

## A NOTE ON INEXTENSIBLE FLOWS OF CURVES IN $E^n$

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**ABSTRACT.** In this paper, we investigate the general formulation for inextensible flows of curves in  $E^n$ . The necessary and sufficient conditions for inextensible curve flow are expressed as a partial differential equation involving the curvatures.

### 1. INTRODUCTION

Flow of curves has a very important place in the field of industry such as modeling ship hulls, buildings, airplane wings, garments, ducts, automobile parts. Moreover Chirikjian and Burdick describe the kinematics of hyperredundant (or "serpentine") robot as the flow of plane curve [1]. The flow of a curve is said to be inextensible if, its arclength is preserved. Firstly, Kwon and Park studied inextensible flows of curves and developable surfaces, which its arclength is preserved, in Euclidean 3-space [10].

Inextensible curve flows conduce to motions in which no strain energy is induced in physical science. For example, the swinging motion of a cord of fixed length can be represented by this type of curve flows. Also inextensible flows of curves have great importance in computer vision and computer animation moreover structural mechanics (see [3], [9], [12]).

There are many studies in the literature on plane curve flows, especially on evolving curves in the direction of their curvature vector field (referred to by various names such as "curve shortening", flow by curvature" and "heat flow"). Among them, perhaps, most important case (but already a very subtle one) is the curve-shortening flow in the plane studied by Gage and Hamilton [4] and Grayson [6]. Another paper about curve flows was studied by Chirikjian [2].

Inextensible flows of curves have been studied in many different spaces. For example, Gürbüz have examined inextensible flows of spacelike, timelike and null curves in [7]. After this work Öğrenmiş et al. have studied inextensible curves

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in Galilean space [13] and Yıldız et al. have studied inextensible flows of curves according to darbox frame in Euclidean 3-space [14], etc.

In the present paper following [10], [7], [13], [14], we study inextensible flows of curves in  $E^n$ . Further, necessary and sufficient conditions for an inextensible curve flow are expressed as a partial differential equation involving the curvatures.

### 2. PRELIMINARY

To meet the requirements in the next sections, the basic elements of the theory of curves in the Euclidean n-space  $E^n$  are briefly presented in this section (A more complete elementary treatment can be found in [5], [8]).

Let  $\alpha : I \subset R \rightarrow E^n$  be an arbitrary curve in  $E^n$ . Recall that the curve  $\alpha$  is said to be a unit speed curve (or parameterized by arclength functions) if  $\langle \alpha'(s), \alpha'(s) \rangle = 1$ , where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product of given by

$$\langle X, Y \rangle = \sum_{i=1}^n x_i y_i,$$

for each  $X = (x_1, x_2, \dots, x_n), Y = (y_1, y_2, \dots, y_n) \in R^n$ . In particular, norm of a vector  $X \in R^n$  is given by  $\|X\| = \sqrt{\langle X, X \rangle}$ . Let  $\{V_1, V_2, \dots, V_n\}$  be the moving Frenet frame along the unit speed curve  $\alpha$ , where  $V_i (i = 1, 2, \dots, n)$  denotes the  $i^{th}$  Frenet vector field. Then Frenet formulas are given by

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ \vdots \\ V_{n-2}' \\ V_{n-1}' \\ V_n' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & k_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -k_{n-2} & 0 & k_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_{n-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{n-2} \\ V_{n-1} \\ V_n \end{bmatrix},$$

where  $k_i (i = 1, 2, \dots, n)$  denotes the  $i^{th}$  curvature function of the curve [5], [8]. If all of the curvatures  $k_i (i = 1, 2, \dots, n)$  of the curve vanish nowhere in  $I \subset R$ , it is called a non-degenerate curve.

### 3. INEXTINSIBLE FLOWS OF CURVE IN $E^n$

Throughout this paper, unless otherwise stated, we assume that

$$\alpha : [0, l] \times [0, w] \rightarrow E^n$$

is a one parameter family of smooth curves in  $E^n$ , where  $l$  is the arclength of the initial curve. Let  $u$  be the curve parameterization variable,  $0 \leq u \leq l$ . If the speed curve  $\alpha$  is denoted by  $v = \left\| \frac{d\alpha}{du} \right\|$  then the arclength of  $\alpha$  is

$$s(u) = \int_0^u \left\| \frac{\partial \alpha}{\partial u} \right\| du = \int_0^u v du.$$

The operator  $\frac{\partial}{\partial s}$  is given by

$$(3.1) \quad \frac{\partial}{\partial s} = \frac{1}{v} \frac{\partial}{\partial u}.$$

In this case; the arclength parameter is  $ds = v du$ .

**Definition 3.1.** Any flow of the curve can be expressed following form:

$$\frac{\partial \alpha}{\partial t} = \sum_{i=1}^n f_i V_i$$

where  $f_i$  denotes the  $i^{\text{th}}$  scalar speed of the curve. Let the arclength variation be

$$(3.2) \quad s(u, t) = \int_0^u v du.$$

In the Euclidean space the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$(3.3) \quad \frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial v}{\partial t} du = 0, \quad u \in [0, l].$$

**Definition 3.2.** A curve evolution  $\alpha(u, t)$  and its flow  $\frac{\partial \alpha}{\partial t}$  are said to be inextensible if

$$(3.4) \quad \frac{\partial}{\partial t} \left\| \frac{\partial \alpha}{\partial u} \right\| = 0.$$

Now, we research the necessary and sufficient condition for inelastic curve flow. For this reason, we need to the following Lemma.

**Lemma 3.1.** Let  $\frac{\partial \alpha}{\partial t} = \sum_{i=1}^n f_i V_i$  be a smooth flow of the curve  $\alpha$ . The flow is inextensible then

$$(3.5) \quad \frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_2 v k_1.$$

*Proof.* Since  $\frac{\partial}{\partial u}$  and  $\frac{\partial}{\partial t}$  are commutative and  $v^2 = \langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \rangle$ , we have

$$\begin{aligned} 2v \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t} \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \right\rangle \\ &= 2 \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial}{\partial t} \left( \sum_{i=1}^n f_i V_i \right) \right\rangle \\ &= 2 \left\langle v V_1, \sum_{i=1}^n \frac{\partial f_i}{\partial u} V_i + \sum_{i=1}^n f_i \frac{\partial V_i}{\partial u} \right\rangle \\ &= 2 \left\langle v V_1, \frac{\partial f_1}{\partial u} V_1 + f_1 \frac{\partial V_1}{\partial u} + \dots + \frac{\partial f_n}{\partial u} V_n + f_n \frac{\partial V_n}{\partial u} \right\rangle \\ &= 2 \left\langle v V_1, \frac{\partial f_1}{\partial u} V_1 + f_1 v k_1 V_2 + \dots + \frac{\partial f_n}{\partial u} V_n - f_n v k_{n-1} V_{n-1} \right\rangle \\ &= 2 \left( \frac{\partial f_1}{\partial u} - f_2 v k_1 \right). \end{aligned}$$

This leads us to the consequence that

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u} - f_2 v k_1.$$

□

**Theorem 3.1.** *Let  $\{V_1, V_2, \dots, V_n\}$  be the moving Frenet frame of the curve  $\alpha$  and  $\frac{\partial \alpha}{\partial t} = \sum_{i=1}^n f_i V_i$  be a smooth flow of  $\alpha$  in  $E^n$ . Then the flow is inextensible if and only if*

$$(3.6) \quad \frac{\partial f_1}{\partial s} = f_2 k_1.$$

*Proof.* Suppose that the curve flow is inextensible. From equations (3.3) and (3.5) for  $u \in [0, l]$ , we see that

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial v}{\partial t} du = \int_0^u \left( \frac{\partial f_1}{\partial u} - f_2 v k_1 \right) du = 0.$$

Thus, it can be see that

$$\frac{\partial f_1}{\partial u} - f_2 v k_1 = 0.$$

Considering the last equation and (3.1), we reach

$$\frac{\partial f_1}{\partial s} = f_2 k_1.$$

Conversely, following similar way as above, the proof is completed.  $\square$

Now, we restrict ourselves to arclength parameterized curves. That is,  $v = 1$  and the local coordinate  $u$  corresponds to the curve arclength  $s$ . We require the following Lemma

**Lemma 3.2.** *Let  $\{V_1, V_2, \dots, V_n\}$  be the moving Frenet frame of the curve  $\alpha$ . Then, the derivative of  $\{V_1, V_2, \dots, V_n\}$  with respect to  $t$  are*

$$\begin{aligned} \frac{\partial V_1}{\partial t} &= \left[ \sum_{i=2}^{n-1} \left( f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) V_i \right] + \left( f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) V_n, \\ \frac{\partial V_j}{\partial t} &= - \left( f_{j-1} k_{j-1} + \frac{\partial f_j}{\partial s} - f_{j+1} k_j \right) V_1 + \left[ \sum_{\substack{k=2 \\ k \neq j}}^n \Psi_{kj} V_k \right], \quad 1 < j < n, \\ \frac{\partial V_n}{\partial t} &= - \left( f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) V_1 + \left[ \sum_{k=2}^{n-1} \Psi_{kn} V_k \right], \end{aligned}$$

where  $\Psi_{kj} = \left\langle \frac{\partial V_j}{\partial t}, V_k \right\rangle$  and  $\Psi_{kn} = \left\langle \frac{\partial V_n}{\partial t}, V_k \right\rangle$ .

*Proof.* For  $\frac{\partial}{\partial t}$  and  $\frac{\partial}{\partial s}$  are commutative, it seen that

$$\begin{aligned} \frac{\partial V_1}{\partial t} &= \frac{\partial}{\partial t} \left( \frac{\partial \alpha}{\partial s} \right) = \frac{\partial}{\partial s} \left( \frac{\partial \alpha}{\partial t} \right) = \frac{\partial}{\partial s} \left( \sum_{i=1}^n f_i V_i \right) = \sum_{i=1}^n \frac{\partial f_i}{\partial s} V_i + \sum_{i=1}^n f_i \frac{\partial V_i}{\partial s} \\ &= \frac{\partial f_1}{\partial s} V_1 + f_1 \frac{\partial V_1}{\partial s} + \frac{\partial f_2}{\partial s} V_2 + f_2 \frac{\partial V_2}{\partial s} + \dots + \frac{\partial f_n}{\partial s} V_n + f_n \frac{\partial V_n}{\partial s} \\ &= \frac{\partial f_1}{\partial s} V_1 + f_1 k_1 V_2 + \frac{\partial f_2}{\partial s} V_2 + f_2 (-k_1 V_1 + k_2 V_3) + \dots + \frac{\partial f_n}{\partial s} V_n - f_n k_{n-1} V_{n-1}. \end{aligned}$$

Substituting the equation (3.6) into the last equation, we have

$$\frac{\partial V_1}{\partial t} = \left[ \sum_{i=2}^{n-1} \left( f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) V_i \right] + \left( f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) V_n.$$

Also, since  $\langle V_1, V_j \rangle = 0$  for  $1 < j < n$ , we get;

$$(3.7) \quad \begin{aligned} 0 &= \frac{\partial}{\partial t} \langle V_1, V_j \rangle = \left\langle \frac{\partial V_1}{\partial t}, V_j \right\rangle + \left\langle V_1, \frac{\partial V_j}{\partial t} \right\rangle \\ &= \left( f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) + \left\langle V_1, \frac{\partial V_j}{\partial t} \right\rangle. \end{aligned}$$

From (3.7), we have obtain

$$\frac{\partial V_j}{\partial t} = - \left( f_{j-1} k_{j-1} + \frac{\partial f_j}{\partial s} - f_{j+1} k_j \right) V_1 + \left[ \sum_{\substack{k=2 \\ k \neq j}}^n \Psi_{kj} V_k \right].$$

Lastly, considering  $\langle V_1, V_n \rangle = 0$  and following similar way as above, we reach

$$\frac{\partial V_n}{\partial t} = - \left( f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) V_1 + \left[ \sum_{k=2}^{n-1} \Psi_{kn} V_k \right].$$

□

**Theorem 3.2.** *Suppose that the curve flow  $\frac{\partial \alpha}{\partial t} = \sum_{i=1}^n f_i V_i$  is inextensible. Then the following system of partial differential equations holds:*

$$\begin{aligned} \frac{\partial k_1}{\partial t} &= f_2 k_1^2 + f_1 \frac{\partial k_1}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - 2 \frac{\partial f_3}{\partial s} k_2 - f_3 \frac{\partial k_2}{\partial s} - f_2 k_2^2 - f_4 k_2 k_3 \\ \frac{\partial k_2}{\partial t} &= \frac{\partial \Psi_{(3)(2)}}{\partial s} - \Psi_{(4)(2)} k_3 + f_2 k_1 k_2 + \frac{\partial f_3}{\partial s} k_1 - f_4 k_1 k_3, \\ \frac{\partial k_i}{\partial t} &= \frac{\partial \Psi_{(i-1)(i)}}{\partial s} - \Psi_{(i+2)(i)} k_{i+1} + \Psi_{(i+1)(i-1)} k_{i-1}, \quad 2 < i < n-1, \\ \frac{\partial k_{n-1}}{\partial t} &= - \frac{\partial \Psi_{(n-1)n}}{\partial s} - \Psi_{(n-2)n} k_{n-2}. \end{aligned}$$

*Proof.* Since  $\frac{\partial}{\partial s} \frac{\partial V_1}{\partial t} = \frac{\partial}{\partial t} \frac{\partial V_1}{\partial s}$ , we get

$$\begin{aligned} \frac{\partial}{\partial s} \frac{\partial V_1}{\partial t} &= \frac{\partial}{\partial s} \left[ \sum_{i=2}^{n-1} \left( f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) V_i + \left( f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) V_n \right] \\ &= \sum_{i=2}^{n-1} \left[ \left( \frac{\partial f_{i-1}}{\partial s} k_{i-1} + f_{i-1} \frac{\partial k_{i-1}}{\partial s} + \frac{\partial^2 f_i}{\partial s^2} - \frac{\partial f_{i+1}}{\partial s} k_i - f_{i+1} \frac{\partial k_i}{\partial s} \right) V_i \right] \\ &\quad + \sum_{i=2}^{n-1} \left[ \left( f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) \frac{\partial V_i}{\partial s} \right] \\ &\quad + \left( \frac{\partial f_{n-1}}{\partial s} k_{n-1} + f_{n-1} \frac{\partial k_{n-1}}{\partial s} + \frac{\partial^2 f_n}{\partial s^2} \right) V_n + \left( f_{n-1} k_{n-1} + \frac{\partial f_n}{\partial s} \right) \frac{\partial V_n}{\partial s} \end{aligned}$$

while

$$\frac{\partial}{\partial t} \frac{\partial V_1}{\partial s} = \frac{\partial}{\partial t} (k_1 V_2) = \frac{\partial k_1}{\partial t} V_2 + k_1 \frac{\partial V_2}{\partial t}.$$

Thus, from the both of above two equations, we reach

$$\frac{\partial k_1}{\partial t} = f_2 k_1^2 + f_1 \frac{\partial k_1}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - 2 \frac{\partial f_3}{\partial s} k_2 - f_3 \frac{\partial k_2}{\partial s} - f_2 k_2^2 - f_4 k_2 k_3.$$

For  $1 < i < n$ , noting that  $\frac{\partial}{\partial s} \frac{\partial V_i}{\partial t} = \frac{\partial}{\partial t} \frac{\partial V_i}{\partial s}$ , it is seen that

$$\begin{aligned} \frac{\partial}{\partial s} \frac{\partial V_i}{\partial t} &= \frac{\partial}{\partial s} \left[ - \left( f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) V_1 + \sum_{k=2}^n \Psi_{kj} V_k \right] \\ &= - \left( \frac{\partial f_{i-1}}{\partial s} k_{i-1} + f_{i-1} \frac{\partial k_{i-1}}{\partial s} + \frac{\partial^2 f_i}{\partial s^2} - \frac{\partial f_{i+1}}{\partial s} k_i - f_{i+1} \frac{\partial k_i}{\partial s} \right) V_1 \\ &\quad + \left( f_{i-1} k_{i-1} + \frac{\partial f_i}{\partial s} - f_{i+1} k_i \right) \frac{\partial V_1}{\partial s} + \sum_{\substack{k=2 \\ k \neq i}}^n \left( \frac{\partial \Psi_{ki}}{\partial s} V_k + \Psi_{ki} \frac{\partial V_k}{\partial s} \right), \end{aligned}$$

while

$$\frac{\partial}{\partial t} \frac{\partial V_i}{\partial s} = \frac{\partial}{\partial t} (-k_{i-1} V_{i-1} + k_i V_{i+1}) = - \frac{\partial k_{i-1}}{\partial t} V_{i-1} - k_{i-1} \frac{\partial V_{i-1}}{\partial t} + \frac{\partial k_i}{\partial t} V_{i+1} + k_i \frac{\partial V_{i+1}}{\partial t}.$$

Thus, we obtain

$$\begin{aligned} \frac{\partial k_2}{\partial t} &= \frac{\partial \Psi_{(3)(2)}}{\partial s} - \Psi_{(4)(2)} k_3 + f_2 k_1 k_2 + \frac{\partial f_3}{\partial s} k_1 - f_4 k_1 k_3 \\ \frac{\partial k_i}{\partial t} &= \frac{\partial \Psi_{(i-1)(i)}}{\partial s} - \Psi_{(i+2)(i)} k_{i+1} + \Psi_{(i+1)(i-1)} k_{i-1}, \quad 2 < i < n - 1 \end{aligned}$$

and

$$\frac{\partial k_{n-1}}{\partial t} = - \frac{\partial \Psi_{(n-1)n}}{\partial s} - \Psi_{(n-2)n} k_{n-2}.$$

No other new formulas are obtained from  $\frac{\partial}{\partial s} \frac{\partial V_n}{\partial t} = \frac{\partial}{\partial t} \frac{\partial V_n}{\partial s}$ . □

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