International Electronic Journal of Geometry
Volume 5 No. 2 pp. 168–182 (2012) ©IEJG

THE TAXICAB HELIX ON TAXICAB CYLINDER

CUMALİ EKİCİ, SİBEL SEVİNÇ, YASEMİN E. CENGİZ

(Communicated by Kazım İLARSLAN)

Abstract. Taxicab geometry which is very close to Euclidean geometry has many areas of application and is easy to be understood. In this paper, we expressed the equation for perpendicular circular cylinder in taxicab space geometry. We also defined taxicab helix on taxicab perpendicular circular cylinder and found the curvature and torsion of this helix.

1. Introduction

Taxicab geometry (known as "Taxi geometry") was considered by Hermann Minkowski in the 19th century. While Euclidean geometry measures distance as the crow flies, Minkowski recognized that this was not necessarily the best model for many real world situations, particularly for problems involving cities where distances are determined along blocks and not as the crow flies. Another valuable aspect of taxicab geometry is its simplicity as a non-Euclidean geometry. It is more easily understandable than many other non-Euclidean geometries.

Taxicab geometry is a form of geometry in which the usual metric of Euclidean geometry is replaced by a new metric in which the distance between two points is the sum of the (absolute) differences of their coordinates [18]. The taxicab distance is defined as the sum of the horizontal and vertical distance of the two points. This is the minimum distance a taxicab would need to travel to reach point B from point A, if all streets are only oriented horizontally and vertically.

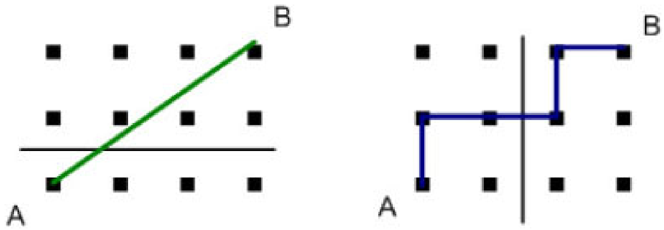
Taxicab plane $\mathbb{R}^2_T$ is almost the same as Euclidean analytical plane $\mathbb{R}^2$. The points are the same, the lines are the same and angles are measured in the same way. More formally, we can define the taxicab distance in the Euclidean space as sum of the lengths of the line segments which each other are the parallel to one of a coordinate axis (see Figure 1). For example, T between the points A and B with coordinates (X_A, Y_A) and (X_B, Y_B) is defined as [4].

$$T = |X_B - X_A| + |Y_B - Y_A| \tag{1}$$

Date: November 16, 2011 and, in revised form, August 03, 2012.
2000 *Mathematics Subject Classification.* 53A35, 51F05, 51K05, 14H50.
Key words and phrases. Non-Euclidean geometry, Taxicab metric, Taxicab geometry, Helix.

while for the Euclidean distance E for the same points is calculated by

$$E^2 = (X_B - X_A)^2 + (Y_B - Y_A)^2. \tag{2}$$



(a) (b)

Figure 1

As shown in Figure 1, the green path (Figure 1(a)) is the Euclidean distance from A to B. The blue path (Figure 1(b)) is one of the taxicab distance from A to B.

The Taxicab geometry is a non-Euclidean geometry. It is, on the contrary, very easy to understand and has many beautiful applications in real life. For instance, it provides us with a non-Euclidean framework for analyzing problems based on blocks - much like the grid of an urban street map - hence the name “Taxicab geometry”.

Some of interesting properties about taxicab geometry are;

(1) Euclidean and Taxicab geometry are very close in the view of axiomatic structure. Taxicab geometry enjoys the twelve properties which constitute a modern day set of axioms for describing the geometry promulgated by Euclid in antiquity [18].

(2) It has significant applications in the real world. (While Euclidean geometry can be considered a good model of the real world, the taxicab geometry is a better model for the world of the cities built by men).

(3) It is understandable by anyone who has gone through a beginning course in Euclidean geometry.

(4) The ratio of the circumference to radius of taxicab circle is 4. That is $\pi_T = 4$ [18].

Since the taxicab plane geometry has a different distance function than the Euclidean geometry, it seems interesting to study the taxicab analogues of the topics that include the concept of distance in the Euclidean geometry. Some of these topics have been studied by some authors [1, 2, 3, 6, 12, 13, 14, 15, 20, 21, 25]. The isometry group of taxicab space was introduced in [7].

The definitions of the inner-product and the norm in the taxicab geometry are given in [6] and [14], and the taxicab trigonometry is introduced in [1] and [25].

We give the following needed definitions which are introduced in [6, 14, 15, 17, 23].

Let $\alpha = (a_1, a_2)$, $\beta = (b_1, b_2) \in \mathbb{R}^2_T$. We define the taxicab inner-product by

$$\langle \alpha, \beta \rangle_T = \begin{cases} |a_1 b_1| + |a_2 b_2| & , \ \alpha, \beta \text{ are in the same octant} \\ -|a_1 b_1| + |a_2 b_2| & , \ \alpha, \beta \text{ are in the neighbour octants} \\ & \text{and } a_1 b_1 < 0, \ a_2 b_2 > 0 \\ |a_1 b_1| - |a_2 b_2| & , \ \alpha, \beta \text{ are in the neighbour octants} \\ & \text{and } a_1 b_1 > 0, \ a_2 b_2 < 0 \\ -|a_1 b_1| - |a_2 b_2| & , \ \alpha, \beta \text{ are in the opposite octants} \end{cases} \tag{3}$$

or

$$\langle \alpha, \beta \rangle_T = \varepsilon_1 |a_1 b_1| + \varepsilon_2 |a_2 b_2|$$

where $\varepsilon_i = \begin{cases} 1 & a_i b_i > 0 \\ -1 & a_i b_i < 0 \end{cases}$, $i = 1, 2$.

Let $\alpha = (a_1, a_2) \in \mathbb{R}^2_T$ be any vector. Then

$$\| \alpha \|_T = \sqrt{\langle \alpha, \alpha \rangle_T + 2 \mid a_1 a_2 \mid} \tag{4}$$

defines the norm of α in taxicab geometry. It is clear that $\|\alpha\|_T = d_T(0, \alpha)$ [17]. Obviously,

$$\| \alpha \|_T = \mid a_1 \mid + \mid a_2 \mid . \tag{5}$$

In this paper we will explore the taxicab helix. Instead of using Euclidean angles measured in radians, we will mirror the usual definition of the radian. Using this definition, we will use taxicab trigonometry functions $cos_T(\theta)$, $sin_T(\theta)$, $tan_T(\theta)$ and $cot_T(\theta)$ [1] and finally calculate curvature and torsion of helix .

2. The Taxicab Helix on Taxicab Cylinder

2.1. **The Euclidean helix.** A "circular helix" is a curve on the perpendicular cylinder such that there is a constant angle between helix's tangents and cylinder's ruling. To obtain this helix, let's take a rectangle $AMNS$ and its diagonal $|AN|$. Wrapping the rectangle $AMNS$ around to form a circular cylinder such that the line segments AM and KN are the diameters of the top and bottom of the cylinder respectively. So, the diagonal of this rectangle forms a helix curve on this cylinder (see Figure 2) [9, 11].

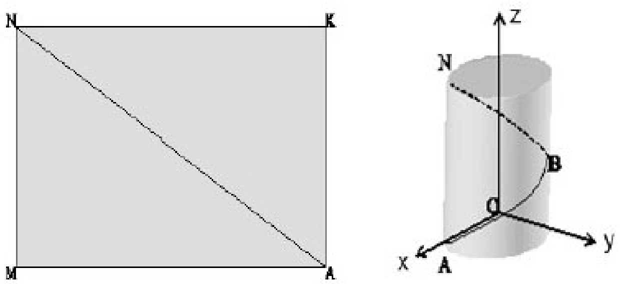


Figure 2

Let $M \subset \mathbb{R}$ be a smooth curve denoted by $\{(I, \alpha)\}$ so that for each $s \in I$, the velocity vector $\alpha'(s)$ constitutes a constant angle with the constant unit vector e_3

on the z axis. Then the curve M is called a helix.

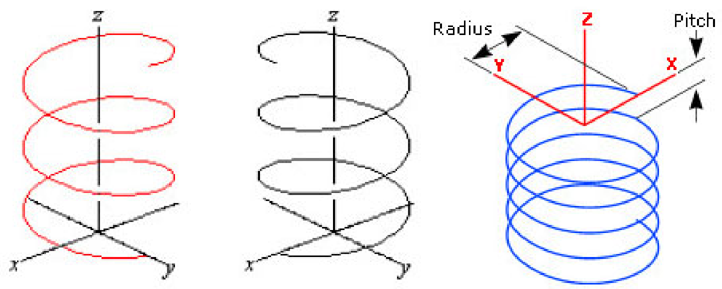


Figure 3

The tangent of α at point $\alpha(u)$ is $\alpha^{'}(u)$, therefore $\langle\frac{\alpha^{'}(u)}{\shortparallel\alpha^{'}(u)\shortparallel}, e_3\rangle = \cos\varphi$, *constant*, and for $k = \pm a\cot\varphi$, k *constant*, the parametric equation for helix is

$$\begin{array}{llll} (6) \qquad \alpha : & [0, 2\pi] & \longrightarrow & E^3 \\ & u & \longrightarrow & \alpha(u) = (a\cos u, a\sin u, ku). \end{array}$$

The graph of this parametric equation has been shown for $a = 1$ in Figure 3.

2.2. **The taxicab cylinder.** A cylindrical surface is the surface generated by a straight line moving always parallel to a given line, and intersecting a given curve (if the curve is a plane curve with its plane parallel to the given line, the cylinder is a plane). The line is called the generator. The curve is called the directrix. A cylinder is either the solid bounded by two parallel planes and a cylindrical surface whose directrix is a closed curve, or the surface consisting of the portion of the cylindrical surface between the planes and the regions of the planes bounded by the cylindrical surface. The elements are the segment of the elements of the cylindrical surface between the planes. Perpendicular cylinder is a cylinder whose directrix lies in the plane and main straight lines are perpendicular to the plane of this directrix [9, 11].

Theorem 2.1. *A given equation represents a cylinder which is perpendicular to any coordinate plane if and only if this equation doesn't have to contain component corresponding to the coordinate axis which is not in this plane.*

For instance, the equation for perpendicular cylinder with directrix $x^2 + y^2 = 4$, $z = 0$ is $x^2 + y^2 = 4$. And this perpendicular cylinder is perpendicular to the plane in which the variable z is zero [11].

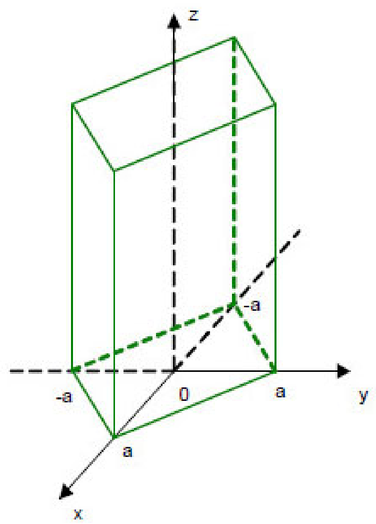


Figure 4

Therefore, for a perpendicular taxicab cylinder in the taxicab space as shown in Figure 4, the directrix is

$$|x| + |y| = a, \ \ z = 0 \tag{7}$$

So for $|x| + |y| = a, z = k$ and $k \in \mathbb{R}$, the parametric equation of taxicab cylinder is

$$X(u, v) = (a \cos_T u, a \sin_T u, v) \, . \tag{8}$$

2.3. **The taxicab helix.** It is well-known that, geometrically, the curve which lies on a right circular cylinder and cuts the elements of the cylinder under constant angle is called a circular helix [9]. So, the parametric equation of the taxicab circular helix is

$$\left\{ \begin{array}{lcl} x & = & a \cos_T u \\ y & = & a \sin_T u \\ z & = & ku = (a \cot_T \theta)\, u \end{array} \right. \tag{9}$$

or

$$X(u) = (a \cos_T u, a \sin_T u, ku) \tag{10}$$

where θ is constant taxicab angle. Here, the taxicab trigonometric functions have been given by [1].

Now, let us calculate $cot_T \theta$ when the taxicab measurement of θ is fixed. Then for some values of angle u, we will find the points of taxicab helix in the octant **I** of taxicab plane:

If $\theta = \dfrac{\pi_T}{6} = \dfrac{2}{3}$, then $\cot_T \theta = 2$. Hence, we find points H_i, $i = 1, 2, 3, 4, 5$ for $u = 0, \dfrac{\pi_T}{6}, \dfrac{\pi_T}{4}, \dfrac{\pi_T}{3}, \dfrac{\pi_T}{2}$, respectively (see Figure 5):

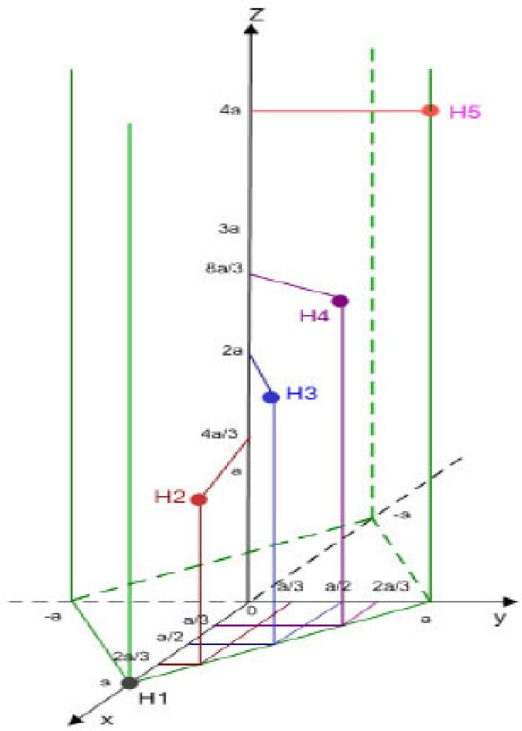


Figure 5

$$
(11) \qquad \begin{aligned} &H_1 = (a, 0, 0), \qquad && H_2 = \left(\frac{2a}{3}, \frac{a}{3}, \frac{4a}{3}\right), \qquad && H_3 = \left(\frac{a}{2}, \frac{a}{2}, 2a\right), \\ &H_4 = \left(\frac{a}{3}, \frac{2a}{3}, \frac{8a}{3}\right), \qquad && H_5 = (0, a, 4a). \end{aligned}
$$

Using the taxicab distance between the points H_1, H_2, H_3, H_4, H_5, we will obtain the minimum distance set about H_i and H_{i+1}.

The set $\{P \mid d_T(A, P) + d_T(P, B) = d_T(A, B)\}$ will be called the taxi minimum distance set. It consists of all points in the interior and on the rectangular determined by lines parallel to the axes drawn at the points A and B which has $|AB|$ as its diagonal. The minimum distance set in taxicab geometry consists of all points in the interior and on the rectangle having sides parallel to the axes and having the segment as a diagonal. In each case, if the two points are on a line parallel to an axis then the minimum distance set is a segment [18, 24].

Now, we want to obtain minimum distance set of H_1 and H_2. Then we have to solve the following equation:

$$
d_T(H_1, P) + d_T(P, H_2) = d_T(H_1, H_2) \tag{12}
$$

where $H_1 = (a, 0, 0)$, $H_2 = \left(\frac{2a}{3}, \frac{a}{3}, \frac{4a}{3}\right)$ and $P = (x, y, z)$.

So we obtain the following equation:

$$(|x - a| + |y| + |z|) + \left(\left|\frac{2a}{3} - x\right| + \left|\frac{a}{3} - y\right| + \left|\frac{4a}{3} - z\right|\right) = 2a. \tag{13}$$

To solve equation (13), we consider the following:

i) If $\frac{2a}{3} \le x < a$, $0 \le y < \frac{a}{3}$, $0 \le z < \frac{4a}{3}$ and $|x| + |y| = a$, we obtain $2a = 2a$. Thus, the solution set is

$\left\{(x, y, z) \mid \frac{2a}{3} \le x < a, 0 \le y < \frac{a}{3}, 0 \le z < \frac{4a}{3}, |x| + |y| = a\right\}$. Shortly, all of the points in this region satisfy the equation.

ii) If $x = a$, $y = 0$, $0 \le z < \frac{4a}{3}$ and $|x| + |y| = a$, we get $2a = 2a$. Thus, the solution set is

$\left\{(x, y, z) \mid x = a, y = 0, 0 \le z < \frac{4a}{3}, |x| + |y| = a\right\}$. Shortly, all of the points in this region satisfy the equation.

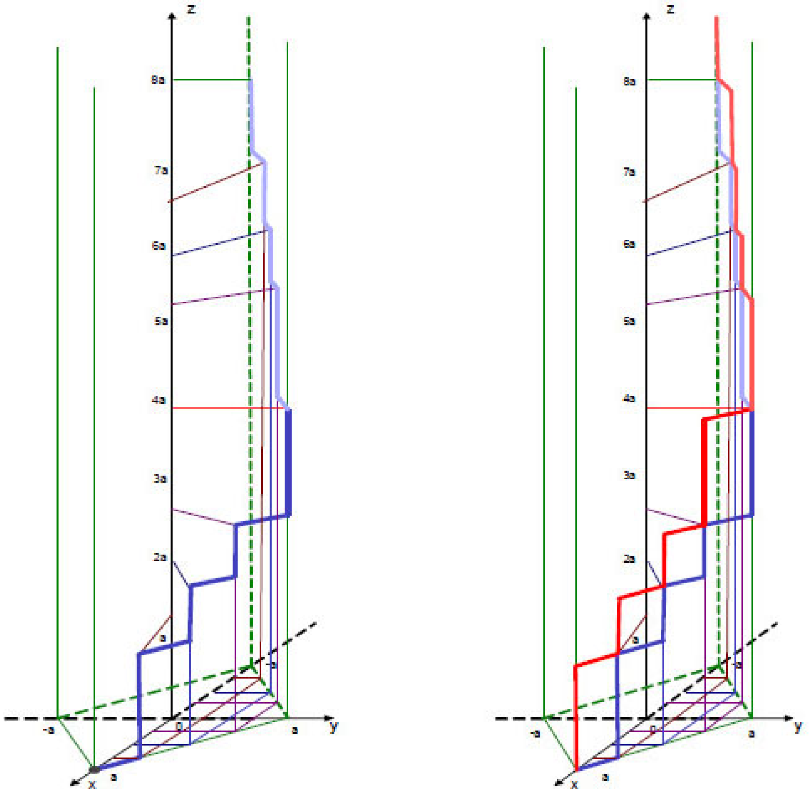
iii) If $x = \frac{2a}{3}$, $y = \frac{a}{3}$, $0 \le z < \frac{4a}{3}$ and $|x| + |y| = a$ then $2a = 2a$. So, all of the points in this region satisfy the equation.

iv) If $\frac{2a}{3} < x < a$, $0 < y < \frac{a}{3}$, $z = \frac{4a}{3}$ and $|x| + |y| = a$, we obtain $2a = 2a$. So, all of the points in this region satisfy the equation.

Other values of x, y, z don't satisfy Equation (13).

For $H_2 = \left(\frac{2a}{3}, \frac{a}{3}, \frac{4a}{3}\right)$, $H_3 = \left(\frac{a}{2}, \frac{a}{2}, 2a\right)$ and $P = (x, y, z)$, we obtain the following equation:

$$\left(\left|x - \frac{2a}{3}\right| + \left|y - \frac{a}{3}\right| + \left|z - \frac{4a}{3}\right|\right) + \left(\left|\frac{a}{2} - x\right| + \left|\frac{a}{2} - y\right| + |2a - z|\right) = a \tag{14}$$



(a) (b)

Figure 6

For $H_3 = \left(\frac{a}{2}, \frac{a}{2}, 2a\right)$, $H_4 = \left(\frac{a}{3}, \frac{2a}{3}, \frac{8a}{3}\right)$ and $P = (x, y, z)$, we obtain the following equation:

$$\left(\left|x - \frac{a}{2}\right| + \left|y - \frac{a}{2}\right| + |z - 2a|\right) + \left(\left|\frac{a}{3} - x\right| + \left|\frac{2a}{3} - y\right| + \left|\frac{8a}{3} - z\right|\right) = a \tag{15}$$

For $H_4 = \left(\frac{a}{3}, \frac{2a}{3}, \frac{8a}{3}\right)$, $H_5 = (0, a, 4a)$ and $P = (x, y, z)$, we obtain the following equation:

$$\left(\left|x - \frac{a}{3}\right| + \left|y - \frac{2a}{3}\right| + \left|z - \frac{8a}{3}\right|\right) + (|0 - x| + |a - y| + |4a - z|) = 2a. \tag{16}$$

Then the solutions of these equations can be obtained similarly.

Using the same techniques above in the second octant of the taxicab plane, similar equalities for x, y, and z can be established (for example; in case $x = \frac{a}{3}$, $y = \frac{2a}{3}$, $\frac{8a}{3} \leq z < 4a$). Then conforming the properties about taxicab geometry; namely, moving around the taxicab unit squares without go back, we can establish the taxicab helix curve. Thus, we can construct "The Main Taxicab Helix Curve".

The main helix curve in octants **I** and **II** is shown in Figure 6. In this figure, both blue and red lines are taken as the main taxicab helix curve. And in Figure 7, the main helix curve with blue line in all octants where $z > 0$ is shown:

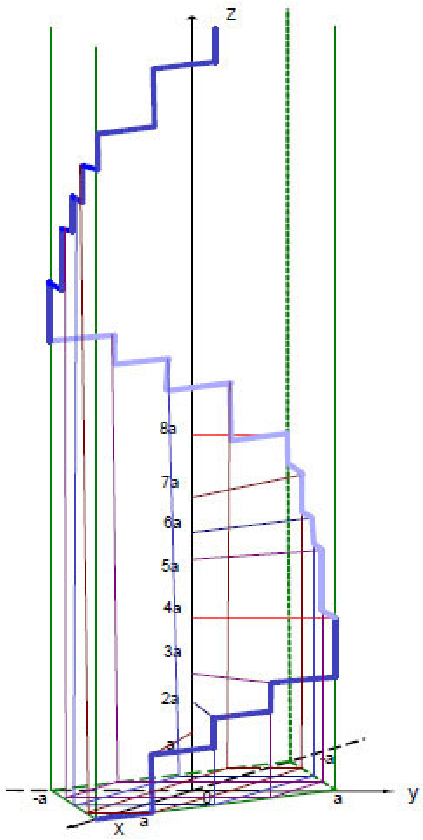
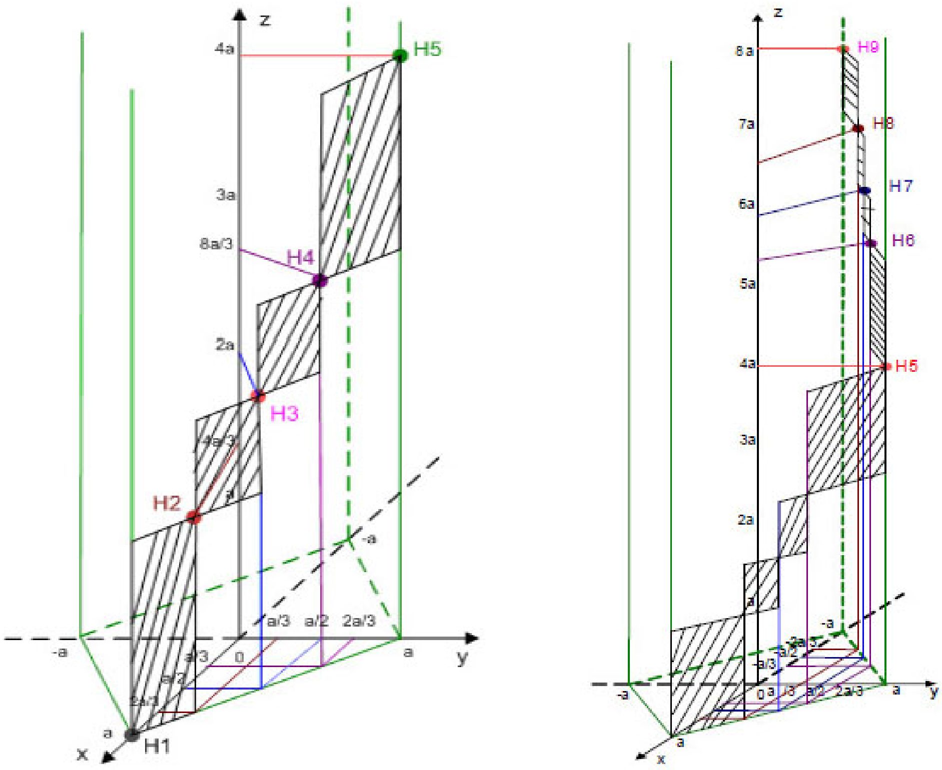


Figure 7

Using the definition for a geodesic in [19], we can call the minimum distance sets of the taxicab helix which we found above in Figure 8 called by "taxicab helical regions" (see Figure 8).



(a) (b)

Figure 8

Now, we find the points of the octant **II** of taxicab plane, as an example:

If we take $\theta = \frac{\pi_T}{6} = \frac{2}{3}$ again (then $\cot_T \theta = 2$) and the values of u from the octant **II**, then we find points H_i, $i = 5, 6, 7, 8, 9$ for $u = \frac{\pi_T}{2}, \frac{2\pi_T}{3}, \frac{3\pi_T}{4}, \frac{5\pi_T}{6}, \pi_T$, respectively. These points are following:

$$H_5 = (0, a, 2a)\,, \qquad H_6 = \left(\frac{-a}{3}, \frac{2a}{3}, \frac{16a}{3}\right), \quad H_7 = \left(\frac{-a}{2}, \frac{a}{2}, 6a\right),$$
$$H_8 = \left(\frac{-2a}{3}, \frac{a}{3}, \frac{20a}{3}\right), \quad H_9 = (-a, 0, 8a)\,. \tag{17}$$

Using the taxicab distance between the points H_5, H_6, H_7, H_8, H_9, we will obtain the minimum distance set about H_i and H_{i+1}.

Now, we want to obtain minimum distance set of H_5 and H_6. Then we have to solve the following equation;

$$d_T\left(H_5, P\right) + d_T\left(P, H_6\right) = d_T\left(H_5, H_6\right) \tag{18}$$

where $H_5 = (0, a, 2a)$, $H_6 = \left(\frac{-a}{3}, \frac{2a}{3}, \frac{16a}{3}\right)$ and $P = (x, y, z)$.

Thus we get the following equation:

$$(|x| + |y-a| + |z-2a|) + \left(\left|\frac{-a}{3} - x\right| + \left|\frac{2a}{3} - y\right| + \left|\frac{16a}{3} - z\right|\right) = 4a. \tag{19}$$

Now we give the generalization about the minimum distance equation:

For $H_i = (x_1, y_1, z_1)\,, H_{i+1} = (x_2, y_2, z_2)$ and $P = (x, y, z)$ the minimum distance equation is

$$d_T\left(H_i, P\right) + d_T\left(P, H_{i+1}\right) = d_T\left(H_i, H_{i+1}\right) \tag{20}$$

and also for the values of $z = ku = (a \cot_T \theta)u$,

$$z_i = z_{i-1} + k(u_i - u_{i-1})z \tag{21}$$

where $\theta = \dfrac{\pi_T}{6}$ is constant (so $a \cot_T \theta = 2a = k$), $u_0 = 0$ and $z_0 = 0$.

Similar generalizations can be found for other θ constant taxicab angles.

Similarly, we can obtain the minimum distance set in all of the regions. For the octant **II** our sets are in the Figure 8.

In the octants **III** and **IV** of taxicab plane, taking $\theta = \dfrac{\pi_T}{6} = \dfrac{2}{3}$ again, we can find similar results. These results are the taxicab helical regions again.

Now, let's take constant angle $\theta = \dfrac{\pi_T}{4} = 1$. Using some u angles in the octant **I** of taxicab plane, we find the points of the helix.

So $\cot_T \theta = 1$. Then the points are

$$\begin{aligned} &H_{11} = (a, 0, 0)\,, \qquad H_{12} = \left(\frac{2a}{3}, \frac{a}{3}, \frac{2a}{3}\right), \quad H_{13} = \left(\frac{a}{2}, \frac{a}{2}, a\right), \\ &H_{14} = \left(\frac{a}{3}, \frac{2a}{3}, \frac{4a}{3}\right), \quad H_{15} = (0, a, 2a)\,. \end{aligned} \tag{22}$$

With these points, we can obtain similar solutions again and these solutions are also taxicab helical regions. Barely, the values of z will change, so the solution sets obtained with these points will also be perpendicular regions. But the only difference is the change of the z values will be the size of the regions to be formed; so the smaller size perpendicular regions will be obtained.

$H_{11} = (a, 0, 0)$, $H_{12} = \left(\dfrac{2a}{3}, \dfrac{a}{3}, \dfrac{2a}{3}\right)$ and $P = (x, y, z)$, the equation

$$(|x-a| + |y| + |z|) + \left(\left|\frac{2a}{3} - x\right| + \left|\frac{a}{3} - y\right| + \left|\frac{2a}{3} - z\right|\right) = \frac{4a}{3} \tag{23}$$

is found.

Similar procedure can be used for octants **II**, **III** and **IV** and the solution sets will be determined according to the change of the taxicab angle θ. Results for all other constant values can be found for all regions by considering $\cot_T \theta$ of taxicab angle θ as defined.

2.4. **Curvature and torsion of the taxicab helix.** The pitch of a helix is the width of one complete helix turn, measured parallel to the axis of the helix. A circular helix, (i.e. one with constant radius) has constant band curvature and constant torsion. If it's circling around the $z-$axis, the radius of it's projection onto the xy-plane is radius of a circle of radius a (see Figure 3) [8, 26].

Corollary 2.1. *Let a be radius and k be the pitch of the taxicab helix and let α_T be a unit speed curve in taxicab geometry. Then α_T is a taxicab circular helix if and only if both its curvature and torsion are nonzero constants.*

Proof. For any number $a > 0$ and $k = a\cot_T\theta$ we compute curvature and torsion of taxicab circular helix. So, if the method in [8] is used, curvature and torsion of α_T become as follow:

$$\kappa = \frac{a}{a^2 + a^2\cot_T^2\theta} \qquad \tau = \frac{a\cot_T\theta}{a^2 + a^2\cot_T^2\theta} \tag{24}$$

and

$$\frac{\tau}{\kappa} = \frac{a\cot_T\theta}{a} = \cot_T\theta = constant. \tag{25}$$

Conversely, suppose that it has constant nonzero. So we get

$$a = \frac{\kappa}{\kappa^2 + \tau^2} \qquad k = a\cot_T\theta = \frac{\tau}{\kappa^2 + \tau^2} \tag{26}$$

□

2.5. **The Gauss curvature of taxicab plane.** The taxicab plane which is equipped with taxicab metric is a metrizable topological space. So, taxicab plane is a Hausdorff space. When the taxicab cylinder is opened, a subplane of taxicab plane in $\mathbb{R}^2$ is obtained. According to the subspace topology this subplane is a topological subspace.

K. O. Sowell has introduced the iso-taxicab geometry in [24]. It is a new non-Euclidean geometry. In [16], the Gauss Curvature of iso-taxicab geometry was defined. So we can define the Gauss curvature of taxicab plane.

We consider the subset $C_T = \{(x, y) : |x| + |y| = 1,\ x, y \in \mathbb{R}\}$ of the analytic plane. This set consists of all points $P = (x, y)$ with taxicab distance 1 from $O = (0, 0)$.

Considering the taxicab circle with radius 1, we can define the following polar parametrization of taxicab plane.

If we change $\cos_T v$ and $\sin_T v$ to $t\cos v$ and $t\sin v$ which is introduced in [3], then $\frac{d}{dv}[t\cos v] = -t\sin v$ because

$$t\cos v = \frac{|OA|\,|OB|}{|OA|_T\,|OB|_T}\cos v \quad \text{and} \quad t\sin v = \frac{|OA|\,|OB|}{|OA|_T\,|OB|_T}\sin v$$

where v is the angle between the vectors OA and OB and $0 \le v \le \pi$. Changing $\cos_T v$ and $\sin_T v$ to $t\cos v$ and $t\sin v$, we can guarantee that Equation (29) is valid. We need to indicate that Equations (30) and (31) are valid because

$$|t\cos v| + |t\sin v| = 1$$

when the reference angle $\alpha = 0$.

Let v be the angle between the vectors given with the reference angle α as in [3] and (u,v) be any point in $\mathbb{R}^2_T$. We define the polar parametrization of (u,v) as

$$\varphi(u,v) = (\varphi_{v_1}, \varphi_{v_2}) = (u.t\cos v, u.t\sin v)\,. \tag{27}$$

Using this parametrization, let's now compute the Gauss curvature.

Theorem 2.2. *The Gauss curvature K of taxicab plane is* 0.

Proof. Let (u,v) be any point in $\mathbb{R}^2_T$. If derivatives of taxicab trigonometric functions in [3] are calculated, then from (27) we get

$$\varphi_u(u,v) = (t\cos v, t\sin v) \tag{28}$$

and

$$\varphi_v(u,v) = (-u.t\sin v, u.t\cos v)\,. \tag{29}$$

Since

$$\begin{aligned} \|\varphi_u\|_T &= d_T(0,\varphi_u) \\ &= |\varphi_{u_1}| + |\varphi_{u_2}| \\ &= |t\cos v| + |t\sin v|\,, \end{aligned}$$

and

$$\begin{aligned} \|\varphi_v\|_T &= d_T(0,\varphi_v) \\ &= |\varphi_{v_1}| + |\varphi_{v_2}| \\ &= |u.t\cos v| + |u.t\sin v| \\ &= |u|\left(|t\cos v| + |t\sin v|\right) \end{aligned}$$

by using (28), (29) and (5), we obtain

$$\begin{aligned} \|\varphi_u\|_T &= |t\cos v| + |t\sin v| = 1, \\ \|\varphi_v\|_T &= |u|\left(|t\cos v| + |t\sin v|\right) = |u|\,. \end{aligned} \tag{30}$$

Moreover, by using (28), (29) and (3) we get

$$\langle \varphi_u, \varphi_v \rangle_T = \varepsilon_1 \left|-u.t\cos v.t\sin v\right| + \varepsilon_2 \left|u.t\cos v.t\sin v\right|.$$

If $(-t\cos v.t\sin v) < 0$ then by (3) $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$ or $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$. Thus, we obtain

$$\langle \varphi_u, \varphi_v \rangle_T = -\left|u.t\cos v.t\sin v\right| + \left|u.t\cos v.t\sin v\right| = 0. \tag{31}$$

Therefore, $\{\varphi_u, \varphi_v\}$ is an orthogonal basis. If

$$E_1 = \varphi_u \tag{32}$$

$$E_2 = \frac{1}{u}\varphi_v \tag{33}$$

then $\{E_1, E_2\}$ becomes orthonormal. Let $\{\theta_1, \theta_2\}$ be the dual base to $\{E_1, E_2\}$. Thus,

$$\begin{aligned} \theta_1 &= du \\ \theta_2 &= udv \end{aligned} \tag{34}$$

By using the first structural equations [22], we obtain connection 1-form as $\omega_{12} = dv$. Hence

$$d\theta_1 = \omega_{12} \wedge \theta_2 \qquad (35)$$
$$d\theta_2 = \omega_{21} \wedge \theta_1.$$

So

$$d\omega_{12} = 0. \qquad (36)$$

We know that

$$d\omega_{12} = -K\theta_1 \wedge \theta_2 \qquad (37)$$

[22]. Using this fact, we get

$$K = 0 \qquad (38)$$

as desired. □

References

[1] Akça, Z. and Kaya, R., *On The Taxicab Trigonometry*, Jour. of Inst. of Math&Comp. Sci(Math.Ser), 10(1997), no. 3, 151-159.

[2] Akça, Z. and Kaya, R., *On The Distance Formulae in Three Dimensional Taxicab Space*, Hadronic Journal, 27(2004), no. 5, 521-532.

[3] Bayar, A., Ekmekçi, S. and Özcan, M., *On Trigonometric Functions and Cosine and Sine Rules in Taxicab Plan*, International Electronic Journal of Geometry (IEJG), 2(2009), no. 1, 17-24.

[4] Caballero, D., *Taxicab Geometry; some problems and solution for square grid-based fire spread simulation*, V International Conference on Forest Fire Research D. X. Viegas (Ed.), 2006.

[5] Divjak, B., *Notes on Taxicab Geometry*, KoG 5(2000), 5-9.

[6] Ekici, C., Kocayusufoğlu, İ. and Akça, Z., *The Norm in Taxicab Geometry*, Tr. Jour. of Mathematics, 22(1998), 295-307.

[7] Gelişgen, Ö. and Kaya, R., *The Taxicab Space Group*, Acta Mathematica Hungarica, 122(2009), no. 1-2, 187-200.

[8] Gray, A., *Modern Differential Geometry of Curves and Surfaces*, CRS Press, Inc. 1993.

[9] Hacısalihoğlu, H. H., *2 ve 3 Boyutlu uzaylarda Analitik Geometri*, Ankara, 1995.

[10] Izquierdo, A. F., *Geometria y Topologia para entender las helices de la Naturaleza*, San Alberto, 2009.

[11] Kaya, R., *Analitik Geometri*, Bilim Teknik Yayınevi, Eskişehir, 2002.

[12] Çolakoğlu, B. H. and Kaya, R., *Volume of a Tetrahedron in the Taxicab Space*, Missouri Journal of Mathematical Sciences, 21(2009), no. 1, 21-27.

[13] Çolakoğlu, B. H. and Kaya, R., *Regular Polygons in Some Models of Protractor Geometry*, International Electronic Journal of Geometry (IEJG), 2(2009), no. 2, 76-87.

[14] Kocayusufoğlu, İ., Akça, Z. and Ekici, C., *The Inner-Product of Taxicab Geometry*, The Scientific J. The Kazakh State National University, on the section Mathematics, Mechanics and Informatics, 20(2000), no. 1, 33-39.

[15] Kocayusufoğlu, İ. and Ekici, C., *Some Area Problems in Taxicab Geometry*, Jour. of Inst. of Math&Comp. Sci(Math. Ser.), 12(1999), no. 2, 95-99.

[16] Kocayusufoğlu, İ. and Özdamar, E., *The Iso-Taxicab Gauss Curvature*, Differential Geometry, Dynamical Systems, 7(2006), 138-143.

[17] Kocayusufoğlu, İ. and Özdamar, E., *Isometries of Taxicab Geometry*, Commun. Fac. Sci. Univ. Ank. Series A1, 47(1998), 73-83.

[18] Krause, E. F., *Taxicab Geometry*, Addison-Wesley, Menlo Park, NJ, 1975.

[19] Morera, D. M., *Computing Geodesic Paths on Manifolds*, 3D Graphic Systems, IMPA, 2003.

[20] Özcan, M. and Kaya, R., *On the Ratio of Driected Lengths in the Taxicab Plane and Related Properties*, Missouri Journal of Mathematical Sciences, 14(2002), 107-117.

[21] Özcan, M. and Kaya, R., *Area of a Triangle in Terms of the Taxicab Distance*, Missouri Jour. of Mathematical Sciences, 15(2003), 178-185.
[22] O' Neill, B., *Elementary Differential Geometry*, Academic PressInc., 1966.
[23] So, S. S., *Recent Developments in Taxicab Geometry*, Cubo Matematica Educacional 4(2002), no. 2, 76-96.
[24] Sowell, K. O., *Taxicab Geometry-A New Slant*, Mathematics Magazine 62(1989), 238-248.
[25] Thompson, K. and Dray, T., *Taxicab Angles and Trigonometry*, Pi Mu Epsilon, 11(2000), 87-96.
[26] *http://en.wikipedia.org/wiki/Helix.*

Department of Mathematics and Computer Sciences, Eskişehir Osmangazi University-TURKEY
E-mail address: cekici@ogu.edu.tr, ssevinc@cumhuriyet.edu.tr, yaseminecengiz@hotmail.com