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GENERALIZED ϕ-RECURRENT LORENTZIAN α-SASAKIAN MANIFOLD

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Abstract. The purpose of this paper is to study generalized ϕ-recurrent Lorentzian α-Sasakian manifolds.

1. Introduction

The notion of generalized recurrent manifolds was introduced by U. C. De and N. Guha [5]. A Riemannian manifold (M^n, g) is called generalized recurrent if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(X)[g(Z,W)Y - g(Y,W)Z]$$

where, A and B are two 1-forms, B is non-zero and these are defined by

$$A(X) = g(X,\rho_1), \ \ B(X) = g(X,\rho_2) \tag{1.1}$$

ρ_1 and ρ_2 are vector fields associated with 1-forms A and B, respectively.

The notion of ϕ-recurrent Sasakian manifolds was introduced by U. C. De, A. A. Shaikh and S. Biswas [4]. This notion generalizes the notion of locally ϕ-symmetric Sasakian manifolds. A Sasakian manifold is said to be a ϕ-recurrent manifold if there exists a non-zero 1-form A such that

$$\phi^2((\nabla_X R)(Y,Z)W) = A(X)R(Y,Z)W$$

for arbitrary vector fields X, Y, Z, W. If the 1-form A vanishes, then the manifold reduces to a ϕ-symmetric manifold.

Generalized ϕ-recurrent (k,μ)-contact metric manifolds were studied by J-B. Jun, A. Yıldız and U. C. De [10]. Also, generalized ϕ-recurrent Sasakian manifolds were studied by D. A. Patil, D. G. Prakasha and C. S. Bagewadi [15]. Motivated by the above studies, in this paper we study generalized ϕ-recurrent Lorentzian α-Sasakian manifolds and obtain some interesting results.

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The paper is organized as follows: After Preliminaries, we give a brief account of Lorentzian α-Sasakian manifolds. In section 4, we study Lorentzian α-Sasakian manifolds satisfying the condition $S(X,\xi)\cdot R=0$, where S and R are the Ricci and Riemannian curvature tensors respectively. Here it is shown that the manifold under this condition is reduced to Einstein one. In Section 5, we show that a generalized ϕ-recurrent Lorentzian α-Sasakian manifold is an Einstein manifold. We also show that in a generalized ϕ-recurrent Lorentzian α-Sasakian manifold the characteristic vector field ξ and the associated vector field $\rho_1\alpha^2+\rho_2$ are in opposite direction. The same section also consists of locally generalized ϕ-recurrent Lorentzian α-Sasakian manifolds and obtained a necessary and sufficient condition for such a manifold to be of locally generalized ϕ-recurrent. In the last section, we show that a 3-dimensional generalized ϕ-recurrent Lorentzian α-Sasakian manifold is of constant curvature.

2. Preliminaries

The product of an almost contact manifold M and the real line $\mathbb{R}$ carries a natural almost complex structure. However if one takes M to be an almost contact metric manifold and supposes that the product metric G on $M\times\mathbb{R}$ is Kaehlerian, then the structure on M is cosymplectic [8] and not Sasakian. On the other hand Oubina [14] pointed out that if the conformally related metric $e^{2t}G$, t being the coordinate on $\mathbb{R}$, is Kaehlerian, then M is Sasakian and conversely.

In [19], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes: (i) homogeneous normal contact Riemannian manifolds with $c>0$, (ii) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c=0$, (iii) a warped product space if $c<0$. It is known that the manifolds of class (i) are characterized by admitting a Sasakian structure.

In the Gray-Hervella classification of almost Hermitian manifolds [7], there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [6]. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [14], [2] if the product manifold $M\times\mathbb{R}$ belongs to the class W_4. The class $C_6\oplus C_5$ [12] coincides with the class of the trans-Sasakian structures of type (α,β). In fact, in [12], local nature of the two subclasses, namely, C_5 and C_6 structures, of trans-Sasakian structures are characterized completely.

We note that trans-Sasakian structures of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are cosymplectic [2], β-Kenmotsu [9] and α-Sasakian [9], respectively. An almost contact metric structure (ϕ,ξ,η,g) on M is called a *trans-Sasakian structure* [14] if ($M\times\mathbb{R}$, J, G) belongs to the class W_4 [7], where J is the almost complex structure on $M\times\mathbb{R}$ defined by

$$J(X,fd/dt)=(\phi X-f\xi,\eta(X)d/dt)$$

for all vector fields X on M and smooth functions f on $M \times \mathbb{R}$, and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [1]

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X) \tag{2.1}$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β). From the formula (2.1) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi) \tag{2.2}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y) \tag{2.3}$$

More generally one has the notion of an α-Sasakian structure [9] which may be defined by

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) \tag{2.4}$$

where α is a non-zero constant. From the condition one may readily deduce that

$$\nabla_X \xi = -\alpha\phi X \tag{2.5}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) \tag{2.6}$$

Thus $\beta = 0$ and therefore a trans-Sasakian structure of type (α, β) with α a non-zero constant is always α-Sasakian [9]. If $\alpha = 1$, then α-Sasakian manifold is a Sasakian manifold.

The relation between trans-Sasakian, α-Sasakian and β-Kenmotsu structures was discussed by Marrero [13].

Proposition 1. [13] *A trans-Sasakian manifold of dimension ≥ 5 is either α-Sasakian, β-Kenmotsu or cosymplectic.*

3. Lorentzian α-Sasakian manifolds

A differentiable manifold M of dimension n is called a Lorentzian α-Sasakian manifold if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy [16, 21]

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0 \tag{3.1}$$

$$\phi^2 X = X + \eta(X)\xi, \quad g(X,\xi) = \eta(X) \tag{3.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{3.3}$$

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi + \eta(Y)X) \tag{3.4}$$

for all $X, Y \in TM$.
Also a Lorentzian α-Sasakian manifold M satisfies

$$\nabla_X \xi = \alpha\phi X \tag{3.5}$$

$$(\nabla_X \eta)(Y) = \alpha g(X, \phi Y) \tag{3.6}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g, then M is called Lorentzian α-Sasakian manifold.

Further, on a Lorentzian α-Sasakian manifold M the following relations hold: [22, 16]

$$\eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \tag{3.7}$$
$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y] \tag{3.8}$$
$$S(X,\xi) = (n-1)\alpha^2\eta(X) \tag{3.9}$$
$$Q\xi = (n-1)\alpha^2\xi \tag{3.10}$$
$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\alpha^2\eta(X)\eta(Y) \tag{3.11}$$

Definition 3.1. *A Lorentzian α-Sasakian manifold (M,g) is said to be Einstein manifold if its Ricci tensor S is of the form*

$$S(X,Y) = ag(X,Y)$$

for any vector fields X and Y, where a is constant on (M,g).

4. Lorentzian α-Sasakian manifold satisfying $S(X,\xi)\cdot R = 0$

Theorem 4.1. *A Lorentzian α-Sasakian manifold (M^n,g), $(n>3)$ satisfying the condition $S(X,\xi)\cdot R = 0$ is an Einstein manifold.*

Proof. Consider a Lorentzian α-Sasakian manifold (M^n,g), $(n>3)$ satisfying the condition

$$(S(X,\xi)\cdot R)(U,V)Z = 0 \tag{4.1}$$

By definition we have

$$\begin{aligned}(S(X,\xi)\cdot R)(U,V)Z &= ((X\wedge_S \xi)\cdot R)(U,V)Z \\ &= (X\wedge_S\xi)R(U,V)Z + R((X\wedge_S\xi)U,V)Z \\ &\quad +R(U,(X\wedge_S\xi)V)Z + R(U,V)(X\wedge_S\xi)Z\end{aligned} \tag{4.2}$$

where the endomorphism $X\wedge_S Y$ is defined by

$$(X\wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y \tag{4.3}$$

Using the definition of (4.3) in (4.2), we get by virtue of (3.9) that

$$\begin{aligned}&(S(X,\xi)\cdot R)(U,V)Z \\ = \; &(n-1)\alpha^2[\eta(R(U,V)Z)X + \eta(U)R(X,V)Z \\ &+\eta(V)R(U,X)Z + \eta(Z)R(U,V)X] \\ &-S(X,R(U,V)Z)\xi - S(X,U)R(\xi,V)Z \\ &-S(X,V)R(U,\xi)Z - S(X,Z)R(U,V)\xi\end{aligned} \tag{4.4}$$

In view of (4.1) and (4.4) we have

$$\begin{aligned}&(n-1)\alpha^2[\eta(R(U,V)Z)X + \eta(U)R(X,V)Z \\ &+\eta(V)R(U,X)Z + \eta(Z)R(U,V)X] \\ &-S(X,R(U,V)Z)\xi - S(X,U)R(\xi,V)Z \\ &-S(X,V)R(U,\xi)Z - S(X,Z)R(U,V)\xi = 0\end{aligned} \tag{4.5}$$

Taking the inner product on both sides of (4.5) with ξ we obtain

$$\begin{aligned}&(n-1)\alpha^2[\eta(R(U,V)Z)\eta(X)+\eta(U)\eta(R(X,V)Z)\\&+\eta(V)\eta(R(U,X)Z)+\eta(Z)\eta(R(U,V)X)]\\&+S(X,R(U,V)Z)-S(X,U)\eta(R(\xi,V)Z)\\&-S(X,V)\eta(R(U,\xi)Z)-S(X,Z)\eta(R(U,V)\xi)=0\end{aligned} \tag{4.6}$$

Putting $U=Z=\xi$ in (4.6) and using (3.7)-(3.11), we get

$$S(X,V)=(n-1)\alpha^2 g(X,V) \tag{4.7}$$

which means that the manifold is an Einstein manifold. This completes the proof of the theorem.

5. Generalized ϕ-recurrent Lorentzian α-Sasakian Manifolds

Definition 5.1. *A Lorentzian α-Sasakian manifold is said to be a generalized ϕ-recurrent if its curvature tensor R satisfies the condition (*[5, 18]*)*

$$\phi^2((\nabla_W R)(X,Y)Z)=A(W)R(X,Y)Z+B(W)[g(Y,Z)X-g(X,Z)Y] \tag{5.1}$$

where, A and B are two 1-forms, B is non-zero and these are defined as in (1.1). If for any vector fields X,Y,Z,W orthogonal to ξ, that is, for any horizontal vector fields X,Y,Z,W, then a generalized ϕ-recurrent manifold reduces to a locally generalized ϕ-recurrent manifold.

We begin with the following:

Theorem 5.2. *A generalized ϕ-recurrent Lorentzian α-Sasakian manifold (M^n,g) $(n>1)$ is an Einstein manifold.*

Proof. Let us consider a generalized ϕ-recurrent Lorentzian α-Sasakian manifold. Then by virtue of (3.2) and (5.1) we have

$$\begin{aligned}&(\nabla_W R)(X,Y)Z+\eta((\nabla_W R)(X,Y)Z)\xi\\=\;&A(W)R(X,Y)Z+B(W)[g(Y,Z)X-g(X,Z)Y]\end{aligned} \tag{5.2}$$

from which it follows that

$$\begin{aligned}&g((\nabla_W R)(X,Y)Z,U)+\eta((\nabla_W R)(X,Y)Z)\eta(U)\\=\;&A(W)g(R(X,Y)Z,U)+B(W)[g(Y,Z)X-g(X,Z)Y]\end{aligned} \tag{5.3}$$

Let $\{e_i\}$, $i=1,2,...,n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (4.2) and taking summation over i, $1\le i\le n$, we get

$$\begin{aligned}&(\nabla_W S)(Y,Z)+\sum_{r=1}^{n}\eta((\nabla_W R)(e_i,Y)Z)\eta(e_i)\\=\;&A(W)S(Y,Z)+(n-1)B(W)g(Y,Z)\end{aligned} \tag{5.4}$$

The second term of (5.4) by putting $Z=\xi$ takes the form $g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi)$ which is denoted by E. In this case E vanishes. Since the following equation is well known

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi),\xi) &= g(\nabla_W R(e_i,Y)\xi,\xi)-g(R(\nabla_W e_i,Y)\xi,\xi) \\ &\quad -g(R(e_i,\nabla_W Y)\xi,\xi)-g(R(e_i,Y)\nabla_W\xi,\xi) \end{aligned}$$

at $p\in M$. Using (3.8), we have

$$g(R(e_i,\nabla_W Y)\xi,\xi)=\alpha^2[g(\nabla_W Y,\xi)g(e_i,\xi)-g(\xi,e_i)g(\nabla_W Y,\xi)]=0$$

Thus we obtain

$$g((\nabla_W R)(e_i,Y)\xi,\xi)=g(\nabla_W R(e_i,Y)\xi,\xi)-g(R(e_i,Y)\nabla_W\xi,\xi)$$

In virtue of $g(R(e_i,Y)\xi,\xi)=g(R(\xi,\xi)Y,e_i)=0$, we have

$$g(\nabla_W R(e_i,Y)\xi,\xi)+g(R(e_i,Y)\xi,\nabla_W\xi)=0$$

which implies

$$g((\nabla_W R)(e_i,Y)\xi,\xi)=-g(R(e_i,Y)\xi,\nabla_W\xi)-g(R(e_i,Y)\nabla_W\xi,\xi)$$

Hence we reach

$$\begin{aligned} E &= -\alpha\sum_{r=1}^{n}\{g(R(\phi W,\xi)Y,e_i)g(\xi,e_i)+g(R(\xi,\phi W)Y,e_i)g(\xi,e_i)\} \\ &= -\alpha\{g(R(\phi W,\xi)Y,\xi)+g(R(\xi,\phi W)Y,\xi)\}=0 \end{aligned}$$

Replacing Z by ξ in (5.4) and using (3.9) we have

$$(\nabla_W S)(Y,\xi)=(n-1)\{A(W)\alpha^2+B(W)\}\eta(Y) \tag{5.5}$$

Now we have $(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi)$.
Using (3.5) and (3.6) in the above relation, it follows that

$$(\nabla_W S)(Y,\xi)=\alpha\{(n-1)\alpha^2 g(W,\phi Y)-S(\phi W,Y)\} \tag{5.6}$$

In view of (5.5) and (5.6), we have

$$\alpha\{(n-1)\alpha^2 g(W,\phi Y)-S(\phi W,Y)\}=(n-1)\{A(W)\alpha^2+B(W)\}\eta(Y) \tag{5.7}$$

Replacing Y by ξ in (5.7) and then using (3.1), we get

$$\alpha^2 A(W)=-B(W) \tag{5.8}$$

So using (5.8) in (5.7) we have

$$(n-1)\alpha^2 g(W,\phi Y)-S(\phi W,Y)=0$$

Replacing Y by ϕY in above and using (3.2) and (3.11) we get

$$S(Y,W)=(n-1)\alpha^2 g(Y,W)$$

for all Y,W. This completes the proof of the theorem.

Theorem 5.3. *In a generalized ϕ-recurrent Lorentzian α-Sasakian manifold (M^n, g) the characteristic vector field ξ and the vector field $\rho_1\alpha^2+\rho_2$ associated to the 1-form $A\alpha^2 + B$ are in opposite direction.*

Proof. Two vector fields P and Q are said to be *codirectional* if $P = fQ$, where f is a non-zero scalar, that is $g(P, X) = fg(Q, X)$ for all X.
Now, from (5.1), we have

$$\begin{aligned}(\nabla_W R)(X, Y)Z &= -\eta((\nabla_W R)(X, Y)Z)\xi + A(W)R(X, Y)Z \\ &\quad +B(W)[g(Y, Z)X - g(X, Z)Y]\end{aligned} \tag{5.9}$$

Then by the use of second Bianchi identity and (5.9), we get

$$\begin{aligned}A(W)\eta(R(X, Y)Z) &+ A(X)\eta(R(Y, W)Z) + A(Y)\eta(R(W, X)Z) \\ &+ B(W)[g(Y, Z)X - g(X, Z)Y] \\ &+ B(X)[g(W, Z)Y - g(Y, Z)W] \\ &+ B(Y)[g(X, Z)W - g(W, Z)X] = 0\end{aligned} \tag{5.10}$$

By virtue of (3.7), we obtain from (5.10) that

$$\begin{aligned}&\{A(W)\alpha^2 + B(W)\}[g(Y, Z)X - g(X, Z)Y] \\ +\ &\{A(X)\alpha^2 + B(X)\}[g(W, Z)Y - g(Y, Z)W] \\ +\ &\{A(Y)\alpha^2 + B(Y)\}[g(X, Z)W - g(W, Z)X] = 0\end{aligned} \tag{5.11}$$

Putting $Y = Z = e_i$ in (5.11) and taking summation over i, $1 \leq i \leq n$, we get

$$\{A(W)\alpha^2 + B(W)\}\eta(X) = \{A(X)\alpha^2 + B(X)\}\eta(W) \tag{5.12}$$

for all vector fields X, W.
Replacing X by ξ in (5.12), it follows that

$$\{A(W)\alpha^2 + B(W)\} = -\eta(W)\{\eta(\rho_1)\alpha^2 + \eta(\rho_2)\} \tag{5.13}$$

for any vector field W, where $A(\xi) = g(\xi, \rho_1) = \eta(\rho_1)$ and $B(\xi) = g(\xi, \rho_2) = \eta(\rho_2)$.
Relation (5.12) and (5.13) completes proof of the theorem.

Theorem 5.4. *A Lorentzian α-Sasakian manifold (M^n, g) is locally generalized ϕ-recurrent if and only if the relation*

$$\begin{aligned}(\nabla_W R)(X, Y)Z &= \alpha\{\alpha^2[g(\phi Y, W)g(X, Z) - g(\phi X, W)g(Y, Z)]\xi \\ &\quad -g(R(X, Y)\phi W, Z)\xi\} + A(W)R(X, Y)Z \\ &\quad +B(W)\{g(Y, Z)X - g(X, Z)Y\}\end{aligned} \tag{5.14}$$

holds for all horizontal vector fields X, Y, Z, W on M.

Proof. By the definition, we have

$$\begin{aligned}g((\nabla_W R)(X, Y)Z, U) &= g(\nabla_W R(X, Y)Z, U) + R(\nabla_W X, Y, U, Z) \\ &\quad +R(X, \nabla_W Y, U, Z) + R(X, Y, U, \nabla_W Z)\end{aligned} \tag{5.15}$$

where $R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the property of curvature tensor have been used. Since ∇ is a metric connection, it follows that

$$g(\nabla_W R(X,Y)Z,U) = g(R(X,Y)\nabla_W U,Z) - \nabla_W g(R(X,Y)U,Z) \tag{5.16}$$

and

$$\nabla_W g(R(X,Y)U,Z) = g(\nabla_W R(X,Y)U,Z) + g(R(X,Y)U,\nabla_W Z) \tag{5.17}$$

From (5.16) and (5.17) we have

$$\begin{aligned} g(\nabla_W R(X,Y)Z,U) &= -g(\nabla_W R(X,Y)U,Z) \\ &\quad -g(R(X,Y)U,\nabla_W Z) + g(R(X,Y)\nabla_W U,Z) \end{aligned} \tag{5.18}$$

Using (5.18) in (5.15), we get

$$g((\nabla_W R)(X,Y)Z,U) = -g((\nabla_W R)(X,Y)U,Z) \tag{5.19}$$

In view of (5.19), it follows from (3.2) and (5.1) that

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= g((\nabla_W R)(X,Y)\xi,Z)\xi \\ &\quad +A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{5.20}$$

By virtue of (3.1), (3.6) and (3.8) we can easily get

$$(\nabla_W R)(X,Y)\xi = \alpha[\alpha^2\{g(\phi Y,W)X - g(\phi X,W)Y\} - R(X,Y,\phi W)] \tag{5.21}$$

Using (5.21) in (5.19) we obtain the relation (5.14). Conversely, if in a Lorentzian α-Sasakian manifold the relation (5.14) holds, then applying ϕ on both sides of (5.14) and keeping mind that X,Y,Z and W are orthogonal to ξ, we obtain (5.1). This completes the proof of the theorem.

Theorem 5.5. *A Lorentzian α-Sasakian manifold is of constant curvature if and only if the relation*

$$\phi^2((\nabla_W R)(X,Y)\xi) = A(W)R(X,Y)\xi + B(W)[g(Y,\xi)X - g(X,\xi)Y] \tag{5.22}$$

holds for all horizontal vector fields X,Y,W.

Proof. With the help of (3.1), the relation (5.22) can be written as

$$\begin{aligned} &(\nabla_W R)(X,Y)\xi + \eta((\nabla_W R)(X,Y)\xi)\xi \\ = \ &A(W)R(X,Y)\xi + B(W)[g(Y,\xi)X - g(X,\xi)Y] \end{aligned} \tag{5.23}$$

By taking account of (3.8) and (5.14) in (5.23), one can get

$$(\nabla_W R)(X,Y)\xi = 0 \tag{5.24}$$

for any horizontal vector fields X,Y,W. By taking account of (5.21) in (5.24) we have

$$R(X,Y,\phi W) = \alpha^2\{g(\phi Y,W)X - g(\phi X,W)Y\} \tag{5.25}$$

for any orthogonal vector fields X,Y,W.

Now assume that X, Y and Z are vector fields such that $(\nabla X)_p = (\nabla Y)_p = (\nabla Z)_p = 0$ for a fixed point p of M^n. By the Ricci identity for ϕ [20]

$$-(R(X,Y)\phi W) = (\nabla_X \nabla_Y \phi)W - (\nabla_Y \nabla_X \phi)W$$

We have at the point p,

$$-R(X,Y,\phi W) + \phi R(X,Y,W) = \nabla_X((\nabla_Y \phi)W) - \nabla_Y((\nabla_X \phi)W)$$

Using (3.4), we have

$$\begin{aligned} -R(X,Y,\phi W) \quad + & \quad \phi R(X,Y,W) \\ = & \quad \alpha \nabla_X \{g(Y,W)\xi + \eta(W)Y\} \\ & \quad -\alpha \nabla_Y \{g(X,W)\xi + \eta(W)X\} \\ = & \quad \alpha\{g(Y,W)\nabla_X \xi + (\nabla_X \eta)(W)Y\} \\ & \quad -\alpha\{g(X,W)\nabla_Y \xi + (\nabla_Y \eta)(W)X\} \end{aligned}$$

In view of (2.5) and (2.6), the above equation becomes

$$\begin{aligned} R(X,Y)\phi W &= \alpha^2\{g(\phi Y,W)X + g(X,W)\phi Y - g(\phi X,W)Y - g(Y,W)\phi X\} \\ &\quad +\phi R(X,Y)W \end{aligned} \tag{5.26}$$

From (5.25) and (5.26), it follows that

$$\phi R(X,Y)W = \alpha^2\{g(Y,W)\phi X - g(X,W)\phi Y\}.$$

Operating ϕ on both sides and using (3.2) we get

$$R(X,Y)W = \alpha^2\{g(Y,W)X - g(X,W)Y\} \tag{5.27}$$

for any vector fields X, Y, W are orthogonal to ξ.

Conversely, if a Lorentzian α-Sasakian manifold is of constant curvature, then from (5.27) it follows that the relation (5.22) holds. This completes the proof of the theorem.

6. 3-dimensional locally generalized ϕ-recurrent Lorentzian α-Sasakian Manifolds

Theorem 6.1. *A 3-dimensional locally generalized ϕ-recurrent Lorentzian α-Sasakian manifold is of constant curvature.*

Proof. In a 3-dimensional Lorentzian α-Sasakian manifold (M^3, g), we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad -S(X,Z)Y + \frac{r}{2}[g(X,Z)Y - g(Y,Z)X] \end{aligned} \tag{6.1}$$

Now putting $Z = \xi$ and using (3.2) and (3.9), we get

$$\begin{aligned} R(X,Y)\xi &= \eta(Y)QX - \eta(X)QY \\ &\quad +2\alpha^2[\eta(Y)X - \eta(X)Y] + \frac{r}{2}[\eta(X)Y - \eta(Y)X] \end{aligned} \tag{6.2}$$

Using (3.6) in (6.2), we have

$$\left(\frac{r}{2}-\alpha^2\right)[\eta(Y)X-\eta(X)Y]=\eta(Y)QX-\eta(X)QY \tag{6.3}$$

Putting $Y=\xi$ in (6.3), we obtain

$$QX=\left(\frac{r}{2}-\alpha^2\right)X+\left(\frac{r}{2}-3\alpha^2\right)\eta(X)\xi \tag{6.4}$$

Therefore, it follows from (6.4) that

$$S(X,Y)=\left(\frac{r}{2}-\alpha^2\right)g(X,Y)+\left(\frac{r}{2}-3\alpha^2\right)\eta(X)\eta(Y) \tag{6.5}$$

Thus from (6.1), (6.4) and (6.5), we get

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2}-2\alpha^2\right)[g(Y,Z)X-g(X,Z)Y] \\ &\quad +\left(\frac{r}{2}-3\alpha^2\right)[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ &\quad +\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y] \end{aligned} \tag{6.6}$$

Taking the covariant differentiation to the both sides of the equation (6.6), we get

$$\begin{aligned} (\nabla_W R)(X,Y)Z &= \frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)Y+g(Y,Z)\eta(X)\xi \\ &\quad -g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X+\eta(X)\eta(Z)Y] \\ &\quad +\left(\frac{r}{2}-3\alpha^2\right)[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\nabla_W\xi \\ &\quad +\left(\frac{r}{2}-3\alpha^2\right)[\eta(Y)X-\eta(X)Y](\nabla_W\eta)(Z) \\ &\quad +\left(\frac{r}{2}-3\alpha^2\right)[g(Y,Z)\xi-\eta(Z)Y](\nabla_W\eta)(X) \\ &\quad -\left(\frac{r}{2}-3\alpha^2\right)[g(X,Z)\xi-\eta(Z)X](\nabla_W\eta)(Y) \end{aligned} \tag{6.7}$$

Noting that we may assume that all vector fields X,Y,Z,W are orthogonal to ξ in the above relation, we have

$$\begin{aligned} &(\nabla_W R)(X,Y)Z=\frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)Y] \\ &+\left(\frac{r}{2}-3\alpha^2\right)[g(Y,Z)(\nabla_W\eta)(X)-g(X,Z)(\nabla_W\eta)(Y)]\xi \end{aligned} \tag{6.8}$$

Applying ϕ^2 to the both sides of (6.8) and using (3.1) and (3.2), we get

$$\phi^2(\nabla_W R)(X,Y)Z=\frac{dr(W)}{2}[g(Y,Z)X-g(X,Z)X] \tag{6.9}$$

By (5.1) the equation (6.9) reduces to

$$A(W)R(X,Y)Z=\left[\frac{dr(W)}{2}-B(W)\right][g(Y,Z)X-g(X,Z)X]$$

Putting $W = e_i$, where $\{e_i\}, i = 1, 2, 3$, is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq 3$, we obtain

$$R(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)X]$$

where $\lambda = \left[\frac{dr(e_i)}{2A(e_i)} + \alpha^2\right]$ is a scalar, since A is a non-zero 1-form. Then by Schur's theorem λ will be a constant on the manifold. Therefore, (M^3, g) is of constant curvature λ. This completes the proof of the theorem.

ÖZET: Bu makalenin amacı genelleștirilmiș ϕ-*recurrent Lorentzian* α-Sasakian manifoldları çalıșmaktır.

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