ (\Rightarrow a = \theta(N) \text{ and } b = \theta(\xi)), \tag{3.2}$$

then from lemma 3.1 we know that $a \neq 0$ and $b \neq 0$.

Substituting (3.2) into $\overline{g}(\zeta, \zeta) = 1$, we have $ab = \frac{1}{2}$. Consider the local unite timelike vector flied V^* on M, the unite timelike vector filed U^* and the local unite spacelike vector filed W^* on $S(TM)$, defined by

$$V^* = -b^{-1}\overline{J}\xi, \quad U^* = -a^{-1}\overline{J}N, \quad W^* = -\overline{J}L, \tag{3.3}$$

then from (3.3) we have $g(U^*, V^*) = 1$. Applying $\overline{J}$ on (3.2) and using the second term of (2.1) and $ab = \frac{1}{2}$, we have

$$0 = a\overline{J}\xi + b\overline{J}N = -\frac{V^* + U^*}{2}, \ \textit{i.e.}, \ U^* = -V^*. \tag{3.4}$$

Thus, we see that $\overline{J}(Rad(TM)) = \overline{J}(ltr(TM))$. Using (3.4) and Lemma 3.1, the tangent bundle TM of M is decomposed as follows:

$$TM = \left\{\overline{J}(Rad(TM)) \oplus_{\text{orth}} \overline{J}(S(TM^{\perp})) \oplus_{\text{orth}} H^*\right\} \oplus_{\text{orth}} Rad(TM). \tag{3.5}$$

Where H^* is a nondegenerate and almost complex distribution on M with respect to the indefinite cosymplectic structure tensor $\overline{J}$.

Denote by $\mathcal{S}^*$ the projection morphism of TM on H^* with respect to the decomposition (3.5), then for any vector field X on M, $\overline{J}V^* = a\xi - bN$, we have

$$
\begin{aligned}
X =&\mathcal{S}^*X + v^*(X)V^* + \omega^*(X)W^* + \eta(X)\xi, \\
\overline{J}X =&JX + av^*(X)\xi - b\eta(X)V^* - bv^*(X)N + \omega^*(X)L,
\end{aligned} \tag{3.6}
$$

where υ^*, u^* and ω^* are 1-forms defined by

$$\upsilon^*(X) = -g(X, V^*), \quad u^*(X) = -g(X, U^*), \quad \omega^*(X) = g(X, W^*), \tag{3.7}$$

where J is a tensor of type $(1,1)$ globally defined on M by $J = \overline{J} \circ \mathcal{S}^*$.

Definition 3.3. *Let $(M, g, S(TM))$ be an ascreen half lightlike submanifold of an indefinite cosymplectic manifold $(\overline{M}, \overline{J}, \zeta, \theta, \overline{g})$. Then M is said to be semi-parallel if its second fundamental form h satisfies*

$$(R(X,Y)\cdot h)(X_1, X_2) = -h(R(X,Y)X_1, X_2) - h(X_1, R(X,Y)X_2) = 0 \tag{3.8}$$

for any $X, Y, X_1, X_2 \in \Gamma(TM)$.

Definition 3.4 (see [6]). *A half lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be irrotational if $\overline{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$.*

From (2.9) and (2.14) we see that a necessary and sufficient condition for a half lightlike submanifold M to be irrotational is $D(X, \xi) = \phi(X) = 0$ for any $X \in \Gamma(TM)$.

Lemma 3.5. *Let $(M, g, S(TM))$ be a ascreen half lightlike submanifold of an indefinite cosymplectic space from $(\overline{M}(c), \overline{g})$, then the curvature tensor of M is given by*

$$\begin{aligned}
&R(X,Y)Z \\
=&\frac{c}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X + \overline{g}(X,Z)\theta(Y)a\xi \\
&- \overline{g}(Y,Z)\theta(X)a\xi + \overline{g}(\overline{J}Y,Z)(JX + a\upsilon^*(X)\xi - b\eta(X)V^*) \\
&- \overline{g}(\overline{J}X,Z)(JY + a\upsilon^*(Y)\xi - b\eta(Y)V^*) \\
&- 2\overline{g}(\overline{J}X,Y)(JZ + a\upsilon^*(Z)\xi - b\eta(Z)V^*)\} \\
&- B(X,Z)A_N Y + B(Y,Z)A_N X - D(X,Z)A_L Y + D(Y,Z)A_L X
\end{aligned} \tag{3.9}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. The proof follows from (2.4), (2.18), (3.2) and (3.6). □

Theorem 3.6. *Let $(M, g, S(TM))$ be a semi-parallel ascreen half lightlike submanifold of an indefinite cosymplectic space from $(\overline{M}(c), \overline{g})$. If M is irrotational, then either $c = 0$ or M is $\big(\overline{J}(Rad(TM)), TM\big)$-mixed geodesic.*

Proof. Putting (3.9) and $h(X,Y) = B(X,Y)N + D(X,Y)L$ into (3.8), using (2.14) and Definition 3.4 we obtain

(3.10)

$$\begin{aligned}
&\frac{c}{4}\{\overline{g}(Y,X_1)B(X,X_2) - \overline{g}(X,X_1)B(Y,X_2) + \overline{g}(Y,X_2)B(X,X_1) \\
&- \overline{g}(X,X_2)B(Y,X_1) + \theta(X)\theta(X_1)B(Y,X_2) - \theta(Y)\theta(X_1)B(X,X_2) \\
&+ \theta(X)\theta(X_2)B(Y,X_1) - \theta(Y)\theta(X_2)B(X,X_1) \\
&+ \overline{g}(\overline{J}Y,X_1)B(JX - b\eta(X)V^*, X_2) \\
&- \overline{g}(\overline{J}X,X_1)B(JY - b\eta(Y)V^*, X_2) + \overline{g}(\overline{J}Y,X_2)B(JX - b\eta(X)V^*, X_1) \\
&- \overline{g}(\overline{J}X,X_2)B(JY - b\eta(Y)V^*, X_1) - 2\overline{g}(\overline{J}X,Y)B(JX_1 - b\eta(X_1)V^*, X_2) \\
&- 2\overline{g}(\overline{J}X,Y)B(JX_2 - b\eta(X_2)V^*, X_1)\} - B(X,X_1)B(A_N Y, X_2) \\
&+ B(Y,X_1)B(A_N X, X_2) - B(X,X_2)B(A_N Y, X_1) + B(Y,X_2)B(A_N X, X_1) \\
&- D(X,X_1)B(A_L Y, X_2) + D(Y,X_1)B(A_L X, X_2) - D(X,X_2)B(A_L Y, X_1) \\
&+ D(Y,X_2)B(A_L X, X_1) = 0.
\end{aligned}$$

and

$$
\begin{aligned}
&\tfrac{c}{4}\big\{\overline{g}(Y,X_1)D(X,X_2)-\overline{g}(X,X_1)D(Y,X_2)+\overline{g}(Y,X_2)D(X,X_1)\\
&-\overline{g}(X,X_2)D(Y,X_1)+\theta(X)\theta(X_1)D(Y,X_2)-\theta(Y)\theta(X_1)D(X,X_2)\\
&+\theta(X)\theta(X_2)D(Y,X_1)-\theta(Y)\theta(X_2)D(X,X_1)\\
&+\overline{g}(\overline{J}Y,X_1)D(JX-b\eta(X)V^*,X_2)\\
&-\overline{g}(\overline{J}X,X_1)D(JY-b\eta(Y)V^*,X_2)+\overline{g}(\overline{J}Y,X_2)D(JX-b\eta(X)V^*,X_1)\\
&-\overline{g}(\overline{J}X,X_2)D(JY-b\eta(Y)V^*,X_1)-2\overline{g}(\overline{J}X,Y)D(JX_1-b\eta(X_1)V^*,X_2)\\
&-2\overline{g}(\overline{J}X,Y)D(JX_2-b\eta(X_2)V^*,X_1)\big\}-B(X,X_1)D(A_NY,X_2)\\
&+B(Y,X_1)D(A_NX,X_2)-B(X,X_2)D(A_NY,X_1)+B(Y,X_2)D(A_NX,X_1)\\
&-D(X,X_1)D(A_LY,X_2)+D(Y,X_1)D(A_LX,X_2)-D(X,X_2)D(A_LY,X_1)\\
&+D(Y,X_2)D(A_LX,X_1)=0.
\end{aligned}
\tag{3.11}
$$

Replacing X by ξ in (3.10) and (3.11) respectively and noting that M is irrotational we obtain

$$
\begin{aligned}
&\tfrac{c}{4}\big\{b\theta(X_1)B(Y,X_2)+b\theta(X_2)B(Y,X_1)+\overline{g}(\overline{J}Y,X_1)B(J\xi-bV^*,X_2)\\
&-bv^*(X_1)B(JY-bV^*(Y),X_2)+\overline{g}(\overline{J}Y,X_2)B(J\xi-bV^*,X_1)\\
&-bv^*(X_2)B(JY-b\eta(Y)V^*,X_1)-2bv^*(Y)B(JX_1-b\eta(X_1)V^*,X_2)\\
&-2bv^*(Y)B(JX_2-b\eta(X_2)V^*,X_1)\big\}+B(Y,X_1)B(A_N\xi,X_2)\\
&+B(Y,X_2)B(A_N\xi,X_1)+D(Y,X_1)B(A_L\xi,X_2)+D(Y,X_2)B(A_L\xi,X_1)\\
&=0.
\end{aligned}
\tag{3.12}
$$

and

$$
\begin{aligned}
&\tfrac{c}{4}\big\{b\theta(X_1)D(Y,X_2)+b\theta(X_2)D(Y,X_1)+\overline{g}(\overline{J}Y,X_1)D(J\xi-bV^*,X_2)\\
&-bv^*(X_1)D(JY-b\eta(Y)V^*,X_2)+\overline{g}(\overline{J}Y,X_2)D(J\xi-bV^*,X_1)\\
&-bv^*(X_2)D(JY-b\eta(Y)V^*,X_1)-2bv^*(Y)D(JX_1-b\eta(X_1)V^*,X_2)\\
&-2bv^*(Y)D(JX_2-b\eta(X_2)V^*,X_1)\big\}+B(Y,X_1)D(A_N\xi,X_2)\\
&+B(Y,X_2)D(A_N\xi,X_1)+D(Y,X_1)D(A_L\xi,X_2)+D(Y,X_2)D(A_L\xi,X_1)\\
&=0.
\end{aligned}
\tag{3.13}
$$

Substituting $X_2=\xi$ into (3.12) and (3.13) respectively and using $J\xi=0$ then we get

$$\frac{b^2c}{4}\{B(Y,X_1)+3v^*(Y)B(V^*,X_1)\}=0. \tag{3.14}$$

and

$$\frac{b^2c}{4}\{D(Y,X_1)+3v^*(Y)D(V^*,X_1)\}=0. \tag{3.15}$$

Finally, substituting $Y=U^*$ into (3.14), (3.15) and using the equation (3.4) implies that $cB(V^*,X_1)=cD(V^*,X_1)=0$ for any $X_1\in\Gamma(TM)$. Thus, it follows that either $c=0$ or $B(V^*,X_1)=D(V^*,X_1)=0$ for any $X_1\in\Gamma(TM)$. Which completes the proof. □

4. Parallel ascreen half lightlike submanifolds

In this section, we mainly prove some properties of parallel ascreen half lightlike submanifolds of indefinite cosymplectic mainfolds.

Lemma 4.1. *Let $(M, g, S(TM))$ be a ascreen half lightlike submanifold of an indefinite cosymplectic space from $(\overline{M}(c), \overline{g})$, then the local second fundamental form B and D are given respectively by*

$$\begin{aligned}&\frac{bc}{4}\{\overline{g}(X,Z)\theta(Y) - \overline{g}(Y,Z)\theta(X) - \overline{g}(\overline{J}Y,Z)v^*(X)\\&\quad + \overline{g}(\overline{J}X,Z)v^*(Y) + 2\overline{g}(\overline{J}X,Y)v^*(Z)\}\\&=(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\\&\quad + \phi(X)D(Y,Z) - \phi(Y)D(X,Z)\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&\frac{c}{4}\{\overline{g}(\overline{J}Y,Z)\omega^*(X) - \overline{g}(\overline{J}X,Z)\omega^*(Y) - 2\overline{g}(\overline{J}X,Y)\omega^*(Z)\}\\&=(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) - \rho(Y)B(X,Z)\end{aligned} \tag{4.2}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. The proof follows from (2.4), (2.18), (3.2), and (3.6). □

Definition 4.2. *Let $(M, g, S(TM))$ be a ascreen half lightlike submanifold of an indefinite cosymplectic manifold $(\overline{M}, \overline{J}, \zeta, \theta, \overline{g})$. Then M is said to be parallel* (*see* [14]) *if its second fundamental form h satisfies*

$$(\nabla_X h)(Y,Z) = 0 \tag{4.3}$$

for any $X, Y, Z \in \Gamma(TM)$.

Using $h(X,Y) = B(X,Y)N + D(X,Y)L$ and (4.3), then a straightforward calculation gives that M is said to be with the parallel second fundamental form h if and only if

$$(\nabla_X B) + \tau(X)B + \phi(X)D = 0 \text{ and } (\nabla_X D) + \rho(X)B = 0 \tag{4.4}$$

for any $X \in \Gamma(TM)$.

Theorem 4.3. *Let $(M, g, S(TM))$ be a parallel ascreen half lightlike submanifold of an indefinite cosymplectic space from $(\overline{M}(c), \overline{g})$, then $c = 0$.*

Proof. Substituting the second term of (4.4) into (4.2) gives

$$\frac{c}{4}\{\overline{g}(\overline{J}Y,Z)\omega^*(X) - \overline{g}(\overline{J}X,Z)\omega^*(Y) - 2\overline{g}(\overline{J}X,Y)\omega^*(Z)\} = 0 \tag{4.5}$$

for any $X, Y, Z \in \Gamma(TM)$. Replacing Y and Z by ξ and U^* respectively in (4.5) we obtain

$$\frac{bc}{4}\omega^*(X) = 0, \quad \forall X \in \Gamma(TM). \tag{4.6}$$

Thus, substituting $X = W^*$ into (4.6) gives $c = 0$. Which completes the proof. □

Let $(M, g, S(TM))$ be a ascreen half lightlike submanifold of an indefinite cosymplectic manifold. We say that the local second fundamental forms B (resp. D) of M is parallel if $\nabla_X B = 0$ (resp. $\nabla_X D = 0$) for any $X \in \Gamma(TM)$.

Lemma 4.4. *Let $(M, g, S(TM))$ be an ascreen half lightlike submanifold of an indefinite cosymplectic space from $(\overline{M}(c), \overline{g})$. If the local second fundamental form D is parallel with respective to ∇, then $c = 0$. Moreover, in this case either $\rho(\xi) = 0$ or $B = 0$.*

Proof. Suppose that the local second fundamental form D is parallel with respective to ∇, that is, $\nabla_X D = 0$ for any $X \in \Gamma(TM)$. Then it follows from (4.2) that

$$\begin{aligned} &\frac{c}{4}\{\overline{g}(\overline{J}Y,Z)\omega^*(X) - \overline{g}(\overline{J}X,Z)\omega^*(Y) - 2\overline{g}(\overline{J}X,Y)\omega^*(Z)\} \\ =&\rho(X)B(Y,Z) - \rho(Y)B(X,Z) \end{aligned} \tag{4.7}$$

for any $X, Y, Z \in \Gamma(TM)$. Substituting $Z = \xi$ into (4.7) and using $B(X, \xi) = 0$ and $\overline{g}(\xi, W^*) = 0$, we obtain

$$\frac{bc}{4}\{-\upsilon^*(Y)\omega^*(X) + \upsilon^*(X)\omega^*(Y)\} = 0 \tag{4.8}$$

for any $X, Y \in \Gamma(TM)$. Putting $Y = U^*$ into (4.8) and using (2.1) give that $\frac{bc}{4}\omega^*(X) = 0$ for any $X \in \Gamma(TM)$, finally, replacing X by W in this equation gives $c = 0$.

Using $c = 0$ in (4.2) and noting that D is parallel with respect to ∇, then we have

$$\rho(X)B(Y,Z) - \rho(Y)B(X,Z) = 0 \tag{4.9}$$

for any $X, Y, Z \in \Gamma(TM)$. Replacing X by ξ in (4.9) and using $B(X, \xi) = 0$ gives $\rho(\xi)B(Y,Z) = 0$ for $Y, Z \in \Gamma(TM)$. Which completes the proof. □

Theorem 4.5. *Let $(M, g, S(TM))$ be a ascreen half lightlike submanifold of an indefinite cosymplectic space from $(\overline{M}(c), \overline{g})$. If the local second fundamental forms B and D are parallel with respective to ∇, then $c = 0$. Moreover, if $\rho(\xi) \neq 0$ and $\phi(\xi) \neq 0$, then M is $S(TM)$-totally geodesic if and only if $\overline{\nabla}_X\xi \in \Gamma(TM)$ for any $X \in \Gamma(S(TM))$.*

Proof. Using the parallelism of two local second fundamental forms B and D and Lemma 4.4, it follows from (4.1) that

$$\tau(X)B(Y,Z) - \tau(Y)B(X,Z) + \phi(X)D(Y,Z) - \phi(Y)D(X,Z) = 0 \tag{4.10}$$

for any $X, Y, Z \in \Gamma(TM)$. If $\rho(\xi) \neq 0$, then from Lemma 4.4 we see that $B = 0$, thus

$$\phi(X)D(Y,Z) - \phi(Y)D(X,Z) = 0 \tag{4.11}$$

for any $X, Y, Z \in \Gamma(TM)$. Replacing X by ξ in (4.11) and using (2.14) we obtain

$$\phi(\xi)D(Y,Z) = -\phi(Y)\phi(Z) \tag{4.12}$$

for any $Y, Z \in \Gamma(TM)$. Then the proof follows from (4.12) and Lemma 4.4. □

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Department of Mathematics, South China University of Technology, Guangzhou 510641, Guangdong, P. R. China
E-mail address: wangwj072@163.com

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, Liaoning, P. R. China
E-mail address: ximinliu@dlut.edu.cn