A Sequence of Kantorovich-Type Operators on Mobile Intervals

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ABSTRACT. In this paper, we introduce and study a new sequence of positive linear operators, acting on both spaces of continuous functions as well as spaces of integrable functions on $[0, 1]$. We state some qualitative properties of this sequence and we prove that it is an approximation process both in $C([0, 1])$ and in $L^p([0, 1])$, also providing some estimates of the rate of convergence. Moreover, we determine an asymptotic formula and, as an application, we prove that certain iterates of the operators converge, both in $C([0, 1])$ and, in some cases, in $L^p([0, 1])$, to a limit semigroup. Finally, we show that our operators, under suitable hypotheses, perform better than other existing ones in the literature.

Keywords: Kantorovich-type operators, Positive approximation processes, Rate of convergence, Asymptotic formula, Generalized convexity.

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1. INTRODUCTION

In [13], the author proposed a modification of the classical Bernstein operators $B_n$ on $[0, 1]$ that, instead of fixing constants and the function $x$, fixes the constants and $x^2$, obtaining, in such a way, an order of approximation at least as good as the order of approximation of the operators $B_n$ in the interval $[0, 1/3]$. More precisely, those operators are defined by setting, for every continuous function on $[0, 1]$, $	ilde{B}_n(f) = B_n(f) \circ r_n$, where, for every $x \in [0, 1]$,

$$r_n(x) = \begin{cases} x^2 & \text{if } n = 1, \\ \frac{1}{2(n-1)} + \sqrt{\frac{nx^2}{n-1} + \frac{1}{4(n-1)^2}} & \text{if } n \geq 2. \end{cases}$$

Subsequently, other modifications of the classical Bernstein operators, as well as of many other well-known operators, that fix suitable functions were introduced (see [2] and the references quoted therein). Here we limit ourselves to mention that, for example, in [9], the authors considered a family of sequences of operators $(B_{n, \alpha})_{n \geq 1}, \alpha \geq 0$, that preserve the constants and the function $x^2 + \alpha x$. A further extension was presented in [12]; in that paper, Gonska, Raşa and Pituţ considered the operators $V_{\tau}^n(f) = B_n(f) \circ \tau_n$ ($f \in C([0, 1])$), where $\tau_n = (B_n(\tau))^{-1} \circ \tau$ and $\tau$ is a strictly increasing function on $[0, 1]$ such that $\tau(0) = 0$ and $\tau(1) = 1$. In particular, the operators $V_{\tau}^n$ preserve the constants and the function $\tau$.

In [10], instead, the authors introduced a modification of Bernstein operators fixing constants and a strictly increasing function $\tau$ in the following way: considering a strictly increasing function $\tau$ which is infinitely many times continuously differentiable on $[0, 1]$ and such that $\tau(0) = 0$...
and \( \tau(1) = 1 \), they introduced the operators

\[
B_n^\tau(f) = B_n(f \circ \tau^{-1}) \circ \tau \quad (n \geq 1, f \in C([0, 1])).
\]

The authors studied shape preserving and approximation properties of the operators \( B_n^\tau \), and compared them, under suitable assumptions, with the \( B_n \)'s and the \( V_n^\tau u \)'s. General sequences of positive linear operators fixing \( \tau \) and \( \tau^2 \) have been recently studied in \([1]\).

In this paper, motivated by works \([7], [4] \) and \([5]\), we present a Kantorovich-type modification of the operators \( B_n^\tau \). In particular, among other things, the authors introduced a sequence of positive linear operators \((C_n^\tau)_{n \geq 1}\) that generalize the classical Kantorovich operators on \([0, 1]\) and present the advantage to reconstruct any integrable function on \([0, 1]\) by means of its mean value on a finite numbers of subintervals of \([0, 1]\) that do not need to be a partition of \([0, 1]\).

Accordingly, in this work, for any integrable function \( f \) on \([0, 1]\) we shall study the operators

\[
C_n^\tau(f) = C_n(f \circ \tau^{-1}) \circ \tau \quad (n \geq 1),
\]

where \( \tau \) is a strictly increasing function that is infinitely many times continuously differentiable on \([0, 1]\) and such that \( \tau(0) = 0 \) and \( \tau(1) = 1 \).

The paper is organized as follows; after giving some preliminaries, we discuss some qualitative properties of the operators \( C_n^\tau \); in particular, we prove that they preserve some generalized convexity. We also prove that the sequence \((C_n^\tau)_{n \geq 1}\) is an approximation process for spaces of continuous as well as integrable functions and we evaluate the rate of convergence in both cases by means of suitable moduli of smoothness. As a byproduct, we obtain a simultaneous approximation result for the operators \( B_n^\tau \).

By using some results of \([5]\), we prove that the operators \( C_n^\tau \) satisfy an asymptotic formula with respect to a second order elliptic differential operator and, as an application, that suitable iterates of the \( C_n^\tau \)'s can be employed in order to constructively approximate strongly continuous semigroups in the function spaces considered in the paper.

Finally, as a further consequence of the above mentioned asymptotic formula, we compare the sequence \((C_n^\tau)_{n \geq 1}\) and the sequence \((C_n)_{n \geq 1}\), showing that, under suitable conditions, the former perform better.

2. Preliminaries

From now on, we denote by \( C([0, 1]) \) the space of all real-valued continuous functions on the interval \([0, 1]\). As usual, \( C([0, 1]) \) will be equipped with the uniform norm \( \| \cdot \|_\infty \).

For every \( i \geq 1 \), the symbol \( e_i \) stands for the functions \( e_i(x) := x^i \) for all \( x \in [0, 1] \); moreover \( 1 \) will indicate the constant function on \([0, 1]\) of constant value 1. If \( X \subset \mathbb{R} \), we denote by \( 1_X \) the characteristic function of \( X \), defined by setting, for every \( x \in \mathbb{R} \),

\[
1_X(x) := \begin{cases} 1 & \text{if } x \in X; \\ 0 & \text{if } x \notin X. \end{cases}
\]

Moreover, for every \( k \in \mathbb{N} \), we denote by \( C^k([0, 1]) \) the space consisting of all real-valued functions which are continuously differentiable up to order \( k \) on \([0, 1]\). In particular, if \( f \in C^k([0, 1]) \), for every \( i = 0, \ldots, k \), \( D^i(f) \) is the derivative of order \( i \) of \( f \). For simplicity, if \( i = 1, 2 \), we might also use the usual symbols \( f' \) and \( f'' \). Further, \( C^\infty([0, 1]) \) is the space of all real-valued functions which are infinitely many times continuously differentiable on \([0, 1]\).

Finally, for every \( p \in [1, +\infty) \), we denote by \( L^p([0, 1]) \) the space of all (the equivalence classes of) Borel measurable real-valued functions on \([0, 1]\) whose \( p \)th power is integrable with respect
to the Borel-Lebesgue measure $\lambda_1$ on $[0, 1]$. The space $L^p([0, 1])$ is endowed with the norm

$$\|f\|_p := \left( \int_0^1 |f(x)|^p \, dx \right)^{1/p} \quad (f \in L^p([0, 1])).$$

In what follows we recall the definition of certain operators acting on the space $L^1([0, 1])$ which represent a generalization of the classical Kantorovich operators on $[0, 1]$. They were studied in [7, Examples 1.2, 1] and subsequently extended to the multidimensional setting in [4, 5].

Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be two sequences of real numbers such that, for every $n \geq 1$, $0 \leq a_n < b_n \leq 1$. Then, consider the positive linear operator $C_n : L^1([0, 1]) \to C([0, 1])$ defined by setting, for any $f \in L^1([0, 1])$, $n \geq 1$ and $x \in [0, 1]$,

$$C_n(f)(x) = \sum_{k=0}^{n} \left( \frac{n+1}{b_n - a_n} \int_{\frac{k+b_n}{n+1}}^{\frac{k+a_n}{n+1}} f(t) \, dt \right) \binom{n}{k} x^k (1-x)^{n-k}. \quad (2.1)$$

Since $C_n(1) = 1$, the restriction to $C([0, 1])$ of each $C_n$ is continuous and we have $\|C_n\| = 1$ for any $n \geq 1$, where $\| \cdot \|$ denotes the usual operator norm on $C([0, 1])$.

We notice that if, in particular, $a_n = 0$ and $b_n = 1$ for any $n \geq 1$, the operators $C_n$ turn into the classical Kantorovich operators on $[0, 1]$.

For every $n \geq 1$,

$$C_n(e_1) = \frac{n}{n+1} e_1 + \frac{a_n + b_n}{2(n+1)} 1, \quad (2.2)$$

$$C_n(e_2) = \frac{1}{(n+1)^2} \left( n^2 e_2 + ne_1 (1-e_1) + n(a_n + b_n)e_1 + \frac{b_n^2 + a_n b_n + a_n^2}{3} 1 \right). \quad (2.3)$$

We also point out that (see [7, Formula (4.2)]), the operators $C_n$ are closely related to the classical Bernstein operators on $[0, 1]$.

In fact, if one denotes by $B_n$ the $n$-th Bernstein operator on $C([0, 1])$, for every $f \in L^1([0, 1])$, considering the function

$$F_n(f)(x) := \frac{n+1}{b_n - a_n} \int_{\frac{a_n + b_n}{n+1}}^{\frac{n+b_n}{n+1}} f(t) \, dt = \int_0^1 f \left( \frac{(b_n-a_n)t + a_n + nx}{n+1} \right) \, dt \quad (x \in [0, 1], n \geq 1)$$

(2.4)

it turns out that

$$C_n(f) = B_n(F_n(f)) \quad (2.5)$$

$(f \in L^1([0, 1]), n \geq 1)$.

As quoted in the Introduction, in [10] the authors introduced a modification of Bernstein operators that fixes suitable functions.

More precisely, consider a function $\tau \in C^\infty([0, 1])$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for every $x \in [0, 1]$.

The operators introduced in [10] are defined by

$$B^\tau_n(f) := B_n(f \circ \tau^{-1}) \circ \tau \quad (n \geq 1, f \in C([0, 1])). \quad (2.6)$$

Namely, for every $f \in C([0, 1]), n \geq 1$ and $x \in [0, 1]$,

$$B^\tau_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1-\tau(x))^{n-k} (f \circ \tau^{-1}) \left( \frac{k}{n} \right). \quad (2.7)$$
After the above preliminaries, we pass to introduce a new sequence of positive linear operators acting on integrable functions on \([0, 1]\), which is a combination of (2.1) and (2.6). More precisely, for any \(f \in L^1([0, 1])\) and \(n \geq 1\), we set
\[
C^n_\tau(f) := C_n(f \circ \tau^{-1}) \circ \tau;
\]
hence, for every \(f \in L^1([0, 1]), n \geq 1\), and \(x \in [0, 1]\),
\[
C^n_\tau(f)(x) = \sum_{k=0}^{n} \left( \frac{n+1}{b_n - a_n} \int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} (f \circ \tau^{-1})(t) \, dt \right) \left( \frac{n}{k} \right) \tau(x)^k(1 - \tau(x))^{n-k},
\]
where we have used the fact that, thanks to the change of variable theorem, \(f \circ \tau^{-1} \in L^1([0, 1])\) provided \(f \in L^1([0, 1])\).

Note that, if \(\tau = e_1\), the operators \(C^n_\tau\) turn into the operators \(C_n\) defined by (2.1), and hence in the classical Kantorovich operators whenever \(a_n = 0\) and \(b_n = 1\) for every \(n \geq 1\).

The operators \(C^n_\tau\) can be viewed as integral modification of Kantorovich-type of the operators \(B^n_\tau\) with mobile intervals.

### 3. Shape Preserving Properties of the \(C^n_\tau\)'s

This section is devoted to show some qualitative properties of the operators \(C^n_\tau\). To this end, we first remark that, taking (2.4), (2.5) and (2.8) into account, the following formula holds true:
\[
C^n_\tau(f) = B_n(F_n(f \circ \tau^{-1})) \circ \tau
\]
\((f \in L^1([0, 1]), n \geq 1)\).

Hence, one can recover some properties of the operators \(C^n_\tau\) by means of the relevant ones held by the \(B_n\)'s.

First off, as \(F_n(f)\) is increasing whenever \(f\) is (continuous and) increasing, the \(B_n\)'s map (continuous) increasing functions into increasing functions (see, e.g., [3, Remark p. 461]), and \(\tau\) is increasing, we have that the operators \(C^n_\tau\) map (continuous) increasing functions into increasing functions.

The \(C^n_\tau\)'s preserve also a particular form of convexity.

We recall (see [17]) that a function \(f \in C([0, 1])\) is said to be convex with respect to \(\tau\) if, for every \(0 \leq x_0 < x_1 < x_2 \leq 1\), one has
\[
\begin{vmatrix}
1 & 1 & 1 \\
\tau(x_0) & \tau(x_1) & \tau(x_2) \\
\tau(x_0) & \tau(x_1) & \tau(x_2)
\end{vmatrix} \geq 0.
\]

In particular, it can be proven that a function \(f\) is convex with respect to \(\tau\) if and only if \(f \circ \tau^{-1}\) is convex.

In [7, Proof of Th. 4.3]) it has been shown that the operators \(C_n\) map (continuous) convex functions into (continuous) convex functions; hence, thanks to (2.8), the operators \(C^n_\tau\) map (continuous) convex functions with respect to \(\tau\) into (continuous) convex functions with respect to \(\tau\).

Moreover, we investigate the monotonicity of the sequence \((C^n_\tau)_{n \geq 1}\) on convex functions with respect to \(\tau\).

**Proposition 3.1.** If \(f \in C([0, 1])\) is convex with respect to \(\tau\) and increasing (resp., decreasing), then, for every \(n \geq 1\),
\[
f \leq C^n_\tau(f) \quad \text{on} \quad [0, \tau^{-1}\left(\frac{a_n + b_n}{2}\right)],
\]
We point out that the operators $k$ and $f$ are the same happens for the $B$ where $B$.

By using the fundamental theorem of calculus, $f$ function $f$ is convex and increasing, so that

$$f \leq C_n^\tau(f) \quad \text{on} \quad \left[\tau^{-1}\left(\frac{a_n + b_n}{2}\right), 1\right].$$

Moreover, if $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ are constant sequences and $f \in C([0, 1])$ is convex with respect to $\tau$, then

$$C_{n+1}^\tau(f) \leq \frac{n+1}{n+2}C_n^\tau(f) + \frac{1}{n+2}B_{n+1}^\tau(f),$$

$B_n^\tau$ being defined by (2.7).

**Proof.** In [7, Proposition 4.5] it has been proven that, if $g$ is convex and increasing, then $g \leq C_n(g)$ on $[0, \frac{a_n + b_n}{2}]$. Hence because $f$ is convex with respect to $\tau$ and increasing, $f \circ \tau^{-1}$ is convex and increasing, so that

$$f \circ \tau^{-1} \leq C_n(f \circ \tau^{-1}) \quad \text{on} \quad \left[0, \frac{a_n + b_n}{2}\right],$$

and from this we get (3.10). Reasoning in the same way, one can establish (3.11).

Moreover, fix $f \in C([0, 1])$ convex function with respect to $\tau$. In [7, Theorem 4.4] it was established that, if $g \in C([0, 1])$ is convex, then, for all $n \geq 1$, $C_{n+1}(g) \leq \frac{n+1}{n+2}C_n(g) + \frac{1}{n+2}B_{n+1}(g)$, so that, by applying this result to $f \circ \tau^{-1}$, we get (3.12).

Besides the convexity with respect to $\tau$, the operators $C_n^\tau$ preserve another type of convexity. More precisely, given $\varphi \in C^\infty([0, 1])$ such that $\varphi'(x) \neq 0$ for all $x \in [0, 1]$ and $\varphi(0) = 0$, and $k \in \mathbb{N}$, a function $f \in C^k([0, 1])$ is said to be $\varphi$-convex of order $k$ if, for every $x \in [0, 1]$,

$$D^{(k)}(f)(x) := D^{(k)}(f \circ \varphi^{-1})(\varphi(x)) \geq 0.$$

For more details about $\varphi$-convex functions of order $k$ see [14].

Since in our case $\tau : [0, 1] \to [0, 1]$ is a bijection and a positive function, it is easy to show that a function $f \in C^k([0, 1])$ is $\tau$-convex of order $k$ if and only if

$$D^{(k)}_\tau(f) := D^{(k)}(f \circ \tau^{-1}) \geq 0.$$

In other words, $f$ is $\tau$-convex of order $k$ iff $f \circ \tau^{-1}$ is $k$-convex. Here we recall that a function $g \in C^k([0, 1])$ is said to be $k$-convex if $D^{(k)}(g) \geq 0$.

By using the fundamental theorem of calculus, $F_n$ maps $k$-convex functions into $k$-convex functions and the same happens for the $B_i$’s (see, for example, [6, Prop. A.2.5]). Then, thanks to (3.9) we have that the $C^\tau_n$’s map $\tau$-convex functions of order $k$ into $\tau$-convex functions of order $k$.

We point out that the operators $C^\tau_n$ do not preserve the convexity. In order to construct an example, we use the following alternative representation for the operators $C^\tau_n$: for every $n \geq 1$ and $f \in L^1([0, 1])$,

$$C^\tau_n(f) = B^\tau_n(G^\tau_n(f \circ \tau^{-1})),$$

where

$$G^\tau_n(f)(x) := \frac{n+1}{b_n - a_n} \int_{a_n}^{b_n} f(t) \, dt.$$

Then, choosing $a_n = 0$, $b_n = 1$ for all $n \geq 1$, $\tau = (e_1 + e_2)/2$ and $f = e_1$,

$$C^\tau_n(e_1) = \frac{n}{n+1}B^\tau_n(e_1) + \frac{1}{2(n+1)}.$$
Recalling that in this case $B_n^*(e_1)$ is not convex for lower $n$ (see [10]), we get that the same happens for $C_n^*(e_1)$.

Now we pass to show that each $C_n^*$ preserves the class of Hölder continuous functions. Given $M > 0$ and $0 \leq \alpha \leq 1$, we shall write $f \in \text{Lip}_M\alpha$ if

$$|f(x) - f(y)| \leq M|x - y|^\alpha \quad \text{for every } 0 \leq x, y \leq 1.$$  

In particular, if $\alpha = 1$, we get the space of all Lipschitz functions of Lipschitz constant $M$.

First observe that, from hypotheses on $\tau$, both $\tau$ and $\tau^{-1}$ are Lipschitz functions. Precisely, $\tau \in \text{Lip}_L 1$ with $L := \|\tau'|_\infty$ and $\tau^{-1} \in \text{Lip}_{N_1} 1$ with $N := (\min_{[0,1]} \tau')^{-1}$. Therefore, by recalling that $C_n(\text{Lip}_M 1) \subset \text{Lip}_{CM} 1$ with $C := \max\{1, |f(0)| + |f(1)|\}$ (see [7, Th. 4.1 and Example n. 1]), from (2.8) it follows that

$$(3.13) \quad C_n^*(\text{Lip}_M 1) \subset \text{Lip}_{CMN} 1 \quad \text{for every } n \geq 1.$$  

On account of [3, Cor. 6.1.20], since $\|C_n^*\| = 1$ and property (3.13) holds, for every $n \geq 1$, $f \in C([0,1])$, $\delta > 0$, $M > 0$ and $0 < \alpha \leq 1$,

$$\omega(C_n^*(f), \delta) \leq (1 + C)\omega(f, \delta) \quad \text{and} \quad C_n^*(\text{Lip}_M\alpha) \subset \text{Lip}_{(CLN)^\alpha M} \alpha.$$  

Finally, for every $k \in \mathbb{N}$, denote by $P_{\tau,k}$ the linear subspace generated by the set $\{\tau^i : i = 0, \ldots, k\}$. $P_{\tau,k}$ is said to be the space of the $\tau$-polynomials of degree $k$. Since both the $B_n^*$’s and the $F_n$’s map polynomials of degree $k$ into polynomials of degree $k$, taking (3.9) into account, we have that

$$C_n^*(P_{\tau,k}) \subset P_{\tau,k} \quad (k \in \mathbb{N}, n \geq 1).$$  

4. Approximation Properties of the $C_n^*$’s

In this section, we prove that $(C_n^*)_{n \geq 1}$ is a positive approximation process both in $C([0,1])$ and in $L^p([0,1])$, $1 \leq p < +\infty$, and we provide some estimates of the rate of convergence, by means of suitable moduli of smoothness. As a byproduct of the uniform convergence, we obtain a property of the operators $B_n^*$ introduced in [10], which seems to be new.

We begin by stating the following result.

**Theorem 4.1.** For every $f \in C([0,1])$, we have that

$$(4.14) \quad \lim_{n \to \infty} C_n^*(f) = f$$

uniformly on $[0,1]$.

**Proof.** From (2.2) and (2.3) it easily follows that

$$(4.15) \quad C_n^*(\tau) = \frac{n}{n+1} \tau + \frac{a_n + b_n}{2(n+1)} 1,$$

$$(4.16) \quad C_n^*(\tau^2) = \frac{1}{(n+1)^2} \left( n^2 \tau^2 + n\tau (1 - \tau) + n(a_n + b_n)\tau + \frac{b_n^2 + a_nb_n + a_n^2}{3} 1 \right);$$

since $C_n^*(1) = 1$ and $\{1, \tau, \tau^2\}$ is an extended Tchebychev system on $[0,1]$, (4.14) comes directly by an application of Korovkin Theorem (see [3, Example 5, p. 246]).

In order to get a quantitative version of the above uniform convergence, we use a result due to Paltanea (see [15]) which involves the usual modulus of continuity of the first and second order, denoted, respectively, by $\omega(f, \delta)$ and $\omega_2(f, \delta)$. To this end, we need some further preliminaries. For $x \in [0,1]$, we denote by $e_{\tau,x}^i$ the function

$$e_{\tau,x}^i(t) = (\tau(t) - \tau(x))^i \quad (i = 0, 1, 2, \ldots).$$
When \( \tau = e_1 \) we shall simply write \( \psi_x^i(t) = (t-x)^i \).
In particular, for any \( n \geq 1 \) and \( x \in [0, 1] \) (see (4.15) and (4.16)),

\[
C_n^\tau(e_n^x, 2)(x) = \frac{1 - n}{(n+1)^2} \tau^2(x) + \frac{n - a_n - b_n}{(n+1)^2} \tau(x) + \frac{b_n^2 + a_nb_n + a_n^2}{3(n+1)^2}. \tag{4.17}
\]

Moreover, we recall the following result (see \([11, \text{Formula (8)}]\)): there exists a constant \( K > 0 \) such that

\[
K \psi^2_x(t) \leq \tau'(x) e^x_{\tau, 2}(t) \quad \text{for every } x, t \in [0, 1]. \tag{4.18}
\]

Obviously, \( K = 1 \) if \( \tau = e_1 \).

**Proposition 4.2.** Consider \( n \geq 1, f \in C([0, 1]), 0 \leq x \leq 1, \) and \( \delta > 0. \) Then

\[
|C_n^\tau(f)(x) - f(x)| \leq \omega(f, \delta_n^\tau(x)) + \frac{3}{2} \omega_2(f, \delta_n^\tau(x)), \tag{4.19}
\]

where

\[
\delta_n^\tau(x) = \frac{\sqrt{n+1}}{n+1}\sqrt{(n-1)\tau(x)(1-\tau(x)) + (1-a_n-b_n)\tau(x) + \frac{b_n^2 + a_nb_n + a_n^2}{3}}.
\]

Moreover,

\[
\|C_n^\tau(f) - f\|_\infty \leq \omega \left( f, \frac{\|\tau'\|^{1/2}_{\infty}}{\sqrt{K}\sqrt{n+1}} \right) + \frac{3}{2} \omega_2 \left( f, \frac{\|\tau'\|^{1/2}_{\infty}}{\sqrt{K}\sqrt{n+1}} \right). \tag{4.20}
\]

**Proof.** Let \( n \geq 1, f \in C([0, 1]), 0 \leq x \leq 1 \) and \( \delta > 0. \) Paltanea’s estimate ([15, Theorem 2.2.1]; see, also, [6, Theorem 1.6.2]) runs as follows:

\[
|C_n^\tau(f)(x) - f(x)| \leq |f(x)||C_n^\tau(1)(x) - 1| + \frac{\delta^{-1}|C_n^\tau(\psi_x^2)(x)|\omega(f, \delta) + \left( C_n^\tau(1)(x) + (2\delta^2)^{-1}C_n^\tau(\psi_x^2)(x) \right)\omega_2(f, \delta)}{1 + (2\delta^2)^{-1}C_n^\tau(\psi_x^2)(x)} \omega_2(f, \delta).
\]

Cauchy-Schwarz inequality yields

\[
|C_n^\tau(\psi_x)| \leq \sqrt{C_n^\tau(\psi_x^2)},
\]

so that

\[
|C_n^\tau(f)(x) - f(x)| \leq \frac{\delta^{-1}}{\sqrt{n+1}} C_n^\tau(\psi_x^2)(x)\omega(f, \delta) + (1 + (2\delta^2)^{-1}C_n^\tau(\psi_x^2)(x))\omega_2(f, \delta).
\]

From (4.18), (4.17) and the positivity of \( C_n^\tau \)'s, we have

\[
KC_n^\tau(\psi_x^2)(x) \leq \tau'(x) C_n^\tau(e_n^x, 2) \leq \frac{\tau'(x)}{(n+1)^2} \left\{ (n-1)\tau(x)(1-\tau(x)) + (1-a_n-b_n)\tau(x) + \frac{b_n^2 + a_nb_n + a_n^2}{3} \right\}.
\]

Therefore,

\[
C_n^\tau(\psi_x^2) \leq \frac{\tau'(x)}{K(n+1)^2} \left\{ (n-1)\tau(x)(1-\tau(x)) + (1-a_n-b_n)\tau(x) + \frac{b_n^2 + a_nb_n + a_n^2}{3} \right\}
\]

and, for \( \delta = \delta_n^\tau(x) \), we get (4.19). Estimate (4.20) follows by noting that

\[
\delta_n^\tau(x) \leq \frac{\|\tau'\|_{\infty}^{1/2}}{\sqrt{K}\sqrt{n+1}}
\]

since \( 0 \leq \tau(x) \leq 1 \). \( \square \)
As a byproduct of Theorem 4.1, we present a simultaneous approximation result for the operators $B_n^\tau$ given by (2.7). As far as we know, this property is new.

**Theorem 4.2.** Suppose that $a_n = 0$ and $b_n = 1$ for every $n \geq 1$. Then, for every $f \in C^1([0, 1])$,
\begin{equation}
(4.22) \quad B_{n+1}^\tau(f)' = \tau' C_n^\tau(f' / \tau').
\end{equation}
Moreover,
\begin{equation}
(4.23) \quad \lim_{n \to \infty} B_n^\tau(f)' = f' \text{ uniformly on } [0, 1].
\end{equation}

**Proof.** Let $x \in [0, 1]$, $f \in C^1([0, 1])$, and $n \geq 1$. From (2.7) it follows that
\[
B_{n+1}^\tau(f)'(x) = \tau'(x) \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1 - \tau(x))^{n-k} \times (n + 1) \left( (f \circ \tau^{-1}) \left( \frac{k + 1}{n + 1} \right) - (f \circ \tau^{-1}) \left( \frac{k}{n + 1} \right) \right) \\
= \tau'(x) \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1 - \tau(x))^{n-k} \left( n + 1 \right) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (f \circ \tau^{-1})'(t) \, dt \\
= \tau'(x) C_n^\tau \left( \frac{f'}{\tau'} \right)(x),
\]
and this completes the proof of (4.22). Formula (4.23) immediately follows from (4.22) and Theorem 4.1, because $\tau'$ is bounded. \hfill \Box

Now we prove that the sequence $(C_n^\tau)_{n \geq 1}$ is a positive approximation process also in $L^p([0, 1])$ for any $p \in [1, +\infty[.

**Theorem 4.3.** Assume that
\[
\sup_{n \geq 1} \frac{1}{b_n - a_n} = M \in \mathbb{R}.
\]
Then, for every $p \in [1, +\infty[$ and $f \in L^p([0, 1]),$
\begin{equation}
(4.24) \quad \lim_{n \to \infty} C_n^\tau(f) = f \text{ in } L^p([0, 1]).
\end{equation}

**Proof.** By Theorem 4.1, for every $f \in C([0, 1])$, \( \lim_{n \to \infty} C_n(f) = f \) in $L^p$-norm, as well. Since $C([0, 1])$ is dense in $L^p([0, 1])$, in order to prove the statement it is sufficient to show, thanks to Banach-Steinhaus theorem, that the sequence of operators $C_n^\tau : L^p([0, 1]) \to L^p([0, 1]) \quad (n \geq 1)$ is equicontinuous, i.e.,
\[
\sup_{n \geq 1} \| C_n^\tau \|_{L^p, L^p} < +\infty.
\]
To this end, for every $n \geq 1$, $f \in L^p([0, 1])$ and $x \in [0, 1]$, we preliminary notice that, since the function $|t|^p (t \in \mathbb{R})$ is convex,
\[
|C_n^\tau(f)(x)|^p \leq \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1 - \tau(x))^{n-k} \left[ \frac{(n + 1)}{(b_n - a_n)} \int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} (f \circ \tau^{-1}) (t) \, dt \right]^p.
\]
By applying Jensen’s inequality (see, e.g., [8, Theorem 3.9]) to the probability measure 

\[ \frac{n+1}{b_n-a_n} 1_{\left[\frac{k+a_n}{n+\tau}, \frac{k+b_n}{n+\tau}\right]} \lambda_1 \] on \([0, 1]\), we get

\[
|C_n^\tau(f)(x)|^p \leq \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1 - \tau(x))^{n-k} \frac{(n+1)}{(b_n-a_n)} \int_{\left[\frac{k+a_n}{n+\tau}, \frac{k+b_n}{n+\tau}\right]} |(f \circ \tau^{-1})(t)|^p \, dt
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1 - \tau(x))^{n-k} \frac{(n+1)}{(b_n-a_n)} \int_{\tau^{-1}(\frac{k+a_n}{n+\tau})}^{\tau^{-1}(\frac{k+b_n}{n+\tau})} |f(y)|^p \, dy
\]

\[
\leq \|\tau'|_{\infty} \| (n+1) \sum_{k=0}^{n} \binom{n}{k} \tau(x)^k (1 - \tau(x))^{n-k} \int_{\tau^{-1}(\frac{k+a_n}{n+\tau})}^{\tau^{-1}(\frac{k+b_n}{n+\tau})} |f(y)|^p \, dy.
\]

We point out that

\[
\int_{0}^{1} \tau(x)^k (1 - \tau(x))^{n-k} \, dx = \int_{0}^{1} t^k(1-t)^{n-k} \frac{\tau'(1-\tau(t))}{\tau'(1-t)} \, dt \leq \frac{1}{\min_{y \in [0,1]} \tau'(y)} \frac{1}{\binom{n}{k}(n+1)}.
\]

Hence, by integrating with respect to \(x\), we obtain

\[
\|C_n^\tau(f)\|_p \leq MN \|f\|_p,
\]

where

\[
N := \frac{\|\tau'|_{\infty}}{\min_{y \in [0,1]} \tau'(y)};
\]

hence \(\|C_n^\tau\|_{L^p,L^p} \leq (MN)^{1/p} < +\infty. \)

An estimate of the convergence in (4.24) can be obtained by using a result due to Swetits and Wood [16, Theorem 1] which involves the second-order integral modulus of smoothness defined, for \(f \in L^p([0,1]), 1 \leq p < +\infty\), as

\[
\omega_{2,p}(f,\delta) := \sup_{0 < t \leq \delta} \|f(\cdot + t) - 2f(\cdot) + f(\cdot - t)\|_p \quad (\delta > 0).
\]

We define

\[
\beta_{n,p,\tau} := \frac{1}{(n+1)\sqrt{K}} \times \left\| \sqrt{\tau'} \left\{ (n-1)\tau(1-\tau) + (1-a_n-b_n)\tau + \frac{b_n^2 + a_nb_n + a_n^2}{3} \right\} \right\|_{p}^{1/2}
\]

and

\[
\gamma_{n,p,\tau} := \frac{1}{(n+1)^{2p/(2p+1)}K^{p/(2p+1)}}
\]

\[
\times \left\| \tau' \left\{ (n-1)\tau(1-\tau) + (1-a_n-b_n)\tau + \frac{b_n^2 + a_nb_n + a_n^2}{3} \right\} \right\|_{p}^{(2p+1)},
\]

where \(K\) is the strictly positive constant in (4.18). Then we can state the following result.

**Proposition 4.3.** Under the hypotheses of Theorem 4.3, for every \(p \in [1, +\infty]\) there exists \(C_p > 0\) such that, for every \(f \in L^p([0,1])\) and for \(n\) sufficiently large,

\[
\|C_n^\tau(f) - f\|_p \leq C_p(\alpha_{n,p,\tau}^2 \|f\|_p + \omega_{2,p}(f,\alpha_{n,p,\tau}))
\]

where \(\alpha_{n,p,\tau} = \max\{\beta_{n,p,\tau}, \gamma_{n,p,\tau}\} \).
Proof. First we introduce the following auxiliary functions:

\[ F^\tau_n(x) := C^\tau_n(x_\tau)(x), \quad G^\tau_n(x) := C^\tau_n(x_\tau^2)(x), \quad x \in [0, 1], n \geq 1. \]

Hence, the result in [16] applied to the uniformly bounded sequence \((C^\tau_n)_{n \geq 1}\) yields that there exists a constant \(C_p > 0\) such that

\[ \|C^\tau_n(f) - f\|_p \leq C_p(\mu_{n,p}^2 \|f\|_p + \omega_{2,p}(f, \mu_{n,p})), \]

where the sequence \(\mu_{n,p} \to 0\) as \(n \to \infty\) and it is defined as follows:

\[
\mu_{n,p} := \max \left\{ \|C^\tau_n(1) - 1\|_p^{1/2}, \|F^\tau_n\|_p^{1/2}, \|G^\tau_n\|_p^{p/(2p+1)} \right\}
\]

By Cauchy-Schwarz inequality we have

\[ |F^\tau_n|^p \leq (\sqrt{G^\tau_n})^p, \]

so

\[ \mu_{n,p} \leq \max \left\{ \|\sqrt{G^\tau_n}\|_p^{1/2}, \|C^\tau_n\|_p^{p/(2p+1)} \right\}. \]

From (4.21) it follows that \(\|\sqrt{G^\tau_n}\|_p^{1/2} \leq \beta_{n,p,\tau}\) and \(\|G^\tau_n\|_p^{p/(2p+1)} \leq \gamma_{n,p,\tau}\) (see (4.25) and (4.26)). Moreover,

\[ \gamma_{n,p,\tau} \leq \frac{\|\tau\|^p/(2p+1)}{(n+1)^{2p/(2p+1)}K^{p/(2p+1)}(n+1)^{p/(2p+1)}} = \frac{\|\tau\|^p/(2p+1)}{(n+1)^{p/(2p+1)}K^{p/(2p+1)}} \to 0 \quad \text{as} \quad n \to \infty. \]

Similarly,

\[ \beta_{n,p,\tau} \leq \frac{\|\sqrt{\tau}\|_\infty^{1/2}}{(n+1)^{1/4} \sqrt{K}}(n+1)^{1/4} = \frac{\|\sqrt{\tau}\|_\infty^{1/2}}{(n+1)^{3/4} \sqrt{K}} \to 0 \quad \text{as} \quad n \to \infty. \]

Therefore, setting \(\alpha_{n,p,\tau} = \max\{\beta_{n,p,\tau}, \gamma_{n,p,\tau}\}\), we have that \(\alpha_{n,p,\tau} \to 0\) as \(n \to \infty\) and this completes the proof.

\[ \square \]

5. Asymptotic formula for the \(C^\tau_n\)'s

In this section we establish an asymptotic formula for the operators \(C^\tau_n\), which, in addition, allows us to derive other properties of them. To this end, from now assume that

\[ (5.27) \quad \lim_{n \to \infty} (a_n + b_n) \in \mathbb{R} \]

and consider the differential operator \((V_l, C^2([0, 1]))\) defined by setting

\[ V_l(u)(x) := \frac{1}{2}x(1-x)u''(x) + \left( \frac{l}{2} - x \right) u'(x), \]

\((u \in C^2([0, 1]), x \in [0, 1])\).
Theorem 5.4. Assume that (5.27) holds true. Then, for each \( f \in C([0, 1]) \), twice differentiable at a certain \( x \in ]0, 1[ \),

\[
\lim_{n \to \infty} n(C_n^\tau(f)(x) - f(x)) = \frac{\tau(x)(1 - \tau(x))}{2} D_\tau^2(f)(x) + \left( \frac{l}{2} - \tau(x) \right) D_\tau(f)(x)
\]

(5.28)

Moreover, for every \( u \in C^2([0, 1]) \)

\[
\lim_{n \to \infty} n(C_n^\tau(u) - u) = V_l(u \circ \tau^{-1}) \circ \tau
\]

uniformly in \([0, 1]\).

Proof. In [5, Theorem 3.1] it was proven that

\[
\lim_{n \to \infty} n(C_n(u) - u) = V_l(u),
\]

for every \( u \in C^2([0, 1]) \) uniformly on \([0, 1]\), but it is easy to prove that the same limit relationship holds pointwise for each \( f \in C([0, 1]) \), twice differentiable at a certain \( x \in ]0, 1[ \). From this, formulas (5.28) and (5.29) easily follow. \( \square \)

5.1. An application to iterates of the operators \( C_n^\tau \). In this subsection we show how iterates of operators \( C_n^\tau \) can be employed in order to approximate constructively certain semigroups of operators. For unexplained terminology concerning Semigroup Theory and its connection with Approximation Theory, we refer, e.g., to [6, Chapter 2].

We begin by recalling that, as shown in [5, Theorem 3.2] the operator \((V_l, C^2([0, 1]))\) is closable and its closure generates a Markov semigroup \((T_l(t))_{t \geq 0}\) on \( C([0, 1]) \) such that, if \( t \geq 0 \) and if \((\rho_n)_{n \geq 1}\) is a sequence of positive integers such that \( \lim_{n \to \infty} \rho_n/n = t \), then

\[
\lim_{n \to \infty} C_n^{\rho_n}(f) = T_l(t)(f) \quad \text{uniformly on } [0, 1]
\]

for every \( f \in C([0, 1]) \), where \( C_n^{\rho_n} \) denotes the iterate of \( C_n \) of order \( \rho_n \).

Moreover (see [5, Theorem 3.4, Remark 3.5,1]), if either \( a_n = 0 \) and \( b_n = 1 \) for every \( n \geq 1 \), or the following properties hold true

\begin{enumerate}
  \item \( 0 < b_n - a_n < 1 \) for every \( n \geq 1 \);
  \item there exist \( \lim_{n \to \infty} a_n = 0 \) and \( \lim_{n \to \infty} b_n = 1 \);
  \item \( M_1 := \sup_{n \geq 1} n(1 - (b_n - a_n)) < +\infty \),
\end{enumerate}

for every \( p \geq 1 \), \((T_l(t))_{t \geq 0}\) extends to a positive \( C_0\)-semigroup \((\tilde{T}(t))_{t \geq 0}\) on \( L^p([0, 1]) \) such that, if \((\rho_n)_{n \geq 1}\) is a sequence of positive integers such that \( \lim_{n \to \infty} \rho_n/n = t \), then for every \( f \in L^p([0, 1]) \),

\[
\lim_{n \to \infty} C_n^{\rho_n}(f) = \tilde{T}(t)(f) \quad \text{in } L^p([0, 1]).
\]

We remark that, for every \( f \in C([0, 1]) \) and \( k \geq 1 \),

\[
(C_n^\tau)^k(f) = C_n^k(f \circ \tau^{-1}) \circ \tau.
\]

From this we get the following result.

Theorem 5.5. Under assumption (5.27), for every \( f \in C([0, 1]) \), \( t \geq 0 \) and for every sequence \((\rho_n)_{n \geq 1}\) of positive integers such that \( \lim_{n \to \infty} \rho_n/n = t \),

\[
\lim_{n \to \infty} (C_n^\tau)^{\rho_n}(f) = T_l(t)(f \circ \tau^{-1}) \circ \tau \quad \text{uniformly on } [0, 1].
\]
Moreover, assume that either \( a_n = 0 \) and \( b_n = 1 \) for every \( n \geq 1 \), or the following properties hold true:

(i) \( 0 < b_n - a_n < 1 \) for every \( n \geq 1 \);
(ii) there exist \( \lim_{n \to \infty} a_n = 0 \) and \( \lim_{n \to \infty} b_n = 1 \);
(iii) \( M_1 := \sup_{n \geq 1} n(1-(b_n-a_n)) < +\infty \).

Then, if \( t \geq 0 \) and \( \rho_n/n \geq A \) is a sequence of positive integers such that \( \lim_{n \to \infty} \rho_n/n = t \), then for every \( f \in L^p([0,1]) \),

\[
\lim_{n \to \infty} (C_n^\tau)^{\rho_n}(f) = \tilde{T}(t)(f \circ \tau^{-1}) \circ \tau \quad \text{in } L^p([0,1]).
\]

5.2. Comparing the operators \( C_n^\tau \) and \( C_n \). The asymptotic formula (5.28) can be also used to prove that, under suitable conditions, the operators \( C_n^\tau \) perform better than the operators \( C_n \) in approximating certain functions. In fact, arguing as in the proof of [10, Theorem 9], we are able to show the following result.

**Theorem 5.6.** Let \( f \in C^2([0,1]) \) and assume that there exists \( n_0 \in \mathbb{N} \) such that, for every \( n \geq n_0 \) and \( x \in ]0,1[ \),

\[
f(x) \leq C_n^\tau(f)(x) \leq C_n(f)(x).
\]

Then, for \( x \in ]0,1[ \),

\[
f''(x) \geq \frac{\tau''(x)}{\tau'(x)} f'(x) + \frac{\tau'(x)(2\tau(x) - l)}{\tau(x)(1 - \tau(x))} f'(x)
\]

\[
\geq \left( 1 - \frac{x(1-x)\tau'(x)^2}{\tau(x)(1 - \tau(x))} \right) f''(x) + \frac{\tau'(x)^2(2x - l)}{\tau(x)(1 - \tau(x))} f'(x).
\]

In particular, \( f'' \geq 0 \) in \( ]0,1/2[ \) (resp., in \( ]1/2,1[ \)) whenever \( f \) is decreasing in \( ]0,1/2[ \) (resp., \( f \) is increasing in \( ]1/2,1[ \)).

Conversely, assume that at a given point \( x_0 \in ]0,1[ \), (5.30) holds with strict inequalities. Then there exists \( n_0 \in \mathbb{N} \) such that, for every \( n \geq n_0 \),

\[
f(x_0) < C_n^\tau(f)(x_0) < C_n(f)(x_0).
\]

**Example 5.1.** Take

\[
\tau = \frac{e_2 + \alpha e_1}{1 + \alpha} \quad (\alpha > 0)
\]

and suppose that \( f \in C^2([0,1]) \) is increasing and strictly convex.

Moreover, assume that the sequences \( (a_n)_{n \geq 1} \) and \( (b_n)_{n \geq 1} \) are such that \( l = \lim_{n \to \infty} (a_n + b_n) = 2 \).

We show that there exist \( x_\alpha \in ]0,1[ \) and \( n_0 \in \mathbb{N} \) such that, for each \( x \in ]x_\alpha,1[ \) and \( n \geq n_0 \),

\[
f(x) < C_n^\tau(f)(x) < C_n(f)(x).
\]

On account of Theorem 5.6, it is sufficient to prove that there exists \( x_\alpha \in ]0,1[ \) such that, for \( x \in ]x_\alpha,1[ \),

\[
f''(x) > \frac{\tau''(x)}{\tau'(x)} f'(x) + \frac{\tau'(x)(2\tau(x) - 2)}{\tau(x)(1 - \tau(x))} f'(x)
\]

\[
> \left( 1 - \frac{x(1-x)\tau'(x)^2}{\tau(x)(1 - \tau(x))} \right) f''(x) + \frac{\tau'(x)^2(2x - 2)}{\tau(x)(1 - \tau(x))} f'(x).
\]

The first inequality in (5.31) is satisfied for \( \alpha > 2f''(1)/M \), where \( M = \min_{[0,1]} f''(x) \). Indeed, for this choice,

\[
f''(x) > \frac{2}{2x + \alpha} f'(x) > \frac{2}{2x + \alpha} f'(x) - 2 \frac{2x + \alpha}{x^2 + \alpha x} f'(x), \quad x \in ]0,1[.
\]
The second inequality in (5.31) is obviously fulfilled for those \( x \) for which

\[
\frac{x(1-x)\tau'(x)^2}{\tau(x)(1-\tau(x))} \geq 1
\]

and

\[
\frac{\tau''(x)}{\tau'(x)} > 2\frac{\tau'(x)^2}{\tau(x)(1-\tau(x))}\left(x - 1 - \frac{\tau(x) - 1}{\tau'(x)}\right).
\]

From one hand (5.32) is verified for \( x \in [y_\alpha, 1] \) where

\[
y_\alpha := \frac{1 - 2\alpha + \sqrt{4\alpha^2 + 8\alpha + 1}}{6}
\]

(see [10, Corollary 11, (iii)]). On the other hand (5.33) is equivalent to solve (with respect to \( x \)) the following inequality:

\[
g_\alpha(x) := (x^2 + \alpha x)(1 + x + \alpha) - (2x + \alpha)^2(1 - x) > 0.
\]

By observing that \( g_\alpha(0) < 0, g_\alpha(1) > 0, \) and evaluating the critical points of \( g_\alpha \) and their position within the interval \([0, 1]\) depending on \( \alpha > 0 \), we can conclude that, for every \( \alpha > 0 \), there exists \( z_\alpha \in [0, 1] \) such that \( g_\alpha(z_\alpha) = 0 \) and \( g_\alpha(x) > 0 \) for every \( z_\alpha < x \leq 1 \). By setting \( x_\alpha = \max\{y_\alpha, z_\alpha\} \) (\( \alpha > 2f'(1)/M \)), we get the claim.

We point out that, in the case \( \alpha = 0, \tau = e_2 \) and the corresponding operators \( C_n^\tau \) are a Kantorovich-type modification on mobile intervals of the operators in [10, p. 159]. On the other hand, \( \tau_\infty = \lim_{\alpha \to +\infty} \tau = e_1 \) uniformly w.r.t. \( x \in [0, 1] \), so that \( C_n^{\tau_\infty} = C_n \) for any \( n \geq 1 \).

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