INVARIANT TENSORS RELATED WITH NATURAL CONNECTIONS FOR A CLASS RIEMANNIAN PRODUCT MANIFOLDS

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ABSTRACT. Some invariant tensors in two Naveira classes of Riemannian product manifolds are considered. These tensors are related with natural connections, i.e. linear connections preserving the Riemannian metric and the product structure.

INTRODUCTION

A Riemannian almost product manifold (M, P, g) is a differentiable manifold M for which almost product structure P is compatible with the Riemannian metric g such that an isometry is induced in any tangent space of M.

The systematic development of the theory of Riemannian almost product manifolds was started by K. Yano in [15].

In [11] A. M. Naveira gave a classification of Riemannian almost product manifolds with respect to the covariant differentiation ∇P, where ∇ is the Levi-Civita connection of g. This classification is very similar to the Gray-Hervella classification in [1] of almost Hermitian manifolds.

M. Staikova and K. Gribachev gave in [13] a classification of the Riemannian almost product manifolds with $\operatorname{tr} P = 0$. In this case the manifold M is even-dimensional.

For the class $\mathcal{W}_1$ of the Staikova-Gribachev classification is valid $\mathcal{W}_1 = \overline{\mathcal{W}}_3 \oplus \overline{\mathcal{W}}_6$, where $\overline{\mathcal{W}}_3$ and $\overline{\mathcal{W}}_6$ are classes of the Naveira classification. In some sense these manifolds have dual geometries.

In [10], a connection ∇' on a Riemannian almost product manifold (M, P, g) is called natural if $\nabla'P = \nabla'g = 0$. In [9], a tensor on such a manifold is called a Riemannian P-tensor if it has properties similar to the properties of the Kähler tensor in Hermitian geometry. In [4], a Riemannian P-tensor K is defined on $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ by the curvature tensor R of ∇ and the structure P.

In the present work[1], we study manifolds (M, P, g) from the class $\overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ for which the curvature tensor of each natural connection is a Riemannian P-tensor.

We consider three tensors $B(L)$, $A(L)$ and $C(L)$ determined by arbitrary Riemannian P-tensor L, where $B(L)$ is the Bochner tensor introduced in [13]. We prove that $B(R') = B(K)$ for arbitrary natural connection ∇' in Theorem 3.1. In Theorem 4.1 we prove that $A(R') = A(K)$ if ∇' is the canonical connection introduced in [10]. In Theorem 5.1 we prove that $C(R') = C(K)$ if ∇' is a natural connection with parallel torsion. Moreover, we consider a tensor $E(L)$ determined by a curvature-like tensor L. In Theorem 6.1 we prove that $E(R') = E(R)$ for the natural connection $\nabla' = D$, considered in [3], in the case when D has a parallel torsion.

1. Preliminaries

Let (M, P, g) be a *Riemannian almost product manifold*, i.e. a differentiable manifold M with a tensor field P of type $(1,1)$ and a Riemannian metric g such that $P^2 x = x$, $g(Px, Py) = g(x, y)$ for any x, y of the algebra $\mathfrak{X}(M)$ of the smooth vector fields on M. Further x, y, z, w will stand for arbitrary elements of $\mathfrak{X}(M)$ or vectors in the tangent space $T_c M$ at $c \in M$.

In [11] A.M. Naveira gives a classification of Riemannian almost product manifolds with respect to the tensor F of type (0,3), defined by $F(x, y, z) = g\left((\nabla_x P) y, z\right)$, where ∇ is the Levi-Civita connection of g.

In this work we consider manifolds (M, P, g) with $\mathrm{tr} P = 0$. In this case M is an even-dimensional manifold. We assume that $\dim M = 2n$.

Using the Naveira classification, in [13] M. Staikova and K. Gribachev give a classification of Riemannian almost product manifolds (M, P, g) with $\mathrm{tr} P = 0$. The basic classes of this classification are $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$. Their intersection is the class $\mathcal{W}_0$ of the *Riemannian P-manifolds* ([12]), determined by the condition $F = 0$. This class is an analogue of the class of Kähler manifolds in the geometry of almost Hermitian manifolds.

The class $\mathcal{W}_1$ from the Staikova-Gribachev classification consists of the Riemannian product manifolds which are locally conformal equivalent to Riemannian P-manifolds. This class plays a similar role of the role of the class of the conformal Kähler manifolds in almost Hermitian geometry. We will say that a manifold from the class $\mathcal{W}_1$ is a $\mathcal{W}_1$-*manifold*.

The characteristic condition for the class $\mathcal{W}_1$ is the following

$$\mathcal{W}_1 : F(x, y, z) = \tfrac{1}{2n}\big\{ g(x, y)\theta(z) - g(x, Py)\theta(Pz) \\ + g(x, z)\theta(y) - g(x, Pz)\theta(Py) \big\},$$

where the associated 1-form θ is determined by $\theta(x) = g^{ij} F(e_i, e_j, x)$. Here g^{ij} will stand for the components of the inverse matrix of g with respect to a basis $\{e_i\}$ of $T_c M$ at $c \in M$. The 1-form θ is *closed*, i.e. $\mathrm{d}\theta = 0$, if and only if $(\nabla_x \theta) y = (\nabla_y \theta) x$. Moreover, $\theta \circ P$ is a closed 1-form if and only if $(\nabla_x \theta) Py = (\nabla_y \theta) Px$.

In [13] it is proved that $\mathcal{W}_1 = \overline{\mathcal{W}}_3 \oplus \overline{\mathcal{W}}_6$, where $\overline{\mathcal{W}}_3$ and $\overline{\mathcal{W}}_6$ are the classes from the Naveira classification determined by the following conditions:

$$\overline{\mathcal{W}}_3: \qquad F(A, B, \xi) = \tfrac{1}{n} g(A, B) \theta^v(\xi), \quad F(\xi, \eta, A) = 0,$$

$$\overline{\mathcal{W}}_6: \qquad F(\xi, \eta, A) = \tfrac{1}{n} g(\xi, \eta) \theta^h(A), \quad F(A, B, \xi) = 0,$$

where $A, B, \xi, \eta \in \mathfrak{X}(M)$, $PA = A$, $PB = B$, $P\xi = -\xi$, $P\eta = -\eta$, $\theta^v(x) = \tfrac{1}{2}(\theta(x) - \theta(Px))$, $\theta^h(x) = \tfrac{1}{2}(\theta(x) + \theta(Px))$. In the case when $\mathrm{tr}P = 0$, the above conditions for $\overline{\mathcal{W}}_3$ and $\overline{\mathcal{W}}_6$ can be written for any x, y, z in the following form:

$$\overline{\mathcal{W}}_3: \qquad F(x, y, z) = \tfrac{1}{2n} \big\{ [g(x, y) + g(x, Py)] \theta(z)$$
$$+ [g(x, z) + g(x, Pz)] \theta(y) \big\}, \quad \theta(Px) = -\theta(x),$$

$$\overline{\mathcal{W}}_6: \qquad F(x, y, z) = \tfrac{1}{2n} \big\{ [g(x, y) - g(x, Py)] \theta(z)$$
$$+ [g(x, z) - g(x, Pz)] \theta(y) \big\}, \quad \theta(Px) = \theta(x).$$

In [13], a tensor L of type (0,4) with properties

$$L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),$$
$$L(x, y, z, w) + L(y, z, x, w) + L(z, x, y, w) = 0$$

is called a *curvature-like tensor*. Such a tensor on a Riemannian almost product manifold (M, P, g) with the property

$$L(x, y, Pz, Pw) = L(x, y, z, w)$$

is called a *Riemannian P-tensor* in [9]. This notion is an analogue of the notion of a Kähler tensor in Hermitian geometry.

Let S be a (0,2)-tensor on a Riemannian almost product manifold. In [13] it is proved that

(1.1)
$$\psi_1(S)(x, y, z, w) = g(y, z)S(x, w) - g(x, z)S(y, w)$$
$$+ S(y, z)g(x, w) - S(x, z)g(y, w)$$

is a curvature-like tensor if and only if $S(x, y) = S(y, x)$, and the tensor

(1.2)
$$\psi_2(S)(x, y, z, w) = \psi_1(S)(x, y, Pz, Pw)$$

is curvature-like if and only if $S(x, Py) = S(y, Px)$. Obviously

$$\psi_2(S)(x, y, Pz, Pw) = \psi_1(S)(x, y, z, w).$$

If $\psi_1(S)$ and $\psi_2(S)$ are curvature-like tensors, then $(\psi_1 + \psi_2)(S)$ is a Riemannian P-tensor. The tensors

(1.3)
$$\pi_1 = \frac{1}{2}\psi_1(g), \qquad \pi_2 = \frac{1}{2}\psi_2(g), \qquad \pi_3 = \psi_1(\widetilde{g}) = \psi_2(\widetilde{g})$$

are curvature-like, where $\widetilde{g}(x, y) = g(x, Py)$, and the tensors $\pi_1 + \pi_2$, π_3 are Riemannian P-tensors.

The curvature tensor R of ∇ is determined by $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z$ and the corresponding tensor of type (0,4) is defined as follows $R(x, y, z, w) = g(R(x, y)z, w)$. We denote the Ricci tensor and the scalar curvature of R by ρ and τ, respectively, i.e. $\rho(y, z) = g^{ij} R(e_i, y, z, e_j)$ and $\tau = g^{ij} \rho(e_i, e_j)$. The associated Ricci tensor ρ^* and the associated scalar curvature τ^* of R are determined by $\rho^*(y, z) = g^{ij} R(e_i, y, z, Pe_j)$ and $\tau^* = g^{ij} \rho^*(e_i, e_j)$. In a similar way there are determined the Ricci tensor $\rho(L)$ and the scalar curvature $\tau(L)$ for any curvature-like tensor L as well as the associated quantities $\rho^*(L)$ and $\tau^*(L)$.

In [10], a linear connection ∇' on a Riemannian almost product manifold (M, P, g) is called a *natural connection* if $\nabla'P = \nabla'g = 0$.

In [2], it is established that the natural connections ∇' on a $\mathcal{W}_1$-manifold (M, P, g) form a 2-parametric family, where the torsion T of ∇' is determined by

$$
\begin{aligned}
T(x, y, z) = {}& \frac{1}{2n}\left\{g(y, z)\theta(Px) - g(x, z)\theta(Py)\right\} \\
& + \lambda\left\{g(y, z)\theta(x) - g(x, z)\theta(y)\right. \\
& \qquad \left. + g(y, Pz)\theta(Px) - g(x, Pz)\theta(Py)\right\} \\
& + \mu\left\{g(y, Pz)\theta(x) - g(x, Pz)\theta(y)\right. \\
& \qquad \left. + g(y, z)\theta(Px) - g(x, z)\theta(Py)\right\},
\end{aligned}
$$

(1.4)

where $\lambda, \mu \in \mathbb{R}$.

Let Q be the tensor determined by

(1.5)
$$
\nabla'_x y = \nabla_x y + Q(x, y).
$$

The corresponding tensor of type (0,3), according to [5], satisfies

(1.6)
$$
Q(x, y, z) = T(z, x, y).
$$

Let us recall the following statement.

Theorem 1.1 ([5]). *Let R' is the curvature tensor of a natural connection ∇' on a $\mathcal{W}_1$-manifold (M, P, g). Then the following relation is valid:*

(1.7)
$$
R = R' - g(p, p)\pi_1 - g(q, q)\pi_2 - g(p, q)\pi_3 - \psi_1(S') - \psi_2(S''),
$$

where

$$
p = \lambda\Omega + \left(\mu + \frac{1}{2n}\right)P\Omega, \quad q = \lambda P\Omega + \mu\Omega, \quad g(\Omega, x) = \theta(x),
$$
$$
\begin{aligned}
S'(y, z) = {}& \lambda\left(\nabla'_y\theta\right)z + \left(\mu + \frac{1}{2n}\right)\left(\nabla'_y\theta\right)Pz \\
& - \frac{1}{2n}\left\{\lambda\theta(y)\theta(Pz) + \mu\theta(y)\theta(z)\right\}, \\
S''(y, z) = {}& \lambda\left(\nabla'_y\theta\right)z + \mu\left(\nabla'_y\theta\right)Pz \\
& + \frac{1}{2n}\left\{\lambda\theta(Py)\theta(z) + \mu\theta(Py)\theta(Pz)\right\}.
\end{aligned}
$$

2. Some properties of the natural connections on the manifolds of the class $\overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$

Let (M, P, g) is a Riemannian product manifold of the class $\overline{\mathcal{W}}_3$ or the class $\overline{\mathcal{W}}_6$, i.e. $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$. Then for the 1-form θ and the vector Ω we have

(2.1)
$$
\theta(Pz) = \varepsilon\theta(z), \qquad P\Omega = \varepsilon\Omega,
$$

where $\varepsilon = 1$ for $(M, P, g) \in \overline{\mathcal{W}}_3$ and $\varepsilon = -1$ for $(M, P, g) \in \overline{\mathcal{W}}_6$.

Let ∇' be a natural connection on $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$.

Using (1.4), (1.6) and (2.1), we obtain for the tensor Q determined by (1.5) the following

$$
\begin{aligned}
Q(x, y) = {}& \left[\lambda + \varepsilon\left(\mu + \frac{1}{2n}\right)\right][g(x, y) - \theta(y)x] \\
& (\mu + \varepsilon\lambda)[g(x, Py) - \theta(y)Px].
\end{aligned}
$$

Now, for the curvature tensors R and R' of ∇ and ∇', it is valid (1.7), where

$$(2.2) \qquad p = \left(\lambda + \varepsilon\mu + \frac{\varepsilon}{2n}\right)\Omega, \qquad q = (\mu + \varepsilon\lambda)\Omega,$$

$$(2.3) \qquad S'(y,z) = \left(\lambda + \varepsilon\mu + \frac{\varepsilon}{2n}\right)\left(\nabla'_y\theta\right)z - \frac{\mu + \varepsilon\lambda}{2n}\theta(y)\theta(z),$$

$$(2.4) \qquad S''(y,z) = (\lambda + \varepsilon\mu)\left(\nabla'_y\theta\right)z + \frac{\mu + \varepsilon\lambda}{2n}\theta(y)\theta(z).$$

Further we consider manifolds $(M,P,g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ with closed 1-form θ. In this case, the tensor K, determined by

$$(2.5) \qquad K(x,y,z,w) = \frac{1}{2}\left[R(x,y,z,w) + R(x,y,Pz,Pw)\right],$$

is a Riemannian P-tensor ([4]).

If $(M,P,g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ has a closed 1-form θ, then the curvature tensor R' of a natural connection ∇' is also a Riemannian P-tensor. Indeed, from (1.7) it is clear, that R' is a Riemannian P-tensor if and only if $\psi_1(S')$ and $\psi_2(S'')$ are curvature-like tensors, i.e. if and only if $S'(y,z) = S'(z,y)$ and $S''(y,Pz) = S''(z,Py)$. According to (2.3) and (2.4), the latter conditions are valid if and only if

$$(2.6) \qquad \left(\nabla'_y\theta\right)z = \left(\nabla'_z\theta\right)y.$$

In [5], it is proved that for any $\mathcal{W}_1$-manifold the following equality is valid:

$$\left(\nabla'_y\theta\right)z - \left(\nabla'_z\theta\right)y = \left(\nabla_y\theta\right)z - \left(\nabla_z\theta\right)y$$

$$- \frac{1}{2n}\left\{\theta(Py)\theta(z) - \theta(y)\theta(Pz)\right\}.$$

Bearing in mind (2.1), the latter equality implies that equality (2.6) is valid on $(M,P,g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ if and only if $\left(\nabla_y\theta\right)z = \left(\nabla_z\theta\right)y$, i.e. if and only if the 1-form θ is closed.

Theorem 2.1. *Let the manifold $(M,P,g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ be with a closed 1-form θ. Then the following equality is valid*

$$(2.7) \qquad K = R' - (\psi_1 + \psi_2)(S),$$

where

$$(2.8) \qquad \begin{aligned} S(y,z) &= \left(\lambda + \varepsilon\mu + \frac{\varepsilon}{4n}\right)\left(\nabla'_y\theta\right)z \\ &+ \frac{g(p,p) + g(q,q)}{4}g(y,z) + \frac{g(p,q)}{2}g(y,Pz). \end{aligned}$$

Proof. According to Theorem 1.1, for (M,P,g) it is valid the equality

$$(2.9) \qquad \begin{aligned} R(x,y,z,w) = \{R' &- g(p,p)\pi_1 - g(q,q)\pi_2 - g(p,q)\pi_3 \\ &- \psi_1(S') - \psi_2(S'')\}(x,y,z,w). \end{aligned}$$

In (2.9), we substitute Pz and Pw for z and w, respectively. We add the obtained equality to (2.9). Then, taking into account (1.1), (1.2), (1.3), (2.3), (2.4), (2.5), (2.8) and the properties of the curvature-like tensors $\psi_1(S')$ and $\psi_2(S'')$, we get (2.7).   □

In Section 3, Section 4 and Section 5, we find some Riemannian P-tensors determined by K on a manifold $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ with a closed 1-form θ. We establish that the found tensors coincide with the corresponding tensors determined by the curvature tensor R' of a natural connection ∇'. In Section 6, we find a curvature-like tensor determined by R on such a manifold and establish that this tensor coincides with the corresponding tensor determined by the curvature tensor R' of the special natural connection D investigated in [5], in the case when D has a parallel torsion.

3. An arbitrary natural connection on a manifold $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ with a closed 1-form θ

In [13], it is defined a Bochner tensor $B(L)$ for an arbitrary Riemannian P-tensor L on a $\mathcal{W}_1$-manifold (M, P, g) ($\dim M \geq 6$) as follows:

$$(3.1) \qquad \begin{aligned} B(L) = L - \frac{1}{2(n-2)} &\left\{ (\psi_1 + \psi_2)(\rho(L)) \right. \\ &\left. - \frac{1}{2(n-1)} \left[\tau(L)(\pi_1 + \pi_2) + \tau^*(L)\pi_3 \right] \right\}. \end{aligned}$$

Let us remark that $B(L)$ is also a Riemannian P-tensor.

Theorem 3.1. *Let the manifold $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ ($\dim M \geq 6$) be with a closed 1-form θ. If R' is the curvature tensor of a natural connection ∇', then $B(R') = B(K)$.*

Proof. Relation (2.7) implies the following equality for the Ricci tensors $\rho(K)$ and ρ' of K and R', respectively:

$$(3.2) \qquad \rho(K) = \rho' - \mathrm{tr}S \; g - \mathrm{tr}\widetilde{S} \; \widetilde{g} - 2(n-2)S,$$

where $\widetilde{S}(y, z) = S(y, Pz)$. Then we get the following equalities for the scalar curvatures:

$$(3.3) \qquad \mathrm{tr}S = \frac{\tau' - \tau(K)}{4(n-1)}, \qquad \mathrm{tr}\widetilde{S} = \frac{\tau'^* - \tau^*(K)}{4(n-1)}.$$

Equalities (3.2) and (3.3) imply

$$(3.4) \qquad S = \frac{1}{2(n-2)} \left\{ \rho' - \rho(K) - \frac{(\tau' - \tau(K))g + (\tau'^* - \tau^*(K))\widetilde{g}}{4(n-1)} \right\}.$$

From (1.1), (1.2), (1.3) and (3.4), we have

$$(3.5) \qquad \begin{aligned} (\psi_1 + \psi_2)(S) &= \\ &= \frac{1}{2(n-2)} \left\{ (\psi_1 + \psi_2)(\rho') - (\psi_1 + \psi_2)(\rho(K)) \right. \\ &\left. - \frac{(\tau' - \tau(K))(\pi_1 + \pi_2) + (\tau'^* - \tau^*(K))\pi_3}{2(n-1)} \right\}. \end{aligned}$$

Using (3.5), (2.7) and the definition (3.1) of the Bochner tensor, we obtain $B(K) = B(R')$. $\qquad \square$

4. THE CANONICAL CONNECTION ON A MANIFOLD $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ WITH A CLOSED 1-FORM θ

The canonical connection on a Riemannian almost product manifold is a natural connection introduced in [10] as an analogue of the Hermitian connection on almost Hermitian manifold. A connection of such a type on almost contact B-metric manifolds is considered in [7], [8].

We define the tensor $A(L)$ for an arbitrary Riemannian P-tensor L by the equality

$$(4.1) \qquad A(L) = L - \frac{\tau(L)(\pi_1 + \pi_2 - \varepsilon\pi_3)}{4n(n-1)}.$$

Obviously, $A(L)$ is also a Riemannian P-tensor.

Theorem 4.1. *Let the manifold $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ be with a closed 1-form θ. If R' is the curvature tensor of the canonical connection, then $A(R') = A(K)$.*

Proof. In [5], it is shown the canonical connection on a $\mathcal{W}_1$-manifold is determined by $\lambda = 0$ and $\mu = -\frac{1}{4n}$. Then, (2.2) implies $p = \frac{\varepsilon\Omega}{4n}$, $q = -\frac{\Omega}{4n}$ and therefore

$$(4.2) \qquad g(p, p) = g(q, q) = -\varepsilon g(p, q) = \frac{\theta(\Omega)}{16n^2}.$$

From (2.8) and (4.2) it is follows $S = \frac{\theta(\Omega)}{32n^2}(g - \varepsilon\widetilde{g})$. Then, because of (1.3), we have $(\psi_1 + \psi_2)(S) = \frac{\theta(\Omega)}{16n^2}(\pi_1 + \pi_2 - \varepsilon\pi_3)$. Thus, (2.7) takes the form

$$(4.3) \qquad K = R' - \frac{\theta(\Omega)(\pi_1 + \pi_2 - \varepsilon\pi_3)}{16n^2}.$$

By virtue of (4.3), we obtain the following equalities

$$(4.4) \qquad \begin{aligned} \rho(K) &= \rho' - \frac{(n-1)\theta(\Omega)(g - \varepsilon\widetilde{g})}{8n^2}, \\ \theta(\Omega) &= \frac{4n(\tau' - \tau(K))}{n-1} = -\frac{4n\varepsilon(\tau'^* - \tau^*(K))}{n-1}. \end{aligned}$$

Bearing in mind (4.3) and (4.4), by suitable calculations we get

$$R' - \frac{\tau'(\pi_1 + \pi_2 - \varepsilon\pi_3)}{4n(n-1)} = K - \frac{\tau(K)(\pi_1 + \pi_2 - \varepsilon\pi_3)}{4n(n-1)}.$$

Then, according to (4.1), we have $A(R') = A(K)$. $\qquad\square$

In [12], a 2-plane $\alpha = (x, y)$ in T_cM is called a totally real 2-plane if α is orthogonal to $P\alpha$. Its sectional curvatures with respect to R'

$$\nu' = \frac{R'(x, y, y, x)}{\pi_1(x, y, y, x)}, \qquad \nu'^* = \frac{R'(x, y, y, Px)}{\pi_1(x, y, y, x)}$$

are called totally real sectional curvatures with respect to R'.

Theorem 4.2. *A manifold $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ with a closed 1-form θ has pointwise constant totally real sectional curvatures*

$$\nu' = \frac{\tau'}{4n(n-1)}, \qquad \nu'^* = -\frac{\varepsilon\tau'}{4n(n-1)}$$

with respect to the curvature tensor R' of the canonical connection if and only if $A(R') = 0$ (or equivalently $A(K) = 0$).

Proof. According to (4.1), the condition for annulment of $A(R')$ is the condition

$$R' = \frac{\tau'(\pi_1 + \pi_2 - \varepsilon\pi_3)}{4n(n-1)}.$$

Then, bearing in mind [12], we establish the truthfulness of the statement. □

5. AN NATURAL CONNECTION WITH PARALLEL TORSION ON A MANIFOLD $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ WITH A CLOSED 1-FORM θ

We define the tensor $C(L)$ for an arbitrary Riemannian P-tensor L by the equality

$$(5.1) \qquad C(L) = L - \frac{\tau(L)(\pi_1 + \pi_2) + \tau^*(L)\pi_3}{4n(n-1)}.$$

Obviously, $C(L)$ is also a Riemannian P-tensor.

Theorem 5.1. *Let the manifold $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ be with a closed 1-form θ. If R' is the curvature tensor of a natural connection with a parallel torsion, then $C(R') = C(K)$.*

Proof. In [5], it is proved that a natural connection ∇' on a $\mathcal{W}_1$-manifold has a parallel torsion if and only if the 1-form θ is also parallel, i.e. $\nabla'\theta = 0$. Then, (2.8) implies

$$S = \frac{g(p,p) + g(q,q)}{4}g + \frac{g(p,q)}{2}\widetilde{g}.$$

Then, because of (1.3), we have

$$(\psi_1 + \psi_2)(S) = \frac{g(p,p) + g(q,q)}{2}(\pi_1 + \pi_2) + g(p,q)\pi_3.$$

Thus, (2.7) takes the form

$$(5.2) \qquad K = R' - \frac{g(p,p) + g(q,q)}{2}(\pi_1 + \pi_2) + g(p,q)\pi_3.$$

By virtue of (5.2), we obtain

$$\rho(K) = \rho' - (n-1)[g(p,p) + g(q,q)]g - 2(n-1)g(p,q)\widetilde{g}),$$

which implies

$$(5.3) \qquad \begin{aligned} \tau(K) &= \tau' - 2n(n-1)[g(p,p) + g(q,q)], \\ \tau^*(K) &= \tau'^* - 4n(n-1)g(p,q). \end{aligned}$$

Bearing in mind (5.2) and (5.3), by suitable calculations we get

$$R' - \frac{\tau'(\pi_1 + \pi_2) + \tau'^*\pi_3}{4n(n-1)} = K - \frac{\tau(K)(\pi_1 + \pi_2) + \tau^*(K)\pi_3}{4n(n-1)}.$$

Then, according to (5.1), we have $C(R') = C(K)$. □

Theorem 5.2. *A manifold $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ with a closed 1-form θ has pointwise constant totally real sectional curvatures*

$$\nu' = \frac{\tau'}{4n(n-1)}, \qquad \nu'^* = \frac{\tau'^*}{4n(n-1)}$$

with respect to the curvature tensor R' of an arbitrary natural connection with parallel torsion if and only if $C(R') = 0$ (or equivalently $C(K) = 0$).

Proof. According to (5.1), the condition for annulment of $C(R')$ is the condition

$$R' = \frac{\tau'(\pi_1 + \pi_2) + \tau'^*\pi_3}{4n(n-1)}.$$

Then, bearing in mind [12], we establish the truthfulness of the statement. $\square$

6. The natural connection D $(\lambda = \mu = 0)$ with parallel torsion on a manifold $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ with a closed 1-form θ

In [3], it is studied the natural connection D determined by $\lambda = \mu = 0$ on a $\mathcal{W}_1$-manifold (M, P, g).

Now we consider the case when $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ is with a closed 1-form θ and the connection $\nabla' = D$ has a parallel torsion. Then, from (2.2), (2.3) and (2.4) we have $p = \frac{\varepsilon\Omega}{2n}$, $q = S' = S'' = 0$ and therefore (1.7) takes the form

$$(6.1) \qquad\qquad R = R' - \frac{\theta(\Omega)\pi_1}{4n^2}.$$

The latter equality implies $\rho = \rho' - \frac{(2n-1)\theta(\Omega)}{4n^2}g$, which gives us

$$(6.2) \qquad\qquad \tau = \tau' - \frac{(2n-1)\theta(\Omega)}{2n}, \qquad \tau^* = \tau'^*.$$

We define the tensor $E(L)$ for an arbitrary curvature-like tensor L by the equality

$$(6.3) \qquad\qquad E(L) = L - \frac{\tau(L)\pi_1}{2n(2n-1)}.$$

Obviously, $E(L)$ is also a curvature-like tensor.

Theorem 6.1. *Let the manifold $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ be with a closed 1-form θ. If R' is the curvature tensor of the connection D with parallel torsion, then $E(R') = E(R)$.*

Proof. Equalities (6.1) and (6.2) imply

$$R - \frac{\tau\pi_1}{2n(2n-1)} = R' - \frac{\tau'\pi_1}{2n(2n-1)}.$$

Then, according to (6.3), we have $E(R') = E(R)$. $\square$

Theorem 6.2. *Let the manifold $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ be with a closed 1-form θ and D be with a parallel torsion. Then D is flat if and only if $E(R') = 0$ (or equivalently $E(R) = 0$).*

Proof. Let $E(R') = 0$ be valid, i.e.

$$(6.4) \qquad\qquad R'(x, y, z, w) = \frac{\tau'}{2n(2n-1)}\pi_1(x, y, z, w).$$

In (6.4), we substitute Pz and Pw for z and w, respectively. Taking into account that R' is a Riemannian P-tensor and $\pi_1(x, y, Pz, Pw) = \pi_2(x, y, z, w)$, we obtain

$$(6.5) \qquad\qquad R' = \frac{\tau'}{2n(2n-1)}\pi_2.$$

From (6.4) and (6.5) it is follows $\tau'(\pi_1 - \pi_2) = 0$ and because of $\pi_1 \neq \pi_2$ we have $\tau' = 0$. Then $R' = 0$, according to (6.4), i.e. D is a flat connection.

Vice versa, let D be flat, i.e. $R' = 0$. Then $\tau' = 0$ and bearing in mind the definition of $E(R')$ we obtain $E(R' = 0$. $\square$

Corollary 6.1. *Let the manifold* $(M, P, g) \in \overline{\mathcal{W}}_3 \cup \overline{\mathcal{W}}_6$ *be with a closed 1-form* θ *and* D *be flat with a parallel torsion. Then* (M, P, g) *is a space form with a negative scalar curvature* τ.

Proof. If D is flat, then by Theorem 6.2 we have $E(R) = 0$, i.e.

$$R = \frac{\tau}{2n(2n-1)}\pi_1.$$

This means that the manifold is a space form. Moreover, $\tau' = 0$ for a flat connection D and therefore $\tau = -\frac{2n-1}{2n}\theta(\Omega)$, because of (6.2). Thus, since $\theta(\Omega) = g(\Omega, \Omega) > 0$, we obtain $\tau < 0$. $\square$

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