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ON QUASI-STATISTICAL CONVERGENCE

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Abstract. The sequence $x = (x_k)$ is quasi-statistically convergent to L provided that for each $\varepsilon > 0$, $\lim_n \frac{1}{c_n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$ where $\lim_n c_n = 0$, $c_n > 0$ for each $n \in \mathbb{N}$ and $\limsup_n \frac{c_n}{n} < \infty$. In this paper quasi-statistical convergence is compared with statistical convergence and other methods. Furthermore a decomposition theorem is proved and a factorization result is also given for quasi-statistical convergence.

1. Introduction

A number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$, $\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$ where the vertical bars indicate the number of elements in the enclosed set. In this case we write $st - \lim x = L$ or $x_k \to L\ (st)$ ([1], [2] and [10]). By S we denote the set of all statistically convergent sequences. This type of convergence method is quite effective, especially when the classical limit does not exist.

In [7] Ganichev and Kadets have defined the quasi-statistical filter. Motivating by their definition of quasi-statistical filter, we introduce quasi-statistical convergence and study the relationship between quasi-statistical convergence and statistical convergence. A decomposition theorem is also proved along with a factorization result for quasi-statistical convergence.

If K is a set of positive integers, $|K|$ will denote the cardinality of K. The natural density of K is given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \ k \in K\}|,$$

if it exists.

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The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set $K_\varepsilon = \{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}$ has natural density zero. In this case we write $st - \lim x = L$.

Throughout the paper we assume that $c := (c_n)$ is a sequence of positive real numbers such that

$$\lim_n c_n = \infty \text{ and } \limsup_n \frac{c_n}{n} < \infty. \tag{1.1}$$

We define the quasi-density of $E \subset \mathbb{N}$ corresponding to the sequence (c_n) by

$$\delta_c(E) := \lim_n \frac{1}{c_n} |\{k \leq n : \ k \in E\}|$$

if it exists.

The sequence $x = (x_k)$ is called quasi-statistically convergent to L provided that for every $\varepsilon > 0$ the set $E_\varepsilon = \{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}$ has quasi-density zero. In this case we write $st_q - \lim x = L$ or $x_k \to L \ (st_q)$.

The next result establishes the relationship between quasi-statistical convergence and statistical convergence.

Lemma 1.1. *If $x = (x_k)$ is quasi-statistically convergent to L then it is statistically convergent to L.*

Proof. Let $st_q - \lim x = L$ and $H := \sup_n \frac{c_n}{n}$. Since

$$\frac{1}{n} |\{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}| \leq \frac{H}{c_n} |\{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}|$$

the proof follows immediately. □

We give an example in order to show that the converse of Lemma 1.1 does not hold.

Example 1.2. Let $c := (c_n)$ be the sequence of positive real numbers such that $\lim_n c_n = \infty$, and $\lim_n \frac{\sqrt{n}}{c_n} = \infty$. We can choose a subsequence $\{c_{n_p}\}$ such that $c_{n_p} > 1$ for each $p \in \mathbb{N}$.

Consider the sequence $x = (x_k)$ defined by

$$x_k := \begin{cases} c_k & ; \quad k \text{ is square and } c_k \in \{c_{n_p} : p \in \mathbb{N}\} \\ 2 & ; \quad k \text{ is square and } c_k \notin \{c_{n_p} : p \in \mathbb{N}\} \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

It is easy to see that x is statistically convergent to zero. Now we show that x is not quasi-statistically convergent to zero.

Let $\varepsilon = 1$.

$$\begin{aligned} \frac{1}{c_n}\left|\{k \in \mathbb{N} : \ |x_k| \geq 1\}\right| &= \frac{1}{c_n}\left[\left|\sqrt{n}\right|\right] \\ &= \frac{1}{c_n}(\sqrt{n} - t_n) \end{aligned} \tag{1.2}$$

where $0 \leq t_n < 1$ for each $n \in \mathbb{N}$. Letting $n \to \infty$ in both sides of (1.2), we observe that x is not quasi-statistically convergent to zero.

The following result relates the statistical convergence to quasi-statistical convergence.

Lemma 1.3. *Let $c := (c_n)$ be the sequence of positive real numbers satisfying (1.1) and*

$$d := \inf_n \frac{c_n}{n} > 0 \tag{1.3}$$

If $x = (x_k)$ is statistically convergent to L then it is quasi-statistically convergent to L.

Proof. The result follows from the inequality:

$$\frac{1}{n}\left|\{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}\right| \geq d\frac{1}{c_n}\left|\{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}\right|.$$

□

Note that the condition given by (1.3) can not be omitted.

By Lemma 1.1 and Lemma 1.3, the next result follows immediately.

Theorem 1.4. *Let $c := (c_n)$ be the sequence of positive real numbers satisfying (1.1) and (1.3). Then $x = (x_k)$ is statistically convergent to L if and only if x is quasi-statistically convergent to L.*

By S_q, we denote the set of all quasi-statistically convergent sequences.

It is easy to see that every convergent sequence is quasi-statistically convergent, i.e., $c \subset S_q$ where c is the set of all convergent sequences.

2. Strong Quasi-Summability

In this section, introducing the strong quasi-summability, one of our purpose is to study inclusion theorems between quasi-statistical convergence and strong quasi-summability. From [3], [4] and [9] we know that there is a natural relationship between statistical convergence, Cesàro summability and strong Cesàro summability.

The sequence $x = (x_k)$ is said to be strongly quasi-summable to L if

$$\lim_n \frac{1}{c_n}\sum_{k=1}^{n}|x_k - L| = 0.$$

The space of all strongly quasi-summable sequences is denoted by N_q.

$$N_q := \left\{ x : \text{for some } L,\ \lim_n \frac{1}{c_n} \sum_{k=1}^{n} |x_k - L| = 0 \right\}.$$

Theorem 2.1. *Let $c := (c_n)$ be the sequence of positive real numbers satisfying (1.1). If x is strongly quasi-summable to L then it is quasi-statistically convergent to L.*

Proof. Let $x = (x_k)$ such that $st_q - \lim x = L$.

$$\frac{1}{c_n} \sum_{k=1}^{n} |x_k - L| \geq \frac{1}{c_n} \sum_{\substack{k=1 \\ |x_k - L| \geq \varepsilon}}^{n} |x_k - L| \geq \frac{\varepsilon}{c_n} \left| \{ k \leq n : |x_k - L| \geq \varepsilon \} \right|$$

which concludes the proof. □

Schoenberg showed that a bounded statistically convergent sequence is Cesàro summable [9]. Combining this result with Lemma 1.1 the following corollary follows easily.

Corollary 1. *Let x be a bounded sequence and a quasi-statistically convergent to L. Then x is Cesàro summable to L.*

Theorem 2.2. *Let x be a bounded sequence and a quasi-statistically convergent to L, and let (1.1) and (1.3) hold. Then x is strongly quasi-summable to L.*

Proof. The result follows from the inequality:

$$\frac{1}{c_n} \sum_{k=1}^{n} |x_k - L| < \varepsilon \frac{n}{c_n} + M \frac{1}{c_n} \left| \{ k \leq n : |x_k - L| \geq \varepsilon \} \right|$$

where $|x_k - L| \leq M$, for every $k \in \mathbb{N}$ since x is bounded. □

The next result is the decomposition theorem for quasi-statistical convergence which is an anolog of the decomposition theorem on statistical convergence ([2], [3], [8]).

Theorem 2.3. *If x is quasi-statistically convergent to L, then there is a sequence y which converges to L and quasi-statistically null sequence z such that $x = y + z$.*

Proof. Let x be a quasi-statistically convergent sequence.

We can find an increasing sequence of positive integers (N_j) such that

$$N_0 = 0 \text{ and } \frac{1}{c_n} \left| \left\{ k \leq n : |x_k - L| \geq \frac{1}{j} \right\} \right| < \frac{1}{j};\ n > N_j\ (j = 1, 2, ...).$$

Let us define $y = (y_k)$ and $z = (z_k)$ as follows;

$$\begin{array}{llll} z_k = 0 & \text{and} \quad y_k = x_k & ; \text{if } N_0 < k \le N_1 & \\ z_k = 0 & \text{and} \quad y_k = x_k & ; \text{if } |x_k - L| < \dfrac{1}{j} & , \quad N_j < k \le N_{j+1}, \text{ for } j \ge 1 \\ z_k = x_k - L & \text{and} \quad y_k = L & ; \text{if } |x_k - L| \ge \dfrac{1}{j} & , \quad N_j < k \le N_{j+1}, \text{ for } j \ge 1 \end{array}$$

It is easy to see that $x = y + z$.

Now we show that y is convergent to L.

Given $\varepsilon > 0$. Let j such that $\varepsilon > \frac{1}{j}$. If $|x_k - L| \ge \frac{1}{j}$; $k > N_j$, then $|y_k - L| = |L - L| = 0$. If $|x_k - L| < \frac{1}{j}$, then $|y_k - L| = |x_k - L| < \frac{1}{j} < \varepsilon$.

Therefore

$$\lim_{k\to\infty} y_k = L.$$

To show that z is quasi-statistically null sequence; it is enough to prove

$$\lim_{n\to\infty} \frac{1}{c_n} |\{k \le n : z_k \ne 0\}| = 0.$$

We know, for $\varepsilon > 0$, that

$$\{k \le n : |z_k| \ge \varepsilon\} \subseteq \{k \le n : z_k \ne 0\}.$$

Thus

$$|\{k \le n : |z_k| \ge \varepsilon\}| \le |\{k \le n : z_k \ne 0\}|.$$

If $\frac{1}{j} < \delta$ for $\delta > 0$ and $j \in \mathbb{N}$, we show that $\frac{1}{n} |\{k \le n : z_k \ne 0\}| < \delta$ for all $n > N_j$.

In this case $z_k \ne 0$ if and only if $|x_k - L| \ge \frac{1}{j}$, $N_j < k \le N_{j+1}$.

If $N_j < k \le N_{j+1}$, then

$$\{k \le n : z_k \ne 0\} = \left\{k \le n : |x_k - L| \ge \frac{1}{j}\right\}.$$

Thus if $N_v < k \le N_{v+1}$ and $v > j$, then

$$\frac{1}{c_n} |\{k \le n : z_k \ne 0\}| \le \frac{1}{c_n} \left|\left\{k \le n : |x_k - L| \ge \frac{1}{v}\right\}\right| < \frac{1}{v} < \frac{1}{j} < \delta$$

which concludes the proof. □

The following result is an immediate consequence of Theorem 2.3.

Corollary 2. *If x is quasi-statistically convergent to L, then x has a subsequence y such that y converges to L.*

The following two Tauberian results follow from Theorems 3 and 5 of [2] and the present Lemma 1.1:

Theorem 2.4. *If x is a sequence such that x is quasi-statistically convergent to L and $\Delta x_k = o(\frac{1}{k})$ then x is convergent to L where $\Delta x_k = x_k - x_{k+1}$.*

Theorem 2.5. *Let* $\{k(i)\}_{i=1}^{\infty}$ *be an increasing sequence of positive integers such that* $\liminf_{i} \dfrac{k(i+1)}{k(i)} > 1$*, and let* x *be a corresponding gap sequence:* $\Delta x_k = 0$ *if* $k \neq k(i)$ *for each* $i \in \mathbb{N}$*, if* x *is quasi-statistically convergent to* L *then* x *is convergent to* L*.*

3. Multipliers

This section is devoted to multipliers and factorization problem. Connor, Demirci and Orhan ([5], [6]) studied multipliers for bounded statistically convergent sequences. Following their idea, we get similar results for quasi-statistically convergent sequences.

Assume that two sequence spaces, E and F are given. A multiplier from E into F is a sequence u such that $ux = (u_n x_n) \in F$ whenever $x \in E$. The linear space of such multipliers will be denoted by $m(E, F)$.

Theorem 3.1. $x \in m(st_q, st_q)$ *if and only if* $x \in st_q$*.*

Proof. Necessity: Let $u \in m(stq, st_q)$. Then we have $ux \in st_q$ for an arbitrary $x \in st_q$. Hence we can choose $x = \chi_{\mathbb{N}} \in st_q$ then $ux = u \in st_q$.
Sufficiency: Let $x \in st_q$, $y \in st_q$. Considering the inequality

$$|\{k \in \mathbb{N} : \ |x_k y_k| \geq \varepsilon\}| \leq \left|\{k \in \mathbb{N} : \ |x_k| \geq \sqrt{\varepsilon}\}\right| + \left|\{k \in \mathbb{N} : \ |y_k| \geq \sqrt{\varepsilon}\}\right|,$$

we obtain $xy \in st_q$, i.e., $x \in m(stq, st_q)$. □

Theorem 3.2. $x \in m(N_q, st_q)$ *if and only if* $x \in st_q$*.*

Proof. Necessity: Let $u \in m(N_q, st_q)$. Then we have $ux \in st_q$ for an arbitrary $x \in N_q$. Hence we can choose $x = \chi_{\mathbb{N}} \in N_q$ then $ux = u \in st_q$.
Sufficiency: Let $x \in st_q$ and $y \in N_q$. Using Theorem 2.1 and Theorem 3.1 we have $x \in m(N_q, st_q)$. □

We shall be interested in sequences x that admit a factorization

$$x = yz$$

in which

$$y \in st_q \text{ and } z \in N_q.$$

Theorem 3.3. x *is a quasi-statistically convergent sequence if and only if there is a strongly quasi-summable sequence* y *and quasi-statistically convergent sequence* z *such that* $x = yz$*.*

Proof. Necessity: Let $x \in st_q$. Since $\chi_{\mathbb{N}} \in N_q$, we have $x = \chi_{\mathbb{N}} x \in N_q.st_q$.
Sufficiency: Let $y \in N_q$ and $z \in st_q$ such that $x = yz$. It follows from Theorem 3.2 that $x \in st_q$ which completes the proof. □

ÖZET: Her $n \in \mathbb{N}$ için $c_n > 0$, $\lim_n c_n = 0$ ve $\limsup_n \frac{c_n}{n} < \infty$ olmak üzere her $\varepsilon > 0$ için $\lim_n \frac{1}{c_n} |\{ k \leq n : |x_k - L| \geq \varepsilon\}| = 0$ ise (x_k) dizisi L sayısına quasi-istatistiksel yakınsaktır denir. Bu çalışmada quasi-istatistiksel yakınsaklık, istatistiksel yakınsaklık ve diğer metodlarla karşılaştırılmıştır. Ayrıca quasi-istatistiksel yakınsaklık için bir ayrıştırma teoremi ve bir faktorizasyon problemi incelenmiştir.

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