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A COMPUTATIONAL METHOD FOR INTEGRO-DIFFERENTIAL HYPERBOLIC EQUATION WITH INTEGRAL CONDITIONS*

AHCENE MERAD AND ABDELFATAH BOUZIANI

Abstract. The subject of this work is to prove existence, uniqueness, and continuous dependence upon the data of solution to integrodifferential hyperbolic equation with integral conditions. The proofs are based on a priori estimates and Laplace transform method. Finally, the solution by using a numerical technique for inverting the Laplace transforms is obtained.

1. Introduction

In this paper we are concerned with the following hyperbolic Integro-differential equation,

$$\frac{\partial^2 v}{\partial t^2}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = g(x,t) + \int_0^t a(t-s)v\,(x,s)\,ds, \tag{1.1}$$
$$0 < x < 1,\ 0 < t \leq T,$$

Subject to the initial conditions

$$\begin{aligned} v\,(x,0) &= \Phi(x),\ 0 < x < 1, \\ \frac{\partial v\,(x,0)}{\partial t} &= \Psi(x),\ 0 < x < 1, \end{aligned} \tag{1.2}$$

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and the integral conditions

$$\begin{aligned}\int_0^1 v(x,t)dx &= r(t), \ 0 < t \leq T, \\ \int_0^1 xv(x,t)dx &= q(t), \ 0 < t \leq T,\end{aligned} \tag{1.3}$$

where v is an unknown function, r, q, and $\Phi(x)$ are given functions supposed to be sufficiently regular, a is suitably defined function satisfying certain conditions to be specified later and T is a positive constant.

Certain problems of modern physics and technology can be effectively described in terms of nonlocal problems for partial differential equations.The linear case of our problem, that is $\int_0^t a(t-s)\, v(x,s)\, ds$, appears, for instance, in the modelling of the quasistatic flexure of a thermoelastic rod, see [4, 6] and has been studied, firstly, by the first author with a more general second-order parabolic equation or a $2m-$parabolic equation in [4, 6, 8] by means of the energy-integrals methods and, secondly, by the Rothe method [22]. For other models, we refer the reader, for instance,to [3], [6], [7], [9], [10]-[13], [14]-[21], [23]-[28], and references therein. Problem (1.1)-(1.3) is studied by the Rothe method [15]. Ang [2] has considered a one-dimensional heat equation with nonlocal (integral) conditions. The author has taken the laplace transform of the problem and then used numerical technique for the inverse laplace transform to obtain the numerical solution.

This paper is organized as follows. In Sect.2, we begin introducing certain function spaces which are used in the next sections, and we reduce the posed problem to one with homogeneous integral conditions. In Sect.3, we first establish the existence of solution by the Laplace transform. In Sect.4, we establish a priory estimates, wich give the uniquenss and continuous dependence upon the data.

2. Statement of the problem and notation

Since integral conditions are inhomogenous, it is convenient to convert problem $(1.1)-(1.3)$ to an equivalent problem with homogenous integral conditions. For this, we introduce a new function $u(x,t)$ representing the deviation of the function $v(x,t)$ from the function

$$u(x,t) = v(x,t) - w(x,t), \quad 0 < x < 1, \ 0 < t \leq T, \tag{2.1}$$

where

$$w(x,t) = 6\left(2q(t) - r(t)\right)x - 2\left(3q(t) - 2r(t)\right). \tag{2.2}$$

Problem $(1.1)-(1.3)$ with inhomogenous integral conditions (1.3) can be equivalently reduced to the problem of finding a function u satisfying

$$\frac{\partial^2 u}{\partial t^2}(x,t)-\frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t)+\int_0^t a(t-s)\,u(x,s)\,ds, \quad 0 < x<1,\ 0<t\leq T, \tag{2.3}$$

$$\begin{aligned} u(x,0) &= \varphi(x), \quad 0<x<1, \\ \frac{\partial u(x,0)}{\partial t} &= \psi(x), \quad 0<x<1, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \int_0^1 u(x,t)dx &= 0, \quad 0<t\leq T, \\ \int_0^1 xv(x,t)dx &= 0, \quad 0<t\leq T, \end{aligned} \tag{2.5}$$

where

$$f(x,t)=g(x,t)-\left(\frac{\partial^2 w}{\partial t^2}(x,t)-\frac{\partial^2 w}{\partial x^2}(x,t)-\int_0^t a(t-s)\,w(x,s)\,ds\right), \tag{2.6}$$

and

$$\begin{aligned} \varphi(x) &= \Phi(x)-w(x,0), \\ \psi(x) &= \Psi(x)-\frac{\partial w(x,0)}{\partial t}. \end{aligned} \tag{2.7}$$

Hence, instead of solving for v, we simply look for u.

The solution of problem $(1.1)-(1.3)$ will be obtained by the relation (2.1) and (2.2). We introduce the appropriate function spaces that will be used in the rest of the note. Let H be a Hilbert space with a norm $\|.\|_H$.

Let $L^2(0,1)$ be the standard function space.

Definition 2.1. (i) Denote by $L^2(0,T,H)$ the set of all measurable abstract functions $u(\cdot,t)$ from $(0,T)$ into H equiped with the norm

$$\|u\|_{L^2(0,T,H)}=\left(\int_0^T \|u(\cdot,t)\|_H^2\,dt\right)^{1/2}<\infty$$

(ii) Let $C(0,T,H)$ be the set of all continuous functions $u(\cdot,t):(0,T)\longrightarrow H$ with

$$\|u\|_{C(0,T,H)}=\max_{0\leq t\leq T}\|u(\cdot,t)\|_H<\infty$$

We denote by $C_0(0,1)$ the vector space of continuous functions with compact support in $(0,1)$. Since such function are Lebesgue integrable with respect to x, we can define on $C_0(0,1)$ the bilinear form given by

$$((u,w)) = \int_0^1 J_x^m u . J_x^m w dx, \quad m \geq 1 \tag{2.8}$$

where

$$J_x^m u = \int_0^x \frac{(x-\zeta)^{m-1}}{(m-1)!} u(\zeta,t) d\zeta; \quad \text{for } m \geq 1 \tag{2.9}$$

The bilinear form (2.8) is considered as a scalar product on $C_0(0,1)$ is not complete.

Definition 2.2. Denote by $B_2^m(0,1)$, the comletion of $C_0(0,1)$ for the scalar product (2.8), which is denoted $(.,.)_{B_2^m(0,1)}$, introduced by [5]. By the norm of function u from $B_2^m(0,1)$, $m \geq 1$, we inderstand the nonnegative number:

$$\|u\|_{B_2^m(0,1)} = \left(\int_0^1 (J_x^m u)^2 dx \right)^{1/2} = \|J_x^m u\|; \text{ for } m \geq 1 \tag{2.10}$$

Lemma 2.3. *For all $m \in \mathbb{N}^*$, the following inequality holds:*

$$\|u\|^2_{B_2^m(0,1)} \leq \frac{1}{2} \|u\|^2_{B_2^{m-1}(0,1)}. \tag{2.11}$$

Proof. See[5]. $\square$

Corollary 1. *For all $m\mathbb{N}^*$, we have the elementary inequality*

$$\|u\|^2_{B_2^m(0,1)} \leq \left(\frac{1}{2}\right)^m \|u\|^2_{L^2(0,1)}. \tag{2.12}$$

Definition 2.4. We denote by $L^2(0,T;B_2^m(0,1))$ the space of functions which are square integrable in the Bochner sense, with the scalar product

$$(u,w)_{L^2(0,T;B_2^m(0,1))} = \int_0^T (u(.,t), w(\cdot,t))_{B_2^m(0,1)} dt. \tag{2.13}$$

Since the space $B_2^m(0,1)$ is a Hilbert space, it can be shown that $L^2(0,T;B_2^m(0,1))$ is a Hilbert space as well. The set of all continuous abstract functions in $[0,T]$ equipped with the norm

$$\sup_{0 \leq t \leq T} \|u(\cdot,t)\|_{B_2^m(0,1)}$$

is denoted $C(0,T;B_2^m(0,1))$.

Corollary 2. *For every* $u \in L^2(0,1)$, *from which we deduce the continuity of the imbedding* $L^2(0,1) \longrightarrow B_2^m(0,1)$, *for* $m \geq 1$.

Lemma 2.5. (***Gronwall Lemma***) *Let* $f_1(t), f_2(t) \geq 0$ *be two integrable functions on* $[0,T]$, $f_2(t)$ *is nondecreasing. If*

$$f_1(\tau) \leq f_2(\tau) + c\int_0^\tau f_1(t)\,dt, \qquad \forall \tau \in [0,T], \tag{2.14}$$

where $c \in \mathbb{R}^+$, *then*

$$f_1(t) \leq f_2(t)\exp(ct), \qquad \forall t \in [0,T]. \tag{2.15}$$

Proof. The proof is the same as that of Lemma 1.3.19 in [19]. □

3. Existence of the Solution

In this section we shall apply the Laplace transform technique to find solutions of partial differential equations, we have the Laplace transform

$$V(x,s) = \mathfrak{L}\{v(x,t); t \longrightarrow s\} = \int_0^\infty v(x,t)\exp(-st)\,dt, \tag{3.1}$$

where s is positive reel parameter. Taking the Laplace transforms on both sides of (1.1), we have

$$\left(s^2 - A(s)\right)V(x,s) - \frac{d^2}{dx^2}V(x,s) = G(x,s) + s\Phi(x) + \Psi(x), \tag{3.2}$$

where $G(x,s) = \mathfrak{L}\{g(x,t); t \longrightarrow s\}$. Similarly, we have

$$\begin{aligned} \int_0^1 V(x,s)\,dx &= R(s), \\ \int_0^1 xV(x,s)\,dx &= Q(s), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} R(s) &= \mathfrak{L}\{r(t); t \longrightarrow s\}, \\ Q(s) &= \mathfrak{L}\{q(t); t \longrightarrow s\}. \end{aligned}$$

Now, we have the following three cases:

Case 1. $s^2 - A(s) > 0$.
Case 2. $s^2 - A(s) < 0$.
Case 3. $s^2 - A(s) = 0$.

We only consider Cases 2 and 3, as Case 1 can be dealt with similarly as in [2]. For $\left(s^2 - A(s)\right) = 0$, we have

$$\frac{d^2}{dx^2}V(x,s) = -G(x,s) - s\Phi(x) - \Psi(x), \tag{3.4}$$

The general solution for case 3 is given by

$$V\left(x,s\right)=-\int_0^x\int_0^y\left[G\left(x,s\right)+s\Phi(x)+\Psi(x)\right]dzdy+C_1\left(s\right)x+C_2(s), \quad (3.5)$$

Putting the integral conditions (3.3) in (3.5) we get

$$\begin{aligned}
&\frac{1}{2}C_1(s)+C_2(s)\\
&=\int_0^1\int_0^x\int_0^y\left[G\left(x,s\right)+s\Phi(x)+\Psi(x)\right]dzdy+R(s),\\
&\frac{1}{3}C_1(s)+\frac{1}{2}C_2(s)\\
&=\int_0^1\int_0^x\int_0^y x\left[G\left(x,s\right)+s\Phi(x)+\Psi(x)\right]dzdy+Q(s),
\end{aligned} \quad (3.6)$$

and

$$\begin{aligned}
C_1(s) &= 12\int_0^1\int_0^x\int_0^y x\left[G\left(x,s\right)+s\Phi(x)+\Psi(x)\right]dzdy-\\
&\quad 6\int_0^1\int_0^x\int_0^y\left[G\left(x,s\right)+s\Phi(x)+\Psi(x)\right]dzdy+\\
&\quad 12Q(s)-6R(s),\\
C_2(s) &= 4\int_0^1\int_0^x\int_0^y\left[G\left(x,s\right)+s\Phi(x)+\Psi(x)\right]dzdy-\\
&\quad 6\int_0^1\int_0^x\int_0^y x\left[G\left(x,s\right)+s\Phi(x)+\Psi(x)\right]dzdy-\\
&\quad 6Q(s)+4R(s).
\end{aligned} \quad (3.7)$$

For case 2, that is, $\left(s^2-A(s)\right)<0$,using the method of variation of parameter, we have the general solution as

$$\begin{aligned}
V\left(x,s\right) &= \frac{1}{\sqrt{A(s)-s^2}}\int_0^x\left(G\left(x,s\right)+s\Phi(x)+\Psi\left(x\right)\right)\sin\left(\sqrt{A(s)-s^2}\right)\left(x-\tau\right)d\tau\\
&\quad +d_1(s)\cos\sqrt{\left(A(s)-s^2\right)}x+d_2(s)\sin\sqrt{\left(A(s)-s^2\right)}x
\end{aligned} \quad (3.8)$$

From the integral conditions (3.3) we get

$$\begin{aligned}
& d_1(s)\int_0^1 \cos\sqrt{(A(s)-s^2)}x dx + d_2(s)\int_0^1 \sin\sqrt{(A(s)-s^2)}x dx \\
= \;& R(s) - \frac{1}{\sqrt{A(s)-s^2}}\int_0^1\int_0^x \left(G(x,s)+s\Phi(x)+\Psi(x)\right)\times \\
& \sin\left(\sqrt{A(s)-s^2}\right)(x-\tau)\, d\tau dx, \\
& d_1(s)\int_0^1 x\cos\sqrt{(A(s)-s^2)}x dx + d_2(s)\int_0^1 x\sin\sqrt{(A(s)-s^2)}x dx \\
= \;& Q(s) - \frac{1}{\sqrt{A(s)-s^2}}\int_0^1\int_0^x x\left(G(x,s)+s\Phi(x)+\Psi(x)\right)\times \\
& \sin\left(\sqrt{A(s)-s^2}\right)(x-\tau)\, d\tau dx. \qquad (3.9)
\end{aligned}$$

Thus d_1, d_2 are given by

$$\begin{pmatrix} d_1(s) \\ d_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \qquad (3.10)$$

where

$$\begin{aligned}
a_{11}(s) &= \int_0^1 \cos\sqrt{(A(s)-s^2)}x dx, \\
a_{12}(s) &= \int_0^1 \sin\sqrt{(A(s)-s^2)}x dx, \\
a_{21}(s) &= \int_0^1 x\cos\sqrt{(A(s)-s^2)}x dx, \\
a_{22}(s) &= \int_0^1 x\sin\sqrt{(A(s)-s^2)}x dx, \\
b_1(s) &= R(s) - \frac{1}{\sqrt{A(s)-s^2}}\int_0^1\int_0^x \left(G(x,s)+s\Phi(x)+\Psi(x)\right)\times \\
& \quad \sin\left(\sqrt{A(s)-s^2}\right)(x-\tau)\, d\tau dx, \\
b_2(s) &= Q(s) - \frac{1}{\sqrt{A(s)-s^2}}\int_0^1\int_0^x x\left(G(x,s)+s\Phi(x)+\Psi(x)\right)\times \\
& \quad \sin\left(\sqrt{A(s)-s^2}\right)(x-\tau)\, d\tau dx. \qquad (3.11)
\end{aligned}$$

If it is not possible to calculate the integrals directly, then we calculate it numerically. We approximate similarly as given in [2]. If the laplace inversion is possible directly for (3.5) and (3.8), in this case we shall get our solution. In another case we use the suitable approximate method and then use the numerical inversion of the Laplace transform. Considering $A(s) - s^2 = k(s)$ and using Gauss's formula given in [1] we have the following approximations of the integrals:

$$
\begin{aligned}
&\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \cos \sqrt{k(s)} x dx \\
\simeq\ & \frac{1}{2} \sum_{i=1}^{N} w_i \begin{pmatrix} 1 \\ \frac{1}{2}[x_i+1] \end{pmatrix} \cos \left(\sqrt{k(s)} \frac{1}{2} [x_i+1] \right), \\
&\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin \sqrt{k(s)} x dx \\
\simeq\ & \frac{1}{2} \sum_{i=1}^{N} w_i \begin{pmatrix} 1 \\ \frac{1}{2}[x_i+1] \end{pmatrix} \sin \left(\sqrt{k(s)} \frac{1}{2} [x_i+1] \right), \\
&\int_0^x \left(G(x,s) + s\Phi(x) + \Psi(x) \right) \sin \left(\sqrt{k(s)} \right) (x-\tau)\, d\tau \\
\simeq\ & \frac{x}{2} \sum_{i=1}^{N} w_i \left[G\left(\frac{x}{2}[x_i+1]; s \right) + s\Phi \left(\frac{x}{2}[x_i+1] \right) + \Psi \left(\frac{x}{2}[x_i+1] \right) \right] \\
& \sin \left(\sqrt{k(s)} \left[x - \frac{x}{2}[x_i+1] \right] \right), \\
&\int_0^1 \left[[G(\tau,s) + s\Phi(\tau) + \Psi(\tau)] \int_\tau^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin \left(\sqrt{k(s)} \right) (x-\tau)\, dx \right] d\tau \\
\simeq\ & \frac{1}{2} \sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2}[x_i+1]; s \right) + s\Phi \left(\frac{1}{2}[x_i+1] \right) + \Psi \left(\frac{1}{2}[x_i+1] \right) \right] \\
& \left(\frac{1 - \frac{1}{2}[x_i+1]}{2} \right) \sum_{i=1}^{N} w_j \begin{pmatrix} 1 \\ \frac{1-\frac{1}{2}[x_i+1]}{2} x_j + \frac{1-\frac{1}{2}[x_i+1]}{2} \end{pmatrix} \times \\
& \sin \left(\sqrt{k(s)} \left[\frac{1 - \frac{1}{2}[x_i+1]}{2} x_j + \frac{1 + \frac{1}{2}[x_i+1]}{2} - \frac{1}{2}(x_i+1) \right] \right), \qquad (3.12)
\end{aligned}
$$

where x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n(x), \ \omega_i \ = 2/\left(1 - x_i^2\right) \left[P_n^{'}(x) \right]^2 .$$

Their tabulated values can be found in [1] for different values of N.
Numerical inversion of Laplace transform. Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; therefore, a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by Hassan Hassanzadeh, Mehran Pooladi-Darvish [18]. In this work we use the Stehfest's algorithm [28] that is easy to implement. This numerical technique was first introduced by Graver [16] and its algorithm then

offered by [28]. Stehfest's algorithm approximates the time domain solution as

$$v(x,t) \approx \frac{\ln 2}{t} \sum_{n=1}^{2m} \beta_n V\left(x; \frac{n \ln 2}{t}\right), \tag{3.13}$$

where, m is the positive integer,

$$\beta_n = (-1)^{n+m} \sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)} \frac{k^m\,(2k)!}{(m-k)!k!\,(k-1)!\,(n-k)!\,(2k-n)!}, \tag{3.14}$$

and $[q]$ denotes the integer part of the real number q.

4. Uniqueness and Continuous dependence of the Solution

We establish an a priori estimate, the uniqueness and continuous dependence of the solution with respect to the data are immediate consequences.

Theorem 4.1. *If $u(x,t)$ is a solution of problem (2.3)-(2.5) and $f \in C(\overline{D})$, then we have a priori estimates:*

$$\begin{aligned} & \|u(\cdot,\tau)\|^2_{L^2(0,1)} \\ \leq\ & c_1 \left(\|f(\cdot,t)\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)} \right) \\ & \left\| \frac{\partial u(\cdot,\tau)}{\partial t} \right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} \\ \leq\ & c_2 \left(\|f(\cdot,t)\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)} \right), \end{aligned} \tag{4.1}$$

where $c_1 = \exp(a_0 T)$, $c_2 = \frac{\exp(a_0T)}{1-a_0}$, $1 < a(x,t) < a_0$, and $0 \leq \tau \leq T$.

Proof. Taking the scalar product in $B_2^1(0,1)$ of equation (2.3) and $\frac{\partial u}{\partial t}$, and integrating over $(0,\tau)$, we have

$$\begin{aligned} & \int_0^\tau \left(\frac{\partial^2 u(\cdot,t)}{\partial t^2}, \frac{\partial u(\cdot,t)}{\partial t} \right)_{B_2^1(0,1)} dt - \\ & \int_0^\tau \left(\frac{\partial^2 u(\cdot,t)}{\partial x^2}, \frac{\partial u(\cdot,t)}{\partial t} \right)_{B_2^1(0,1)} dt \\ = \ & \int_0^\tau \left(f(\cdot,t), \frac{\partial u\,(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt + \\ & \int_0^\tau \left(\int_0^t a\,(t-s)\,u\,(x,s)\,ds, \frac{\partial u(\cdot,t)}{\partial t} \right)_{B_2^1(0,1)} dt. \end{aligned} \tag{4.2}$$

By integrating by parts on the left-hand side of (4.2) we obtain

$$\begin{aligned} & \frac{1}{2}\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{B_2^1(0,1)} - \frac{1}{2}\left\|\psi\right\|^2_{B_2^1(0,1)} + \\ & \frac{1}{2}\left\|u(\cdot,\tau)\right\|^2_{L^2(0,1)} - \frac{1}{2}\left\|\varphi\right\|^2_{L^2(0,1)} \\ = \; & \int_0^\tau \left(f(\cdot,t), \frac{\partial u\,(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt + \\ & \int_0^\tau \left(\int_0^t a\,(t-s)\,u\,(x,s)\,ds, \frac{\partial u(\cdot,t)}{\partial t}\right)_{B_2^1(0,1)} dt. \end{aligned} \tag{4.3}$$

By the Cauchy inequality, the first term in the right-hand side of (4.3) is bounded by

$$\frac{1}{2}\left\|f(\cdot,t)\right\|^2_{L^2\left(0,T;\;B_2^1(0,1)\right)} + \frac{1}{2}\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{L^2\left(0,T;\;B_2^1(0,1)\right)} \tag{4.4}$$

and second term in the right-hand side of (4.3) is bounded by

$$\frac{a_0}{2}\int_0^t \left\|u\,(x,s)\right\|^2_{L^2\left(0,T;\;B_2^1(0,1)\right)} ds + \frac{a_0}{2}\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{L^2\left(0,T;\;B_2^1(0,1)\right)} \tag{4.5}$$

Substitution of (4.4) and (4.5) into (4.3) yields

$$\begin{aligned} & (1-a_0)\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{L^2\left(0,T;\;B_2^1(0,1)\right)} + \left\|u(\cdot,\tau)\right\|^2_{L^2(0,1)} \leq \\ & \left(\left\|f(\cdot,t)\right\|^2_{L^2\left(0,T;\;B_2^1(0,1)\right)} + \left\|\varphi\right\|^2_{L^2(0,1)} + \left\|\psi\right\|^2_{B_2^1(0,1)}\right) + \\ & \frac{a_0}{2}\int_0^t \left\|u\,(x,s)\right\|^2_{L^2\left(0,T;\;B_2^1(0,1)\right)} ds. \end{aligned} \tag{4.6}$$

By Gronwall Lemma we have

$$\begin{aligned} & (1-a_0)\left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{L^2\left(0,T;\;B_2^1(0,1)\right)} + \left\|u(\cdot,\tau)\right\|^2_{L^2(0,1)} \\ \leq \; & \exp\left(a_0 T\right)\left(\left\|f(\cdot,t)\right\|^2_{L^2\left(0,T;\;B_2^1(0,1)\right)} + \left\|\varphi\right\|^2_{L^2(0,1)} + \left\|\psi\right\|^2_{B_2^1(0,1)}\right). \end{aligned} \tag{4.7}$$

From (4.7), we obtain estimates (4.1). □

Corollary 3. *If problem (2.3)-(2.5) has a solution, then this solution is unique and depends continuously on* (f,φ,ψ).

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Current address: A. Merad and A. Bouziani; Department of Mathematics, Larbi Ben M'hidi University, 04000, ALGERIA

E-mail address: merad_ahcene@yahoo.fr, aefbouziani@yahoo.fr

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