Commun.Fac.Sci.Univ.Ank.Series A1
Volume 60, Number 1, Pages 15–25 (2011)
ISSN 1303–5991

INVERSE SPECTRAL PROBLEMS FOR DISCONTINUOUS STURM-LIOUVILLE OPERATOR WITH EIGENPARAMETER DEPENDENT BOUNDARY CONDITIONS.

BAKI KESKIN, A. SINAN OZKAN AND NUMAN YALÇIN

Abstract. In this study, Sturm–Liouville problem with discontinuities in the case when an eigenparameter linearly appears not only in the differential equation but it also appears in both of the boundary conditions is investigated.

1. Introduction.

Let us consider the boundary value problem L :

$$\ell y := -y'' + q(x)y = \lambda y, \quad x \in (0,d) \cup (d,\pi) \tag{1}$$

$$U(y) := \lambda(y'(0) + h_0 y(0)) - h_1 y'(0) - h_2 y(0) = 0 \tag{2}$$

$$V(y) := \lambda(y'(\pi) + H_0 y(\pi)) - H_1 y'(\pi) - H_2 y(\pi) = 0 \tag{3}$$

$$\begin{cases} y(d+0) = \alpha y(d-0) \\ y'(d+0) = \alpha^{-1} y'(d-0) \end{cases} \tag{4}$$

where α, d, h_i, H_i, $i = 0,1,2$, are real numbers, $\alpha \in \mathbb{R}^+$, $\rho_1 := h_2 - h_0 h_1 > 0$, $\rho_2 := H_0 H_1 - H_2 > 0$, $d \in (0,\pi)$, $q(x)$ is real valued functions in $\mathcal{L}_2(0,\pi)$, λ is spectral parameter.

Spectral problems for Sturm-Liouville operator with eigenvalue dependent boundary conditions were studied extensively. [2] and [3] are well known example for works with boundary conditions depend on eigenvalue parameter linearly. Moreover, [5], [16] and [18] are also interested in linearly conditions. Nonlinearly conditions were considered in [4], [7], [9]- [12] and [17]. These works have been on Hilbert and Pontryagin space formulations, expansion theory, direct and inverse spectral theory. In [2] and [20] an operator-theoretic formulation of the problems without discontinuity conditions (4) and with spectral parameter contained in only one of the boundary conditions has been given.

Received by the editors Nov. 22, 2010, Accepted: May 13, 2011.
2000 *Mathematics Subject Classification.* 34A55, 34B24, 34L05.
Key words and phrases. Inverse Problem, Weyl Function, Spectrum, Sturm-Liouville Operator.

Boundary value problems with discontinuities inside the interval are extensively studied ([6], [13]) These kinds of problems are often appear in mathematics, mechanics, physics, geophysics and other branches of natural properties. Discontinuous inverse problems also appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [14],[24] and [19]. It must be noted that some special cases of the considered problem (1)–(4) arise after an application of the method of separation of variables to the varied assortment of physical problems. For example, some boundary-value problems with transmission conditions arise in heat and mass transfer problems (see, for example, [21]), in vibrating string problems when the string loaded additionally with point masses (see, for example, [22]) and in diffraction problems (see, for example, [23]).

Inverse problem for discontinious Sturm-Liouville operator with Dirichlet conditions were given in [15]. In the present paper, not only differential equation, but also both of the boundary conditions of the problem L contain spectral parameter. In this case, inverse problem according to the Weyl functions [8] and the spectral data, i.e. (i): the sets of eigenvalues and norming constants; (ii): two different eigenvalues sets, is studied.

2. **Operator Treatment.**

Let the inner product in the Hilbert Space $\mathcal{H} = \mathcal{L}_2(0,\pi) \oplus \mathbb{C}^2$ be defined by

$$\langle F, G\rangle := \int_0^\pi F_1(x)\overline{G_1(x)}dx + \frac{1}{\rho_1}F_2\overline{G_2} + \frac{1}{\rho_2}F_3\overline{G_3}$$

for

$$F = \begin{pmatrix} F_1(x) \\ F_2 \\ F_3 \end{pmatrix}, \quad G = \begin{pmatrix} G_1(x) \\ G_2 \\ G_3 \end{pmatrix} \in \mathcal{H}.$$

Define an operator T acting in $\mathcal{H}$ with the domain $D(T) = \{F \in \mathcal{H} : F_1(x)$ and $F_1'(x)$ are absolutely continuous in $[0,d) \cup (d,\pi]$, $\ell F_1 \in \mathcal{L}_2(0,\pi)$, $F_1(d+0) = \alpha F_1(d-0)$, $F_1'(d+0) = \alpha^{-1}F_1'(d-0)$, $F_2 = F_1'(0) + h_0F_1(0)$, $F_3 = F_1'(\pi) + H_0F_1(\pi)\}$ such that,

$$T(F) := \begin{pmatrix} -F_1''(x) + q(x)F_1(x) \\ h_1F_1'(0) + h_2F_1(0) \\ H_1F_1'(\pi) + H_2F_1(\pi) \end{pmatrix}$$

Theorem 2.1. *The linear operator T is symmetric.*

Proof. Let $F, G \in D(T)$. Since,

$$\langle TF, G\rangle - \langle F, TG\rangle = \int_0^d TF_1(x)\overline{G_1}(x)dx + \int_d^\pi TF_1(x)\overline{G_1}(x)dx$$

$$+\frac{1}{\rho_1}TF_2\overline{G_2} + \frac{1}{\rho_2}TF_3\overline{G_3} - \int_0^d F_1(x)\overline{TG_1}(x)dx - \int_d^\pi F_1(x)\overline{TG_1}(x)dx$$

$-\dfrac{1}{\rho_1}F_2\overline{TG_2}-\dfrac{1}{\rho_2}F_3\overline{TG_3}$

by two partial integration, we get

$$\langle TF,G\rangle-\langle F,TG\rangle=\left(-F_1'(x)\overline{G_1}(x)+F_1(x)\overline{G_1'}(x)\right)\left(|_0^d+|_d^\pi\right)$$
$$+\frac{1}{\rho_1}\left(h_1F_1'(0)+h_2F_1(0)\right)\left(\overline{G_1'}(0)+h_0\overline{G_1}(0)\right)$$
$$-\frac{1}{\rho_1}\left(h_1\overline{G_1'}(0)+h_2\overline{G_1}(0)\right)\left(F_1'(0)+h_0F_1(0)\right)$$
$$+\frac{1}{\rho_2}\left(H_1F_1'(\pi)+H_2F_1(\pi)\right)\left(\overline{G_1'}(\pi)+H_0\overline{G_1}(\pi)\right)$$
$$-\frac{1}{\rho_2}\left(H_1\overline{G_1'}(\pi)+H_2\overline{G_1}(\pi)\right)\left(F_1'(\pi)+H_0F_1(\pi)\right)$$

Use the domain of the operator T and $\rho_1>0$, $\rho_2>0$ to see that, $\langle TF,G\rangle=\langle F,TG\rangle$. So T is symmetric. □

Corollary 1. *All eigenvalues of the operator T (or the problem L) are real and two eigenfunctions $\varphi(x,\lambda_1)$, $\varphi(x,\lambda_2)$ corresponding to different eigenvalues λ_1 and λ_2 are orthogonal, i.e.,*

$$\int_0^\pi\varphi(x,\lambda_1)\overline{\varphi(x,\lambda_2)}dx+\frac{1}{\rho_1}\left(\varphi'(0,\lambda_1)+h_0\varphi(0,\lambda_1)\right)\left(\overline{\varphi'(0,\lambda_2)+h_0\varphi(0,\lambda_2)}\right)$$
$$+\frac{1}{\rho_2}\left(\varphi'(\pi,\lambda_1)+H_0\varphi(\pi,\lambda_1)\right)\left(\overline{\varphi'(\pi,\lambda_2)+H_0\varphi(\pi,\lambda_2)}\right)=0$$

Let us denote the solutions of (1) by $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ satisfying the initial conditions

$$\begin{pmatrix}\varphi\\ \varphi'\end{pmatrix}(0,\lambda)=\begin{pmatrix}-\lambda+h_1\\ \lambda h_0-h_2\end{pmatrix},\begin{pmatrix}\psi\\ \psi'\end{pmatrix}(\pi,\lambda)=\begin{pmatrix}-\lambda+H_1\\ \lambda H_0-H_2\end{pmatrix}\tag{5}$$

respectively and the jump conditions (4). These solutions satisfy the relation $\psi\left(x,\lambda_n\right)=\beta_n\varphi\left(x,\lambda_n\right)$ for any eigenvalue λ_n, where $\beta_n=\dfrac{\psi'\left(0,\lambda_n\right)+h_0\psi\left(0,\lambda_n\right)}{\rho_1}$.

Theorem 2.2. *The following asymptotics hold for sufficienly large $|k|$*

$$\frac{\varphi(x,k)}{k^2}=\begin{cases}-\cos kx+O\left(\dfrac{1}{|k|}\exp|\tau|\,x\right), & x<d,\\ -\alpha^+\cos kx+\alpha^-\cos k(2d-x)+O\left(\dfrac{1}{|k|}\exp|\tau|\,x\right), & x>d\end{cases}\tag{6}$$

$$\frac{\psi(x,k)}{k^2} = \begin{cases} -\alpha^+ \cos k(\pi - x) - \alpha^- \cos k(x + \pi - 2d) + \\ \quad +O\left(\frac{1}{|k|} \exp|\tau|(\pi - x)\right), \quad x < d, \\ -\cos k(\pi - x) + O\left(\frac{1}{|k|} \exp|\tau|(\pi - x)\right), \quad x > d \end{cases} \tag{7}$$

where, $k = \sqrt{\lambda}$, $\tau = \operatorname{Im} k$ *and* $\alpha^{\pm} = \frac{1}{2}\left(\alpha \pm \frac{1}{\alpha}\right)$.

Proof. It is clear that the functions $\varphi(x,k)$ and $\psi(x,k)$ satisfy the following integral equations,

$$\begin{aligned} \varphi(x,k) &= -\frac{h_2 - k^2 h_0}{k} \sin kx + (h_1 - k^2) \cos kx \\ &+ \frac{1}{k} \int_0^x \sin k(x-t) q(t) \varphi(t,k) dt \qquad x < d, \\ \varphi(x,k) &= -\frac{h_2 - k^2 h_0}{k} (\alpha^+ \sin kx + \alpha^- \sin k(2d - x)) \\ &+ (h_1 - k^2)(\alpha^+ \cos kx + \alpha^- \cos k(2d - x)) \\ &+ \frac{1}{k} \int_0^d (\alpha^+ \sin k(x-t) - \alpha^- \sin k(x+t-2d)) q(t) \varphi(t,k) dt \\ &+ \frac{1}{k} \int_d^x \sin k(x-t) q(t) \varphi(t,k) dt \qquad x > d. \\ \psi(x,k) &= \frac{H_2 - k^2 h_0}{k} (\alpha^+ \sin k(\pi - x) + \alpha^- \sin k(x + \pi - 2d)) \\ &+ (H_1 - k^2)(\alpha^+ \cos k(\pi - x) - \alpha^- \cos k(x + \pi - 2d)) \\ &+ \frac{1}{k} \int_d^\pi (\alpha^+ \sin k(t - x) + \alpha^- \sin k(x + t - 2d)) q(t) \psi(t,k) dt \\ &+ \frac{1}{k} \int_x^d \sin k(t-x) q(t) \psi(t,k) dt \qquad x < d, \\ \psi(x,k) &= \frac{H_2 - k^2 H_0}{k} \sin k(\pi - x) + (H_1 - k^2) \cos k(\pi - x) \\ &+ \frac{1}{k} \int_x^\pi \sin k(t-x) q(t) \psi(t,k) dt \qquad x > d. \end{aligned}$$

Since the proof of the equalities (6) and (7) are similar, let us prove only (7). Divide both sides of last integral equality by k^2 and put

$$\Psi_{(0)}(x,k) = \frac{H_1 - k^2}{k^2} (\alpha^+ \cos k(\pi - x) - \alpha^- \cos k(x + \pi - 2d))$$

$$+\frac{H_2-k^2H_0}{k^3}(\alpha^+\sin k(\pi-x)+\alpha^-\sin k(x+\pi-2d))$$

$$\Psi_{(n+1)}(x,k)=\frac{1}{k}\int_d^\pi(\alpha^+\sin k(t-x)+\alpha^-\sin k(x+t-2d))q(t)\Psi_{(n)}(t,k)dt$$

$$+\frac{1}{k}\int_x^d\sin k(t-x)q(t)\Psi_{(n)}(t,k)dt,\qquad n\geq 1.$$

If we use the Picard's iteration method to above integral equations, we get

$$|\Psi(x,k)|=\left|\sum_{n=0}^{\infty}\Psi_{(n)}(x,k)\right|\leq\sum_{n=0}^{\infty}\left|\Psi_{(n)}(x,k)\right|\leq C\exp\left\{C\sigma(x)\right\},$$

where $C=(\alpha^++|\alpha^-|)(|H_0|+|H_1|+|H_2|+1)$, $\sigma(x)=\int_x^\pi|q(t)|\,dt$

Last inequality gives $\psi(x,k)=O\left(k^2\exp|\tau|(\pi-x)\right)$. From this equality and integral equations of $\psi(x,k)$, we get equality (7) □

3. **Properties of Spectrum.**

In the present section, properties of eigenvalues, eigenfunctions, and the resolvent operator of the problem L are investigated.

It is obvious that the characteristic function $\Delta(\lambda)$ of the problem L is as follows

$$\Delta(\lambda)\;=\;W(\varphi,\psi)=\lambda(\varphi'(\pi,\lambda)+H_0\varphi(\pi,\lambda))-H_1\varphi'(\pi,\lambda)-H_2\varphi(\pi,\lambda)\quad(8)$$

The roots of $\Delta(\lambda)=0$ coincide with the eigenvalues of problem L.

We define norming constants by

$$\alpha_n\;:\;=\int_0^\pi\varphi^2(x,\lambda_n)dx+\frac{1}{\rho_1}\left(\varphi'(0,\lambda_n)+h_0\varphi(0,\lambda_n)\right)^2\quad(9)$$

$$+\frac{1}{\rho_2}\left(\varphi'(\pi,\lambda_n)+H_0\varphi(\pi,\lambda_n)\right)^2$$

Lemma 3.1. *The eigenvalues of the problem L are simple and seperated.*

Proof. Let us write the following equations,

$$-\psi''(x,\lambda)+q(x)\psi(x,\lambda)=\lambda\psi(x,\lambda),\quad-\varphi''(x,\lambda_n)+q(x)\varphi(x,\lambda_n)=\lambda_n\varphi(x,\lambda_n).$$

If we multiply the first equation by $\varphi(x,\lambda_n)$, second by $\psi(x,\lambda)$ and subtract, after integrating from 0 to π we obtain

$$\left[\varphi'(x,\lambda_n)\psi(x,\lambda)-\psi'(x,\lambda)\varphi(x,\lambda_n)\right]\left(|_0^d+|_d^\pi\right)=(\lambda-\lambda_n)\int_0^\pi\psi(x,\lambda)\varphi(x,\lambda_n)\,dx.$$

Let $\varphi(x,\lambda_n)$ be an eigenfunction and use initial conditions (5), to get

$$\frac{\Delta(\lambda)}{\lambda-\lambda_n}=\int_0^{\pi}\psi(x,\lambda)\varphi(x,\lambda_n)dx+(\varphi'(\pi,\lambda_n)+H_0\varphi(\pi,\lambda_n))-\left(\psi'(0,\lambda_n)+h_0\psi(0,\lambda_n)\right)$$

Pass through the limit as $\lambda\to\lambda_n$ and using the equality $\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n)$, to get $\Delta'(\lambda_n)=\beta_n\alpha_n$. It is obvious that $\Delta'(\lambda_n)\neq 0$. So eigenvalues of the problem L are simple.

Since, $\Delta(\lambda)$ is an entire function of λ, the zeros of $\Delta(\lambda)$ are seperated. So lemma is proven. □

One can easily prove the following theorem using same methods in [1].

Theorem 3.2. *The operator T (or problem L) has a discrete spectrum. Moreover, the resolvent operator of T is defined as follows ;*

$$R_\lambda(T):=(T-\lambda I)^{-1}\begin{pmatrix}G_1(x)\\ G_2\\ G_3\end{pmatrix}=\begin{pmatrix}F_1(x)\\ F_2\\ F_3\end{pmatrix},\ \text{with } F_1=\int_0^{\pi}G(x,t,\lambda)G_1(t)dt,$$

$F_2=F_1'(0)+hF_1(0)$ *and* $F_3=F_1'(\pi)+HF_1(\pi)$

where $G(x,t,\lambda)$ *is defined by* $G(x,t,\lambda)=\dfrac{-1}{\Delta(\lambda)}\begin{cases}\varphi(x,\lambda)\psi(t,\lambda), & x\leq t\\ \varphi(t,\lambda)\psi(x,\lambda), & t\leq x\end{cases}$

The following theorem gives knowledge about the behaviour at infinity of the eigenvalues, eigenfunctions and normalizing numbers of L.

Theorem 3.3. *The eigenvalues λ_n, eigenfunctions $\varphi(x,\lambda_n)$ and normalizing numbers α_n of the problem L have the following asymptotic estimates for large n;*

$$\sqrt{\lambda_n}=k_n=k_{n-3}^0+\frac{\delta_n}{n}+\frac{\zeta_n}{n} \tag{10}$$

$$\varphi(x,k_n)=\begin{cases}-\left(k_{n-3}^0\right)^2\cos k_{n-3}^0x+O\left(\dfrac{1}{n}\right), & x<d,\\ \left(k_{n-3}^0\right)^2\left[-\alpha^+\cos k_{n-3}^0x+\alpha^-\cos k_{n-3}^0(2d-x)\right]+O\left(\dfrac{1}{n}\right), & x>d\end{cases} \tag{11}$$

$$\alpha_n=\left(k_{n-3}^0\right)^4\left(\frac{\pi-d}{2}\left(\left(\alpha^+\right)^2+\left(\alpha^-\right)^2\right)+\frac{d}{2}\right)+O\left(n^3\right) \tag{12}$$

where, $\zeta_n=o(1)$, $\delta_n\in\ell_\infty$, k_n^0 *are zeros of*

$\Delta_0(\lambda):=\lambda^3\left(\alpha^+\dfrac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}+\alpha^-\dfrac{\sin\sqrt{\lambda}(2d-\pi)}{\sqrt{\lambda}}\right)$ *and* $k_n^0=n+h_n$, $h_n\in\ell_\infty$.

Proof. Using (6) in (8), we get,

$$\Delta(\lambda)=\lambda^3\left(\alpha^+\frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}}+\alpha^-\frac{\sin\sqrt{\lambda}(2d-\pi)}{\sqrt{\lambda}}\right)+O\left(\lambda^2\exp|\tau|\pi\right)$$

for sufficiently large value of $|\lambda|$. Denote

$$\Delta_0(\lambda) := \lambda^3 \left(\alpha^+ \frac{\sin \sqrt{\lambda}\pi}{\sqrt{\lambda}} + \alpha^- \frac{\sin \sqrt{\lambda}(2d-\pi)}{\sqrt{\lambda}} \right),$$

$$G_n := \{\lambda \in C : \lambda = k^2, |k| = \left|k_n^0\right| - \delta\},$$

where δ is sufficiently small and k_n^0 are the zeros of $\Delta_0(\lambda)$ except 0.

Since, $|\Delta_0(\lambda)| \geq C \exp(\lambda^{\frac{5}{2}} \exp|\tau|\pi)$ and $|\Delta(\lambda) - \Delta_0(\lambda)| = O(\lambda^2 \exp|\tau|\pi)$ for $\lambda \in G_n$ and large values n, using the Rouche's theorem, we establish that, the functions $\Delta_0(\lambda)$ and $\Delta(\lambda)$ have the same number of zeros inside the contour G_n. Consequently, in the annulus between G_n and G_{n+1}, Δ has precisely one zero, namely k_n^2. Therefore, for the eigenvalue λ_n, the equality $\lambda_{n+3} = k_n^2$ is true. On the other hand, by using again the Rouche's theorem in $\gamma_\varepsilon := \left\{\lambda : \left|\lambda - k_n^0\right| < \varepsilon\right\}$, for sufficiently small ε, we get the asymptotic formulae $k_n = k_n^0 + \varepsilon_n, (\varepsilon_n = o(1))$ is valid for large n. Finally, the equality $\varepsilon_n = O(\frac{1}{n})$ is taken from the well known formulae $\Delta_0(k_n^0 + \varepsilon_n) = \Delta_0'(k_n^0).\varepsilon_n + o(\varepsilon_n)$. This fact proves the equality (10). By using (10) in (6) and (11) in (9) we get (11) and (12) , respectively. □

4. **Inverse Problems.**

In the present section, we study the inverse problem recovering the boundary value problem L from its spectral data. We consider three statements of the inverse problem of the reconstruction of the boundary-value problem L fom the Weyl function, from the spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$ and from two spectra $\{\lambda_n, \mu_n\}_{n\geq 0}$.

Let the functions $\chi(x,k)$ and $\Phi(x,k)$ denote solutions of (1) that satisfy the conditions $\chi(0,k) = -1$, $\chi'(0,k) = h_0$ and $U(\Phi) = 1$, $V(\Phi) = 0$, respectively and the jump conditions (4). Since $W[\chi, \varphi] = \rho_1 \neq 0$, it is clear that the functions $\psi(x,k)$ can be represented as follows

$$\psi(x,k) = \frac{\Delta(k)}{\rho_1}\chi(x,k) - \frac{\psi'(0,k) + h_0\psi(0,k)}{\rho_1}\varphi(x,k)$$

or

$$\frac{\psi(x,k)}{\Delta(k)} = \frac{\chi(x,k)}{\rho_1} - \frac{\psi'(0,k) + h_0\psi(0,k)}{\rho_1\Delta(k)}\varphi(x,k) \tag{13}$$

Denote

$$M(k) := \frac{\psi'(0,k) + h_0\psi(0,k)}{\rho_1\Delta(k)} \tag{14}$$

It is clear that.

$$\Phi(x,k) = \frac{\chi(x,k)}{\rho_1} - M(k)\varphi(x,k) \tag{15}$$

The function $\Phi(x,k)$ and $M(k)$ are called the Weyl solution and theWeyl function, respectively for the boundary value problem L.

The Weyl function is meromorphic in λ with simple poles in the points λ_n. Relations (13) and (15) yield

$$\Phi(x,k) = \frac{\psi(x,k)}{\Delta(k)} \tag{16}$$

To study the inverse problem, we agree that together with L we consider a boundary value problem $\widetilde{L}$ of the same form but with different coefficients $\widetilde{q}(x), \widetilde{\alpha}, \widetilde{d}, \widetilde{h}_i, \widetilde{H}_i$, $i = 0, 1, 2$.

Theorem 4.1. *If $M(k) = \tilde{M}(k)$, then $L = \widetilde{L}$, i. e., $q(x) = \widetilde{q}(x)$, a.e. and $d = \widetilde{d}$, $\alpha = \widetilde{\alpha}$, $h_i = \widetilde{h}_i$, $H_i = \widetilde{H}_i$, $i = 0, 1, 2$.*

Proof. Let us define the matrix $P(x,k) = [P_{ij}(x,k)]_{i,j=1,2}$ by the formula

$$\begin{pmatrix} P_{11}(x,k) & P_{12}(x,k) \\ P_{21}(x,k) & P_{22}(x,k) \end{pmatrix} = \begin{pmatrix} \varphi\tilde{\Phi}' - \Phi\tilde{\varphi}' & \Phi\tilde{\varphi} - \varphi\tilde{\Phi} \\ \varphi'\tilde{\Phi}' - \Phi'\tilde{\varphi}' & \Phi'\tilde{\varphi} - \varphi'\tilde{\Phi} \end{pmatrix} \tag{17}$$

and inversion formula

$$\begin{pmatrix} \varphi(x,k) \\ \Phi(x,k) \end{pmatrix} = \begin{pmatrix} P_{11}(x,k)\tilde{\varphi}(x,k) + P_{12}(x,k)\tilde{\varphi}'(x,k) \\ P_{11}(x,k)\tilde{\Phi}(x,k) + P_{12}(x,k)\tilde{\Phi}'(x,k) \end{pmatrix} \tag{18}$$

It is easy to see that the functions $[P_{ij}(x,k)]_{i,j=1,2}$ are meremorphic in k . Moreover, if $M(k) = \tilde{M}(k)$ then from (15) and (17) $P_{11}(x,k)$ and $P_{12}(x,k)$ are entire functions in k. Denote

$$G_\delta = \{k : |k - k_n| > \delta, \ n = 1, 2, ...\}$$

and

$$\widetilde{G}_\delta = \left\{k : \left|k - \widetilde{k}_n\right| > \delta, \ n = 1, 2, ...\right\},$$

where δ is sufficiently small number and k_n and $\widetilde{k}_n$ are square roots of eigenvalues of problem L and $\widetilde{L}$, respectively. $\Phi^{(\nu)}(x,k) \le C_\delta |k|^{\nu-3} \exp(-|\tau| x)$ is valid for $k \in G_\delta$, $\nu = 0, 1$, Thus we get

$$|P_{11}(x,k)| \le C_\delta, \ |P_{12}(x,k)| \le C_\delta |k|^{-1}, \ k \in G_\delta \cap \widetilde{G}_\delta \tag{19}$$

from (17). According to the last inequalities, if $M(k) = \tilde{M}(k)$ then from well known Liouville's theorem $P_{11}(x,k) = A(x)$, $P_{12}(x,k) \equiv 0$. Using (18), we take

$$\varphi(x,k) = A(x)\widetilde{\varphi}(x,k), \ \Phi(x,k) = A(x)\widetilde{\Phi}(x,k) \tag{20}$$

From

$$\begin{aligned} W\left[\Phi(x,\lambda),\varphi(x,\lambda)\right] &= \Phi(0,\lambda)\left(\lambda h_0-h_2\right)-\Phi'(0,\lambda)\left(h_1-\lambda\right) \\ &= \frac{\psi(0,\lambda)\left(\lambda h_0-h_2\right)-\psi'(0,\lambda)\left(h_1-\lambda\right)}{\Delta(\lambda)}\equiv 1 \end{aligned}$$

and similarly $W\left[\widetilde{\Phi}(x,\lambda),\widetilde{\varphi}(x,\lambda)\right]\equiv 1$, we have $A(x)\equiv 1$, i.e. $\varphi(x,k)\equiv\widetilde{\varphi}(x,k)$, $\Phi(x,k)\equiv\widetilde{\Phi}(x,k)$ and $\psi(x,k)\equiv\widetilde{\psi}(x,k)$. Therefore we get from (1), (4) and (5) that $q(x)=\widetilde{q}(x)$, a.e. and $d=\widetilde{d}$, $\alpha=\widetilde{\alpha}$, and $h_i=\widetilde{h}_i$, $H_i=\widetilde{H}_i$, $i=0,1,2$. Consequently $L=\widetilde{L}$. □

Lemma 4.2. *The following equality holds:*

$$M(\lambda) = \sum_{n=0}^{\infty}\frac{1}{\alpha_n(\lambda_n-\lambda)} \tag{21}$$

Proof. It follows from (14) the residues of $M(\lambda)$ at λ_n are $\operatorname*{Res}_{\lambda=\lambda_n}M(\lambda)=\dfrac{\psi'(0,\lambda_n)+h_0\psi(0,\lambda_n)}{\rho_1\dot{\Delta}(\lambda_n)}$. If we use equalities $\beta_n=-\dfrac{\psi'\left(0,\lambda_n\right)+h_0\psi\left(0,\lambda_n\right)}{\rho_1}$ and $\Delta'(\lambda_n)=\beta_n\alpha_n$,

$$\operatorname*{Res}_{\lambda=\lambda_n}M(\lambda) = -\frac{1}{\alpha_n} \tag{22}$$

is obtained. Denote $\Gamma_n=\{k:|k|=k_n+\varepsilon\}$, ε is sufficiently small number. Consider the counter integral $F_n(\lambda)=\dfrac{1}{2\pi i}\displaystyle\int_{\Gamma_n}\frac{M(k)}{(k-\lambda)}dk$, $\lambda\in int\Gamma_n$, Since, the estimate $\Delta(k)\geq|k|^5 C_\delta\exp(|\tau|\pi)$ holds for $k\in G_\delta$, using this inequality and (14) we get, $|M(k)|\leq\dfrac{C_\delta}{|k|^2}$, $k\in G_\delta$. Hence, $\lim\limits_{n\to\infty}F_n(\lambda)=0$, consequently, $M(\lambda)=\displaystyle\sum_{n=0}^{\infty}\frac{1}{\alpha_n(\lambda_n-\lambda)}$ is obtained by residue theorem and (22). □

Theorem 4.3. *If* $k_n=\widetilde{k}_n$ *and* $\alpha_n=\tilde{\alpha}_n$ *for* $n=0,1,\ldots$, *then* $L=\widetilde{L}$, *i. e.,* $q(x)=\widetilde{q}(x)$, *a.e. and* $d=\widetilde{d}$, $\alpha=\widetilde{\alpha}$, $h_i=\widetilde{h}_i$, $H_i=\widetilde{H}_i$, $i=0,1,2$. *Thus, the problem* L *is uniquely defined by spectral data.*

Proof. If $k_n=\widetilde{k}_n$ and $\alpha_n=\tilde{\alpha}_n$ for $n=0,1,...$, then from Lemma2, we get $M(\lambda)=\widetilde{M}(\lambda)$. Hence, Theorem6 gives $L=\widetilde{L}$. □

Let us consider the boundary value problem L_1 which is the problem that we take the condition $y'(0,k)-h_0y(0,k)=0$ instead of the condition (2) in L and $\{\mu_n^2\}_{n\geq 0}$ be the eigenvalues of the problem L_1.

Theorem 4.4. *If* $k_n = \widetilde{k}_n$ *and* $\mu_n = \tilde{\mu}_n$ *for all* $n \in \mathbb{N}$ *, then*

$$L\left(q, d, \alpha, h_0, h_1, h_2, H_i\right) = L(\widetilde{q}, \widetilde{d}, \widetilde{\alpha}, \widetilde{h}_0, h_1, \widetilde{h}_2, \widetilde{H}_i), i = 0, 1, 2.$$

Proof. Since, $\Delta(\lambda)$ is entire in λ of order $\frac{1}{2}$, by Hadamard's factorization theorem

$$\Delta(\lambda) \quad = \quad C \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right) \tag{23}$$

Where, C is a constant which depends only on $\{\lambda_n\}_{n \geq 0}$. It follows from (23) that the specification of the spectrum $\{\lambda_n\}_{n \geq 0}$ uniquely determines the characteristic function $\Delta(\lambda)$. Analogously, the function $\psi'(0, k) - h_0\psi(0, k)$ is uniquely determined by $\{\mu_n\}_{n \geq 0}$. Thus $\Delta(\lambda) = \widetilde{\Delta}(\lambda)$, $\psi'(0, k) - h_0\psi(0, k) = \widetilde{\psi}'(0, k) - \widetilde{h}_0\widetilde{\psi}(0, k)$ and consequently by (14), $M(\lambda) \equiv \widetilde{M}(\lambda)$. Hence from Theorem6 , the proof is completed. □

ÖZET: Bu çalışmada, sadece denklemin değil, her iki sınır koşulunun da spektral parametreye bağlı olduğu, aralıkta süreksizliğe sahip Sturm–Liouville problemi ele alınmıştır. Öncelikle bu probleme karşılık gelen operatör tanımlanmış ve bu operatör yardımıyla problemin spektral özellikleri araştırılmıştır. Daha sonra, spektral karakteristiklere göre problemin tek olarak belirlenebileceğini gösteren ters problemin teklik teoremleri ispatlanmıştır. Bu teoremler çalışmanın temel sonuçlarıdır.

References

[1] B. M. Levitan, and I. S. Sargsyan, Sturm-Liouville and Dirac Operators [in Russian], Nauka, Moscow (1988).

[2] C. T. Fulton, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, Proc. R. Soc. Edinburgh, A77 (1977), 293-308.

[3] C. T. Fulton, Singular eigenvalue problems with eigenvalue parameter contained in the boundary conditions, Proc. R. Soc. Edinburgh, A87 (1980), 1-34.

[4] E. M. Russakovskii, Operator treatment of boundary problems with spectral parameters entering via polynomials in the boundary conditions, Funct. Anal. Appl. 9 (1975) 358–359.

[5] N. J. Guliyev, Inverse eigenvalue problems for Sturm-Liouville equations with spectral parameter linearly contained in one of the boundary conditions, Inverse Problems, 21(2005), 1315-1330.

[6] G. Freiling, V. Yurko, Inverse Sturm–Liouville Problems and their Applications, Nova Science, New York, 2001.

[7] H. Schmid and C. Tretter, Singular Dirac Systems and Sturm–Liouville Problems Nonlinear in the Spectral Parameter, Journal of Differential Equations, Volume 181, Issue 2, 20 May 2002, Pages 511-542.

[8] H. Weyl, Über gewohnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen, Math. Ann. 68 (1910) 220–269.

[9] P. A. Binding, P. J. Browne, K. Seddighi, Sturm–Liouville problems with eigenparameter dependent boundary conditions, Proc. Edinburgh Math. Soc. (2) 37 (1993) 57–72.
[10] P. A. Binding, P. J. Browne, B. A. Watson, Inverse spectral problems for Sturm–Liouville equations with eigenparameter dependent boundary conditions, J. London Math. Soc. 62 (2000) 161–182.
[11] P. A. Binding, P. J. Browne and B. A. Watson,. Equivalence of inverse Sturm–Liouville problems with boundary conditions rationally dependent on the eigenparameter, J. Math. Anal. Appl. 291 (2004) 246–261.
[12] P. A. Binding and P. J. Browne, Oscillation theory for indefinite Sturm-Liouville problems with eigenparameter-dependent boundary conditions, Proc. R. Soc. Edinburgh A 127 (1997), 1123-1136.
[13] O. H. Hald, Discontinuous inverse eigenvalue problems, Comm. Pure Appl. Math., 37, (1984), 539-577.
[14] O. N. Litvinenko, V. I. Soshnikov, The theory of Heteregeneous Lines and Their Applications in Radio Engineering, Radio, Moscow, 1964 (in Russian).
[15] R. Kh. Amirov, On Sturm-Liouville operators with discontinuity conditions inside an interval, J. Math. Anal. Appl. 317 (2006) 163-176.
[16] R. Kh. Amirov, A. S. Ozkan, B. Keskin, Inverse Problems for Impulsive Sturm-Liouville Operator with Spectral Parameter Linearly Contained in Boundary Conditions, Integral Transforms and Special Functions, Vol. 20, 8(2009), 607–618.
[17] R. Mennicken, H. Schmid, A. A. Shkalikov, On the eigenvalue accumulation of Sturm-Liouville problems depending nonlinearly on the spectral parameter, Math. Nachr. 189, 157-170 (1998).
[18] N. Altinisik, M. Kadakal, O. Sh. Mukhtarov, Eigenvalues and eigenfunctions of discontinuous Sturm-Liouville problems with eigenparameter dependent boundary conditions, Acta Math. Hungar. 102 (1-2), 2004, 159-175.
[19] V. P. Meschanov, A. L. Feldstein, Automatic Design of Directional Couplers, Sviaz, Moscow, 1980.
[20] J. Walter, Regular eigenvalue problems with eigenvalue parameter in the boundary condition, Math.Z. 133(1973), 301-312.
[21] A. V. Likov and Yu. A. Mikhailov, The Theory of Heat and Mass Transfer, Gosenergoizdat.
[22] A. N. Tikhonov and A. A. Samarskii, Equations of Mathematical Physics, Pergamon, Oxford.
[23] N. N. Voitovich, B. Z. Katsenelbaum, and A. N. Sivov, Generalized Method of Eigen-vibration in the Theory of Diffraction [M]. Nauka, Moskov, 1997(Russian).
[24] J. McLaughlin and P. Polyakov, On the uniqueness of a spherical symmetric speed of sound from transmission eigenvalues, J. Diff. Eqn. 107(1994),351-382.

Current address: Baki Keskin and Numan Yalçın, Department of Mathematics Faculty of Art and Sci. Cumhuriyet University 58140, Sivas, TURKEY

E-mail address: `bkeskin,bsozkan@cumhuriyet.edu.tr`

URL: `http://communications.science.ankara.edu.tr`