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CONTACT CR-SUBMANIFOLDS OF $N(k)-$CONTACT METRIC MANIFOLDS

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Abstract. The object of the present paper is to study contact CR-submanifolds of $N(k)-$contact metric manifolds. Contact CR-submanifold is a generalization of invariant and anti-invariant submanifolds. The integrability criterions of the distributions have been investigated. Finally it has been shown that a totally contact umbilical submanifold becomes totally contact geodesic under certain circumstances.

1. Introduction

As a generalization of invariant and anti-invariant submanifolds of contact manifolds, contact CR-submanifolds have been introduced by A. Bejancu, N. Papaghiuc[1] and simultaneously by Yano, Kon[12]. Since then many results have been obtained on geometry of CR-submanifolds. C. Calin extensively studied integrability and geodesic properties of the distributions of contact CR-submanifold of quasi-Sasakian manifolds([4][5][6]), quasi K-Sasakian manifolds[8], cosymplectic manifolds[3], trans-Sasakian manifolds[7] and various other manifolds.

Invariant submanifolds always play an important role in studying various other subjects, like dynamical systems, linear and nonlinear autonomous systems etc. and so is anti-invariant submanifolds of higher codimension. CR-submanifolds being a generalization of these two, make the study more interesting.

On the other hand, through the works of Ch. Baikoussis, D. E. Blair and Th. Koufogiorgos [9] a new class of non-Sasakian contact manifolds has evolved which is termed as $N(k)-$contact metric manifolds. In the present paper we have studied contact CR-submanifolds of $N(k)-$contact manifolds. The paper is organized as follows: After Preliminaries in Section 3 we have discussed integrability criterions of the distributions. It has been proved that if TM is invariant under h, then the normal space and also all the distributions remains invariant under h. A Lemma has been proved which is often used in later results. It has been shown that D is not integrable, but $D^{\perp}$ and $D^{\perp} \oplus \xi$ are integrable under certain conditions. In Section 4, totally contact umbilical contact CR-submanifolds have been considered, and it has been proved that they reduce to totally contact geodesic submanifolds.

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2. Preliminaries

An $(2n+1)$-dimensional manifold M^{2n+1} is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$(a)\ \phi^2 = -I + \eta \otimes \xi,\ \ (b)\ \eta(\xi) = 1,\ \ (c)\ \phi\xi = 0,\ \ (d)\ \eta \circ \phi = 0. \tag{2.1}$$

An almost contact structure is said to be normal if the corresponding almost complex structure J on the product manifold $M^{2n+1} \times \mathbf{R}$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ is integrable, where X is tangent to M, t is the coordinate of $\mathbf{R}$ and f is a smooth function on $M \times \mathbf{R}$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (2.2) it can be easily seen that

$$(a) g(X, \xi) = \eta(X), (b) g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

for all vector fields X, Y. An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{2.4}$$

for all vector fields X, Y. The 1-form η is then a contact form and ξ is its characterstic vector field. We define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie-differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. A normal contact metric manifold is a Sasakian manifold. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ [2]. On the other hand, on a Sasakian manifold the following holds:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.5}$$

The k-nullity distribution $N(k)$ of a Riemannian manifold M^{2n+1} [2] is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

k being a constant. If the characterstic vector field $\xi \in N(k)$, then we call a contact metric manifold an $N(k)$-contact metric manifold. If $k = 1$, then $N(k)$-contact metric manifold is Sasakian and if $k = 0$, then $N(k)$-contact metric manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$. If $k < 1$, the scalar curvature is $r = 2n(2n - 2 + k)$.

In [9], $N(k)$-contact metric manifold were studied in some detail. In $N(k)$-contact metric manifold the following relations hold:

$$h^2 = (k-1)\phi^2, \quad k \leq 1, \tag{2.6}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.7}$$

$$\nabla_X \xi = -X - \phi hX. \tag{2.8}$$

Let $\bar{M}$ be an almost contact metric manifold and M be a submanifold of $\bar{M}$ such that $\xi \in TM$. We say that M is a CR-submanifold of $\bar{M}$ if there exists two distributions D and $D^\perp$ such that

$$TM = D \oplus D^\perp \oplus < \xi >$$

and $\phi X \in TM, \phi Y \in T^\perp M$, for all $X \in D, Y \in D^\perp$, where TM and $T^\perp M$ denote the tangent and normal space of M respectively.

If $\bar{\nabla}$ and ∇ denote the Levi-Civita connections in $\bar{M}$ and M respectively, then for $X, Y \in TM, N \in T^{\perp}M$we have the Gauss and Weingarten formulae as

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y) \tag{2.9}$$

$$\bar{\nabla}_X N = \nabla_X N - A_N X, \tag{2.10}$$

where $B(X, Y), A_N X$ are second fundamental forms connected by the relation

$$g(B(X, Y), N) = g(A_N X, Y). \tag{2.11}$$

Definition 2.1. *We say that a CR-submanifold M is totally contact umbilical if there exists a normal vector field H such that*

$$B(X, Y) = g(\phi X, \phi Y)H + \eta(X)B(Y, \xi) + \eta(Y)B(X, \xi), \forall X, Y \in TM. \tag{2.12}$$

We say that M is totally contact geodesic if $H = 0$.

3. Integrability of the Distributions

It is obvious that $\phi D^{\perp}$ is anti-invariant part in $T^{\perp}M$. We can easily verify that

$$T^{\perp}M = \phi D^{\perp} \oplus \mu,$$

where by μ we denote the invariant part of $T^{\perp}M$.

Proposition 3.1. *If TM is invariant under h, then $D, D^{\perp}, T^{\perp}M, \mu$ and $\phi D^{\perp}$ all are invariant under h.*

Proof. Let $X \in D, Y \in D^{\perp}, Z \in T^{\perp}M, W \in \mu, N \in \phi D^{\perp}$.

Then, $\phi hX = -h\phi X \in TM$. And $g(hX, \xi) = g(X, h\xi) = 0$, for all $X \in D$. Hence, D is invariant under h.

Again, $g(hY, X) = g(Y, hX) = 0$ and $g(hY, \xi) = g(Y, h\xi) = 0$ together imply $D^{\perp}$ is invariant under h.

Since, TM is invariant under h, we obtain, $g(hZ, T) = g(Z, hT) = 0$, for all $T \in TM$. So, $T^{\perp}M$ is invariant under h.

Now, $\phi hW = -h\phi W \in T^{\perp}M$. Hence, μ is invariant under h.

Finally, $g(W, hN) = g(N, hW) = 0$, since μ is invariant under h.

Hence we have the result. □

Lemma 3.1. *Let $X \in D, Z \in D^{\perp}$. If TM is invariant under h, then the followings hold:*

(i) $\nabla_X \xi = -\phi X - \phi hX, B(X, \xi) = 0,$
(ii) $\nabla_Z \xi = 0, B(Z, \xi) = -\phi Z - \phi hZ,$
(iii) $\nabla_\xi \xi = 0, B(\xi, \xi) = 0,$
(iv) $\nabla_\xi X \in D,$
(v) $\nabla_\xi Z \in D^{\perp}.$

Proof. We have, $-\phi T - \phi hT = \bar{\nabla}_T \xi = \nabla_T \xi + B(T, \xi)$, for all $T \in TM$.

Hence, (i),(ii) and (iii) are obvious from Proposition 3.1.

Now,

$$\begin{aligned} \phi\nabla_\xi X &= \phi(\bar{\nabla}_\xi X - B(X, \xi)) \\ &= \bar{\nabla}_\xi \phi X, [since, (\bar{\nabla}_\xi \phi)X = 0] \\ &= \nabla_\xi \phi X + B(\xi, \phi X) \\ &= \nabla_\xi \phi X. \end{aligned} \tag{3.1}$$

From 3.1 we obtain, $Q\nabla_\xi X = 0$.

Also, $g(\nabla \xi X, \xi) = -g(X, \nabla_\xi \xi) = 0$.

Hence, $\nabla_\xi X \in D$.

Now, $g(\nabla_\xi Z, X) = -g(Z, \nabla_\xi X) = 0$, since $\nabla_\xi X \in D$.
Also, $g(\nabla_\xi Z, \xi) = -g(Z, \nabla_\xi \xi) = 0$.
Hence, $\nabla_\xi Z \in D^\perp$. □

Lemma 3.2. *In a contact CR-submanifold of a $N(k)-$contact manifold, for $Z, W \in D^\perp$,*

$$A_{\phi W} Z = A_{\phi Z} W.$$

Proof. Let $X \in TM$. Then, by (2.11)

$$\begin{aligned} g(A_{\phi Z} W, X) &= g(B(X, W), \phi Z) \\ &= g(\overline{\nabla}_X W, \phi Z) \\ &= -g(\phi \overline{\nabla}_X W, Z) \\ &= -g(\overline{\nabla}_X \phi W, Z) \\ &= g(A_{\phi W} X, Z) \\ &= g(B(X, Z), \phi W) \\ &= g(A_{\phi W} Z, X), \end{aligned} \tag{3.2}$$

which proves the Lemma. □

Theorem 3.1. *D is not integrable in a proper contact CR-submanifold M of $N(k)-$contact manifolds provided TM remains invariant under h.*

Proof. Let $X, Y \in D$. Then,

$$\begin{aligned} g([X, Y], \xi) &= g(\nabla_X Y - \nabla_Y X, \xi) \\ &= -g(Y, \nabla_X \xi) + g(X, \nabla_Y \xi) \\ &= -g(Y, -\phi X - \phi h X) + g(X, -\phi Y - \phi h Y), by\ Lemma\ 3.1 \\ &= g(Y, \phi X) + g(Y, \phi X) + g(Y, \phi h X) - g(X, \phi h Y) \\ &= 2g(Y, \phi X) + g(Y, \phi h X) - g(Y, \phi h X) \\ &= 2g(Y, \phi X) \end{aligned} \tag{3.3}$$

Now, if D is to be integrable, then $[X, Y] \in D$. So, from (3.3) we obtain, $g(Y, \phi X) = 0$, which implies $D = \{0\}$. Hence the theorem is proved. □

Theorem 3.2. *In a contact CR-submanifold M of a $N(k)-$contact manifold $D^\perp$ is integrable provided TM is invariant under h.*

Proof. Let $Z, W \in D^\perp, X \in D$.

$$\begin{aligned} g([Z, W], \xi) &= g(\nabla_Z W - \nabla_W Z, \xi) \\ &= -g(W, \nabla_Z \xi) + g(Z, \nabla_W \xi) \\ &= 0, by\ Lemma\ 3.1 \end{aligned} \tag{3.4}$$

Also,

$$\begin{aligned} g([Z, W], \phi X) &= g(\bar{\nabla}_Z W, \phi X) - g(\bar{\nabla}_W Z, \phi X) \\ &= -g(W, \bar{\nabla}_Z \phi X) + g(Z, \bar{\nabla}_W \phi X) \\ &= -g(W, \phi \bar{\nabla}_Z X) + g(Z, \phi \bar{\nabla}_W X) \\ &= g(\phi W, \bar{\nabla}_Z X) - g(\phi Z, \bar{\nabla}_W X) \\ &= g(\phi W, B(X, Z)) - g(\phi Z, B(X, W)) \\ &= g(A_\phi W Z, X) - g(A_\phi Z W, X) \\ &= 0, by\ Lemma\ 3.2 \end{aligned} \tag{3.5}$$

Hence the theorem is proved. □

Theorem 3.3. *In a contact CR-submanifold M of a $N(k)$-contact manifold, if TM remains invariant under h, then $D^{\perp}\oplus <\xi>$ is integrable.*

Proof. Let $Z \in D^{\perp}, X \in D$. Then,

$$\begin{aligned} g([Z,\xi],X) &= g(\nabla_Z\xi,X) - g(\nabla_\xi Z,X) \\ &= 0, by\ Lemma\ 3.1. \end{aligned} \tag{3.6}$$

Thus with the help of Theorem 3.2, we conclude that $D^{\perp}\oplus <\xi>$ is integrable. □

4. Totally contact Umbilical and Totally Contact Geodesic Submanifolds

Lemma 4.1. *If M is a totally contact umbilical proper contact CR-submanifold of a $N(k)-$contact manifold, then either dim $D^{\perp} = 1$ or the normal vector $H \in \mu$.*

Proof. Let $X \in TM, Y \in D^{\perp}$.

Since M is totally contact umbilical, we obtain,

$$g(B(X,X),\phi Y) = g(\phi X,\phi X)g(H,\phi Y) + 2\eta(X)g(B(X,\xi),\phi Y). \tag{4.1}$$

Now, if $X \in <\xi>$, then by Lemma 3.1 $B(X,\xi) = 0$, and if $X \notin <\xi>$, then $\eta(X) = 0$. So, from (4.1), we obtain,

$$g(B(X,X),\phi Y) = [g(X,X) - \eta(X)^2]g(H,\phi Y). \tag{4.2}$$

If dim $D^{\perp} > 1$, there exists unit vector $Z \in D^{\perp}$ orthogonal to Y.

Then, from (4.1) we get

$$\begin{aligned} g(H,\phi Y) &= g(B(Z,Z),\phi Y) \\ &= g(A_{\phi Y}Z,Z) \\ &= -g(\bar{\nabla}_Z\phi Y,Z) \\ &= -g(Y,\phi\bar{\nabla}_Z Z) \\ &= -g(Y,\bar{\nabla}_Z\phi Z) \\ &= g(Y,A_{\phi Z}Z) \\ &= g(B(Y,Z),\phi Z) \\ &= g(\phi Y,\phi Z)g(H,\phi Z), by (4.1) \\ &= 0, since\ Z \perp Y. \end{aligned} \tag{4.3}$$

Hence the result is proved. □

Theorem 4.1. *Let M be a totally contact umbilical proper contact CR-submanifold of a $N(k)-$contact manifold with dim $D^{\perp} > 1$. Then M is totally contact geodesic.*

Proof. Since dim $D^{\perp} > 1$, from Lemma 4.1, we have $H \perp \phi D^{\perp}$, for all $X \in TM$.

Now,

$$\begin{aligned} 0 &= g((\bar{\nabla}_X\phi)\phi X,H) \\ &= -g(\phi X,(\bar{\nabla}_X\phi)H), [since,\ H \perp \phi D^{\perp}] \\ &= -g(\phi X,\bar{\nabla}_X\phi H - \phi\bar{\nabla}_X H) \\ &= g(\phi X,A_{\phi H}X) - g(X,A_H X) \\ &= g(\phi H,B(X,\phi X)) - g(H,B(X,X)) \\ &= -g(H,B(X,X)) \\ &= -g(X,X)g(H,H), \forall X \in D\oplus D^{\perp}. \end{aligned} \tag{4.4}$$

Hence, $H = 0$. Therefore, M is totally contact geodesic. □

References

[1] Bejancu A. and Papaghiuc N., Semi-invariant submanifolds of a Sasakian manifold, Ann. St. Univ. "Al.I.Cuza" Iasi 27(1981), 163-170.

[2] Blair D.E., Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics, 509 Springer-Verlag, Berlin, 2003.

[3] Calin C., CR-submanifolds of a nearly cosymplectic manifold, Publ. Math. Debrecen 45 (1994), 225-237.

[4] Calin C., Contact CR-submanifold of quasi-Sasakian manifolds, Bull. Math. Soc. Sci. Math. Roumanie 36(84)(1992),217-226.

[5] Calin C., Geometry of leaves on a CR-submanifold of a quasi-Sasakian manifold, Bul. I. P. Iasi Tom XL(XLIV)Fasc.1-4(1994),37-44.

[6] Calin C., Normal contact CR-submanifolds of a quasi-Sasakian manifold, Publ. Math. Debrecen 53/3-4 (1998),257-270.

[7] Calin C., Normal contact CR-submanifolds of a trans-Sasakian manifold,Bul. Inst. Poli. Din Iasi Tom XLII(XLVI) Fasc. 1-2,1996,9-15.

[8] Calin C., Semi-invariant submanifolds of a quasi K-Sasakian manifolds,Ann. St. Univ. "Al.I.Cuza" Iasi Tom XLII,(1996),95-103.

[9] Ch. Baikoussis, D. E. Blair and Th. Koufogiorgos, A decomposition of the curvature tensor of a contact manifold satisfying $R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y)$, Mathematics Technical Report, University of Ioanniana, No.204,June 1992.

[10] Chen B.Y., Geometry of submanifolds, Pure and Applied Mathematics, No. 22. Marcel Dekker, Inc. New York, 1973.

[11] Tanno S., Ricci curvatures of contact Riemannian manifolds, Tohoku Mathematical Journal 40(1988), 441-448.

[12] Yano K. and Kon M., CR-submanifolds of Kaehler and Sasakian manifolds, Birkhauser,Boston,1983.

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