CONE CONVERGENCE FOR MULTIPLE SEQUENCES*

AHMET ŞAHINER

ABSTRACT. The aim of this paper is to introduce a new type convergence which is useful when a $d$-multiple sequence is not convergent in some usual senses.

1. INTRODUCTION

Main purpose of this paper is to introduce a new type convergence which especially can be thought to be useful when a multidimensional sequence is not convergent. Though the new idea could be, explained in and applied to many subjects of functional analysis including multiple sequences related to convergence types such as statistical convergence, ideal convergence and to matrix transformations between sequence spaces and so on. For the sake of clarity we introduce this notion in some plain part of the notion of statistical convergence.

Let $\mathbb{N}^d$ be the set of $d$-tuples $\mathbf{k} := (k_1, k_2, \ldots, k_d)$ with nonnegative integers for coordinates $k_j$, where $d$ is a fixed positive integer. Note that $\mathbb{N}^d$ is partially ordered by agreeing $\mathbf{k} \leq \mathbf{n}$ if and only if $k_j \leq n_j$ for each integer $j$ (see [8]). A function $x : \mathbb{N}^d \to \mathbb{R} (\mathbb{C})$ is called a real (complex) $d$-multiple sequence. If $d = 2$ then a function $x : \mathbb{N}^2 \to \mathbb{R} (\mathbb{C})$ is called a real (complex) double sequence. The definition of the convergence of double sequences was given by Pringsheim in [3]. Remember that a double sequence $(x_{nk})$ is said to be convergent to $L$ in Pringsheim’s sense if for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that $|x_{nk} - L| < \varepsilon$ whenever $n, k \geq N(\varepsilon)$ [2, 4, 5, 6].

The idea of statistical convergence was first presented by Fast in [1]. The notion of statistically convergent double sequences has been studied by many authors (see for instance, [5, 6, 7, 8]). Regarding these works, to be adopted to the definition of
the density of a subset $E$ of $\mathbb{N}^2$, the density $\rho(E)$ of any subset of $E \subseteq \mathbb{N}^d$ can be given by

$$
\rho(E) = \lim_{\min K_i \to \infty} \frac{1}{K_1 \cdots K_d} \sum_{k_1 \leq K_1} \cdots \sum_{k_d \leq K_d} \chi_E(k_1, \ldots, k_d), \quad (i = 1, 2, \ldots, d)
$$

provided the limit exists.

Now to recall the definition of a cone let $\mathbb{R}^d_{\geq}$ denote the set of $d$-tuples $x : = (x_1, x_2, \ldots, x_d)$ with nonnegative reals for coordinates $x_j$. Suppose that $x_1 : = (x_{11}, x_{12}, \ldots, x_{1d})$, $x_2 : = (x_{21}, x_{22}, \ldots, x_{2d})$, ..., $x_d : = (x_{d1}, x_{d2}, \ldots, x_{dd}) \in \mathbb{R}^d_{\geq}$ are given such that $x_i$ and $x_j$ are not co-linear for $i \neq j$. Then the set

$$
\sigma = \mathbb{R}^d_{\geq} x_1 + \cdots + \mathbb{R}^d_{\geq} x_d = \{ \alpha_1 x_1 + \cdots + \alpha_d x_d : \alpha_i \in \mathbb{R}^1_{\geq} \} \quad \text{for } i = 1, \ldots, d
$$

is called the cone generated by $x_1, x_2, \ldots, x_d$. A cone is said to be pointed if it includes the null vector $0$.

For a given cone $\sigma = \mathbb{R}^d_{\geq} x_1 + \cdots + \mathbb{R}^d_{\geq} x_d$ and $u : = (u_1, u_2, \ldots, u_d) \in \mathbb{R}^d_{\geq}$, the shift of $\sigma$ with respect to $u$ is defined to be the set $u + \sigma$.

2. Main results

Definition 2.1. Let $x = (x_k : k \in \mathbb{N}^d)$ be a $d$-multiple sequence of (real or complex) numbers and $\sigma = \mathbb{R}^d_{\geq} s_1 + \cdots + \mathbb{R}^d_{\geq} s_d$ be a fixed cone. Then the $d$-multiple sequence $(x_k : k \in \mathbb{N}^d)$ is called $\sigma$-Cauchy sequence if for each $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that $|x_n - x_k| < \varepsilon$ whenever $n \geq k \geq N$ and $n, k \in \sigma$.

Definition 2.2. A $d$-multiple sequence $x = (x_k : k \in \mathbb{N}^d)$ is said to be $\sigma$-bounded if there exists $M > 0$ such that $|x_k| < M$ for all $k \in \sigma$.

Definition 2.3. Let $x = (x_k : k \in \mathbb{N}^d)$ be a $d$-multiple sequence of (real or complex) numbers and $\sigma = \mathbb{R}^d_{\geq} s_1 + \cdots + \mathbb{R}^d_{\geq} s_d$ be a fixed cone. Then the $d$-multiple sequence $(x_k : k \in \mathbb{N}^d)$ is called $\sigma$-convergent to a number $L$ if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}^d$ such that $|x_k - L| < \varepsilon$ whenever $k (\in \sigma) \geq N$.

If $(x_k : k \in \mathbb{N}^d)$ is $\sigma$-convergent to a real number $L$ we denote this by $\sigma - \lim (x_k : k \in \mathbb{N}^d) = L$ or $(x_k : k \in \mathbb{N}^d) \sigma \rightarrow L$.

Note that every double sequence, which is convergent in Pringsheim’s sense, is convergent with respect to the fixed cone $\sigma = \mathbb{R}^2_{\geq} (1, 0) + \mathbb{R}^2_{\geq} (0, 1)$. More generally, every $d$-multiple sequence, which is convergent in Pringsheim’s sense, is convergent with respect to the fixed cone $\sigma = \mathbb{R}^d_{\geq} (1, 0, \ldots, 0) + \mathbb{R}^d_{\geq} (0, 1, 0, \ldots, 0) + \cdots + \mathbb{R}^d_{\geq} (0, 0, 0, \ldots, 1)$.

Example 2.4. Let $\sigma = \mathbb{R}^2_{\geq} (1, 0) + \mathbb{R}^2_{\geq} (1, 1)$ and

$$
x_{k_1, k_2} : = \begin{cases} 
1, & k_1 \leq k_2, \\
0, & \text{otherwise}.
\end{cases}
$$
Then \((x_k : k \in \mathbb{N}^2) \xrightarrow{\sigma} 0\). On the other hand it is obvious that this double sequence is not convergent in Pringsheim’s sense.

Due to simplicity, the proofs of the following proposition and some of the theorems are omitted.

**Proposition 1.** If \((x_k : k \in \mathbb{N}^d)\) is \(\sigma\)-convergent then its limit is unique.

**Theorem 2.5.** If a \(d\)-multiple sequence is \(\sigma\)-convergent then it is \(\sigma\)-bounded. But, the converse of this is not true in general.

**Theorem 2.6.** Let \(\sigma - \lim (x_k : k \in \mathbb{N}^d) = L_1\) and \(\sigma - \lim (y_k : k \in \mathbb{N}^d) = L_2\). Then, \(\sigma - \lim (x_k + y_k : k \in \mathbb{N}^d) = L_1 + L_2\) and \(\sigma - \lim (c(x_k : k \in \mathbb{N}^d)) = cL\) for all scalars \(c\).

**Lemma 2.7.** If \(\sigma_1\) and \(\sigma_2\) are any two pointed cones and \(\sigma_3 = \sigma_1 \cap \sigma_2 \neq \emptyset\) then \(\sigma_3\) is also a pointed cone.

Using Lemma 2.7, we have the following:

**Theorem 2.8.** If \((x_k : k \in \mathbb{N}^d) \xrightarrow{\sigma_1} a\), \((y_k : k \in \mathbb{N}^d) \xrightarrow{\sigma_2} b\) and \(\sigma_3 = \sigma_1 \cap \sigma_2 \neq \emptyset\) has non-empty interior then \((x_k + y_k : k \in \mathbb{N}^d) \xrightarrow{\sigma_3} a + b\).

**Remark 2.9.** If \((x_k : k \in \mathbb{N}^d) \xrightarrow{\sigma_1} a\), \((y_k : k \in \mathbb{N}^d) \xrightarrow{\sigma_2} b\) and \(\sigma_3 = \sigma_1 \cap \sigma_2\) has an empty interior then we may not have \((x_k + y_k : k \in \mathbb{N}^d) \xrightarrow{\sigma_3} a + b\) in general for any pointed cone \(\sigma_3\).

We can see this by the following example.

**Example 2.10.** Let \(\sigma_1 = \mathbb{R}_{\geq}^2 (1, 2) + \mathbb{R}_{\geq}^2 (0, 1)\), \(\sigma_2 = \mathbb{R}_{\geq}^2 (1, 1) + \mathbb{R}_{\geq}^2 (1, 0)\) and

\[
x_{k:=(k_1,k_2)} = \begin{cases} 1, & k_1 \geq 2k_2, \\ 0, & \text{otherwise}, \\ \end{cases} \quad y_{k:=(k_1,k_2)} = \begin{cases} 2, & k_1 \leq k_2, \\ 0, & \text{otherwise}. \\ \end{cases}
\]

Then

\[
x_{k:=(k_1,k_2)} + y_{k:=(k_1,k_2)} = \begin{cases} 1, & k_1 \geq 2k_2 \\ 0, & k_2 < k_1 < 2k_2 \\ 2, & k_1 \leq k_2 \end{cases}
\]

and we have \((x_k + y_k : k \in \mathbb{N}^2) \xrightarrow{\sigma_3} 1\), \((x_k + y_k : k \in \mathbb{N}^2) \xrightarrow{\sigma_2} 2\) and \((x_k + y_k : k \in \mathbb{N}^2) \xrightarrow{\sigma_3} 0\), where \(\sigma_3 = \mathbb{R}_{\geq}^2 (1, 2) + \mathbb{R}_{\geq}^2 (1, 1)\).

**Definition 2.11.** A subset \(E\) of \(\mathbb{N}^d\) is said to have density \(\rho_{\sigma}(E)\) with respect to the fixed cone \(\sigma = \mathbb{R}_{\geq}^d x_1 + \cdots + \mathbb{R}_{\geq}^d x_d\) if the following limit exists.

\[
\rho_{\sigma}(E) = \lim_{\min K_i \to \infty \min K_d} \frac{1}{K_1 \cdots K_d} \sum_{k_1 \leq K_1} \cdots \sum_{k_d \leq K_d} \chi_E(k_1, \ldots, k_d);
\]

where \(K_i, k_i \in \sigma\) with \(i = 1, 2, \ldots, d\).
Definition 2.12. A $d$-multiple sequence $x = (x_k : k \in \mathbb{N}^d)$ is said to be $\sigma$-statistically convergent to $L$ if for every $\varepsilon > 0$, $\rho_\sigma \left( \{(k_1, \ldots, k_d) : |x_{k_1, \ldots, k_d} - L| \geq \varepsilon \} \right) = 0$.

Definition 2.13. Let $x = (x_k : k \in \mathbb{N}^d)$ and $y = (y_k : k \in \mathbb{N}^d)$ be two $d$-multiple sequences and $\sigma = \mathbb{R}_d x_1 + \cdots + \mathbb{R}_d x_d$ be a fixed cone. Then we say that $(x_k : k \in \mathbb{N}^d) = (y_k : k \in \mathbb{N}^d)$ for almost all $k \in \sigma$ if
\[
\delta_d \left( \{k \in \mathbb{N}^d \cap \sigma : (x_k : k \in \mathbb{N}^d) \neq (y_k : k \in \mathbb{N}^d) \} \right) = 0.
\]

Definition 2.14. Let $x = (x_k : k \in \mathbb{N}^d)$ be a $d$-multiple sequence. A subset $D$ of $\mathbb{R}^d$ is said to contain a $\sigma$-multiple sequence if for almost all $k \in \mathbb{N}^d$, $x_k$ contains a point of $D$. Take $\delta_d \left( \{k \in \mathbb{N}^d \cap \sigma : (x_k : k \in \mathbb{N}^d) \notin D \} \right) = 0$.

Theorem 2.15. A $d$-multiple sequence $(x_k : k \in \mathbb{N}^d)$ is $\sigma$-statistically convergent if and only if it is $\sigma$-statistically Cauchy.

Proof. Since the necessity is obvious, we only prove the sufficiency. Let $(x_k : k \in \mathbb{N}^d)$ be a $\sigma$-statistically Cauchy sequence. Choose $\varepsilon = 1$, then there exist $k_1, k_2, \ldots, k_d$ such that the closed circle $U_1$ of diameter 2 units with center at $k_1, k_2, \ldots, k_d$ contains $x_k$ for almost all $k \in \sigma$. Now for $\varepsilon = 1/2$ there exist $k_1', k_2', \ldots, k_d'$ such that the closed circle $U_2$ of diameter 1 unit with center at $k_1', k_2', \ldots, k_d'$ contains $x_k$ for almost all $k \in \sigma$. Take $U_3 = U_2 \cap U_2$ and $U_3$ is closed subset of $\mathbb{R}^d$ with diameter less than or equal to 1 unit such that $U_3$ contains $x_k$ for almost all $k \in \sigma$. If we choose $U_4 = U_3 \cap U_3$ then $U_4$ is closed subset of $\mathbb{R}^d$ with diameter less than or equal to 1 unit such that $U_4$ contains $x_k$ for almost all $k \in \sigma$. Following this way, we have a sequence $(U_n)$ of closed subsets of $\mathbb{R}^d$ such that

\begin{itemize}
  \item[(i)] $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$.
  \item[(ii)] diam$U_n \leq 2^{-n}$ for all $n \in \mathbb{N}$.
\end{itemize}

Then $\bigcap_{n=1}^{\infty} U_n$ contains one point. Let us call this point as $L$. Then $L \in U_n$ for all $n \in \mathbb{N}$, If we choose $m$ such that $2^{-m} < \varepsilon$ then $U_n$ contains $x_k$ for almost all $k \in \sigma$. This means $(x_k : k \in \mathbb{N}^d)$ is statistically convergent to $L$.

Now we are ready to give the following cone $d$-multiple analogues of the result in [7].

Theorem 2.16. Let $x = (x_k : k \in \mathbb{N}^d)$ is a $d$-multiple sequence and $\sigma = \mathbb{R}_d x_1 + \cdots + \mathbb{R}_d x_d$ be a fixed cone. Then the following statements are equivalent:

\begin{itemize}
  \item[(i)] $(x_k : k \in \mathbb{N}^d)$ is $\sigma$-statistically convergent to $L$.
  \item[(ii)] $(x_k : k \in \mathbb{N}^d)$ is $\sigma$-statistically Cauchy.
  \item[(iii)] There exists a subsequence $(y_k : k \in \mathbb{N}^d)$ of $(x_k : k \in \mathbb{N}^d)$ such that $\sigma - \lim (y_k : k \in \mathbb{N}^d) = \ell$.
\end{itemize}
CONCLUSION

As is mentioned at the beginning of the article this new type convergence can be applied to many subjects of functional analysis including multiple sequences related to convergence types such as statistical convergence, ideal convergence and so on and to matrix transformations between sequence spaces including multiple sequences. So, application area of this new type convergence is enormous for further works.

REFERENCES


Current address: Department of Mathematics, Süleyman Demirel University, 32260, Isparta, TURKEY

E-mail address: ahmetsahiner@sdu.edu.tr