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MONOTONE ITERATIVE TECHNIQUE FOR A COUPLED SYSTEM OF FIRST ORDER DIFFERENTIAL EQUATIONS

ELIF DEMIRCI AND NURI OZALP

Abstract. By the help of upper and lower solutions, the monoton iterative technique is applied to a coupled system of first order ordinary differential equations with initial conditions depending on a function of end points. Some existence and uniqueness results are obtained. An example for a predator-prey system is given.

1. Introduction

It is well known that one of the most effective methods of estimating the solutions of differential equations and systems with initial conditions is monotone iterative technique (for details see [2]). In [1], an existence result is given for the problem

$$\begin{aligned} x^{'}(t) &= f(t,x(t)),\ t\in J=[0,T],\ T>0,\\ x(0) &= g(x(T)). \end{aligned}$$

by using a monotone technique. Here $f\in C(J\times\mathbb{R},\mathbb{R}),\ g\in C(\mathbb{R},\mathbb{R})$. This technique was also applied to above problem with special cases of the function g (See, [3]$-$[9]). For application of monotone iterative techniques to higher order equations see, for example, [2] and [10]. In this paper, we consider the following coupled system of differential problem.

$$\begin{cases} u'=\begin{pmatrix} u_1\\ u_2\end{pmatrix}'=\begin{pmatrix} f_1(t,u_1,u_2)\\ f_2(t,u_1,u_2)\end{pmatrix}=f(t,u), \quad t\in J=[0,T], \quad T>0,\\ \\ u(0)=\begin{pmatrix} u_1(0)\\ u_2(0)\end{pmatrix}=\begin{pmatrix} g_1(u_1(T),u_2(T))\\ g_2(u_1(T),u_2(T))\end{pmatrix}=g(u(T)), \end{cases} \tag{1}$$

where $f\in C(J\times\mathbb{R}^2,\mathbb{R}^2)$, $g\in C(\mathbb{R}^2,\mathbb{R}^2)$.
The purpose of this paper is to prove that monotone technique can be applied

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successfully to problems of type (1) with some assumptions on f and g. A predator-prey example satisfying the conditions given on f and g is also stated.

2. Existence and Uniqueness Results

Theorem 2.1. *Let $f \in C(\Omega, \mathbb{R}^2)$, $g \in C(\Delta, \mathbb{R}^2)$. Moreover, we assume that there exists functions $v,\ w \in C^1(J, \mathbb{R}^2)$ such that*

$$\begin{array}{rcl} v_i(t) & \leq & w_i(t), \quad v_i'(t) \leq f_i(t, v(t)), \quad w_i'(t) \geq f_i(t, w(t)), \quad t \in J,\ i = 1, 2, \\ v_i(0) & \leq & g_i(s) \leq w_i(0) \quad for \quad v_i(T) \leq s_i \leq w_i(T), \quad i = 1, 2, \end{array}$$

where

$$\begin{array}{rcl} \Omega & = & \{(t, u) : v_i(t) \leq u_i(t) \leq w_i(t),\ t \in J,\ i = 1, 2\}, \\ \Delta & = & \{u \in C^1(J, \mathbb{R}^2) : v_i(t) \leq u_i(t) \leq w_i(t),\ t \in J,\ i = 1, 2\}. \end{array}$$

Then problem (1) *has at least one solution in* Δ.

Proof. Let $P : J \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$P(t, u(t)) = \begin{pmatrix} P_1(t, u(t)) \\ P_2(t, u(t)) \end{pmatrix} = \begin{pmatrix} \max\{v_1(t), \min(u_1(t), w_1(t))\} \\ \max\{v_2(t), \min(u_2(t), w_2(t))\} \end{pmatrix}$$

Then $f(t, P(t, u(t)))$ defines a continuous extension of f to $J \times \mathbb{R}^2$. Because of the fact that f is bounded on Ω, $f(t, P(t, u(t)))$ is bounded on $J \times \mathbb{R}^2$. Similarly, $g(P(t, u(t)))$ is a continuous extension of $g(u(t))$ to $\mathbb{R}^2$. Therefore, the problem

$$\begin{array}{rcl} u' & = & f(t, P(t, u)) \\ u(0) & = & g(P(T, u(T))) \end{array}$$

has a solution defined on J (see [2]). For $\varepsilon_i > 0$, $i = 1, 2$, we consider

$$\begin{array}{rcl} (w_{\varepsilon_i})_i(t) & = & w_i(t) + \varepsilon_i(1 + t) \\ (v_{\varepsilon_i})_i(t) & = & v_i(t) - \varepsilon_i(1 + t). \end{array}$$

Let $v_i(T) \leq u_i(T) \leq w_i(T)$. We have

$$(v_{\varepsilon_i})_i(T) = v_i(T) - \varepsilon_i(1 + T) < v_i(T) \leq u_i(T) \leq w_i(T) < (w_{\varepsilon_i})_i(T) \quad , \quad i = 1, 2.$$

Then, $(v_{\varepsilon_i})_i(0) \leq u_i(0) \leq (w_{\varepsilon_i})_i(0)$, $i = 1, 2$. We want to show that

$$(v_{\varepsilon_i})_i(t) < u_i(t) < (w_{\varepsilon_i})_i(t) \ , \quad i = 1, 2, \quad t \in J.$$

Suppose that $t_i \in (0, T]$ is such that, for $i = 1, 2$,

$$(v_{\varepsilon_i})_i(t) < u_i(t) < (w_{\varepsilon_i})_i(t) \qquad \text{for } t \in [0, t_i)$$

and $u_i(t_i) = (w_{\varepsilon_i})_i(t_i)$. Then, $u_i(t_i) > w_i(t_i)$ and so, $P_i(t_i, u(t_i)) = w_i(t_i)$, $i = 1, 2$. We know that

$$v_i(t_i) \leq P_i(t_i, u(t_i)) \leq w_i(t_i), \qquad i = 1, 2$$

from the definition of P. We can also write

$$w_i'(t_i) \geq f_i(t_i, w(t_i)) = f_i(t_i, P(t_i, u(t_i))) = u_i'(t_i), \quad i = 1, 2.$$

Since $(w_{\varepsilon_i})_i'(t_i) > w_i'(t_i)$, we have $(w_{\varepsilon_i})_i'(t_i) > u_i'(t_i)$, $i = 1,2$. If we set $z_i = (w_{\varepsilon_i})_i - u_i$, this gives

$$z_i'(t_i) \geq 0 \quad \text{and} \quad z_i(t_i) = 0, \quad i = 1,2.$$

By using the definition of derivative, we have

$$z_i'(t_i) = \lim_{h\to 0^+} \frac{z_i(t_i) - z_i(t_i - h)}{h} = \lim_{h \to 0^+} \frac{-z_i(t_i - h)}{h}.$$

Since for $h > 0$ small enough, $z_i(t_i - h) > 0$, we have $z_i'(t_i) < 0$ which contradicts the assumption $z_i'(t_i) \geq 0$, $i = 1,2$. So, $u_i(t) \leq (w_{\varepsilon_i})_i(t)$ on J for $i = 1,2$. Similarly, it can be shown that $(v_{\varepsilon_i})_i(t) \leq u_i(t)$. Consequently, $(v_{\varepsilon_i})_i(t) \leq u_i(t) \leq (w_{\varepsilon_i})_i(t)$ on J for $i = 1,2$. Letting $\varepsilon_i \to 0$, we get $v_i(t) \leq u_i(t) \leq w_i(t)$ on J for $i = 1,2$. □

The functions $v,\ w \in C^1(J, \mathbb{R}^2)$ are said to be a lower and an upper solution of problem (1), respectively, if

$$\begin{aligned} v_i'(t) &\leq f_i(t, v(t)) \\ v_i(0) &\leq g_i(v(T)), \qquad t \in J,\ i = 1,2 \end{aligned}$$

and

$$\begin{aligned} w_i'(t) &\geq f_i(t, w(t)) \\ w_i(0) &\geq g_i(w(T)), \qquad t \in J,\ i = 1,2. \end{aligned}$$

Theorem 2.2. *Let $f \in C\left(J \times \mathbb{R}^2, \mathbb{R}^2\right)$, $g \in C\left(\mathbb{R}^2, \mathbb{R}^2\right)$, v_0, w_0 be lower and upper solutions of* (1) *such that $v_{0i} \leq w_{0i}$ on J for $i = 1,2$ and let g_i be nondecreasing on J for $i = 1,2$. Suppose further that*

$$f_i(t,u) - f_i(t,\bar{u}) \geq -M_i(u_i - \bar{u}_i) \quad \text{for } v_{0i} \leq \bar{u}_i \leq u_i \leq w_{i0}\ , \quad M_i \geq 0\ , \quad i = 1,2.$$

Then there exists monotone sequences $\{v_n\}$, $\{w_n\}$ such that $v_n \to v$, $w_n \to w$ as $n \to \infty$ monotonically and uniformly on J and that v and w are minimal and maximal solutions of (1)*, respectively.*

Proof. For any $\eta \in C(J, \mathbb{R}^2)$ such that $v_{0i} \leq \eta_i \leq w_{0i}$, $i = 1,2$, we consider the following problem:

$$u_i' = f_i(t,\eta) - M_i(u_i - \eta_i), \qquad u_i(0) = g_i(\eta(T)). \tag{2}$$

For every such η, problem (2) has a unique solution u on J. Define a mapping A as $A_i\eta = u_i$, $i = 1,2$. This mapping will be used to define the sequences $\{v_{ni}\}$ and $\{w_{ni}\}$, $i = 1,2$. First, we will prove that
(a) $v_{0i} \leq A_i v_0$, $w_{0i} \geq A_i w_0$, $i = 1,2$.
(b) A_i are monotone operators on Δ, $i = 1,2$.
To prove (a), set $A_i v_0 = v_{1i}$ where v_{1i} is the unique solution of (2) for $\eta_i = v_{0i}$, $i = 1,2$. Setting $p_i = v_{1i} - v_{0i}$, we have

$$p_i' = v_{1i}' - v_{0i}' \geq f_i(t, v_0) - M_i(v_{1i} - v_{0i}) - f_i(t, v_0) = -M_i p_i \quad , \quad i = 1,2,$$

and

$$p_i(0) = v_{1i}(0) - v_{0i}(0) \geq g_i(v_0(T)) - g_i(v_0(T)) = 0.$$

This gives us that $p_i(t) \geq 0$, so $v_{0i} \leq A_i v_0$, $i = 1,2$. Similarly, it can be proven that $w_{0i} \geq A_i w_0$.

To prove (b), let $\bar{\eta}_i,\ \tilde{\eta}_i \in [v_{0i}, w_{0i}]$ such that $\bar{\eta}_i \leq \tilde{\eta}_i$, $i = 1,2$. Suppose that

$$u_{1i} = A_i\bar{\eta} \quad and \quad u_{2i} = A_i\tilde{\eta}.$$

Here, u_{1i} and u_{2i} are the unique solutions of (2) for $\bar{\eta}$ and $\tilde{\eta}$, respectively. Set $p_i = u_{2i} - u_{1i}$, $i = 1,2$, then,

$$\begin{aligned} p_i' &= u_{2i}' - u_{1i}' = f_i(t,\tilde{\eta}) - M_i(u_{2i} - \tilde{\eta}_i) - f_i(t,\bar{\eta}) + M_i(u_{1i} - \bar{\eta}_i) \\ &\geq -M_i(\tilde{\eta}_i - \bar{\eta}_i) - M_i(u_{2i} - u_{1i} - \tilde{\eta}_i + \bar{\eta}_i) = -M_i p_i \end{aligned}$$

and

$$\begin{aligned} p_i(0) &= u_{2i}(0) - u_{1i}(0) \\ &= g_i(\tilde{\eta}(T)) - g_i(\bar{\eta}(T)) \geq 0, \end{aligned}$$

since g_i are nondecreasing for $i = 1,2$. This gives us that $p_i(t) \geq 0$ so, $u_{2i} \geq u_{1i}$, $i = 1,2$. Since $A_i\tilde{\eta} \geq A_i\bar{\eta}$, A_i are monotone operators on Δ for $i = 1,2$.

As a result of (a) and (b), the sequences $v_{ni} = A_i v_{n-1}$ and $w_{ni} = A_i w_{n-1}$ can be defined. We can also show, by using mathematical induction, that

$$v_{0i} \leq v_{1i} \leq ... \leq v_{ni} \leq w_{ni} \leq ... \leq w_{2i} \leq w_{1i} \leq w_{0i} \quad on\ J\ for\ i = 1,2.$$

It then follows

$$\lim_{n\to\infty} v_{ni} = v_i \qquad and \qquad \lim_{n\to\infty} w_{ni} = w_i$$

monotonically and uniformly on J, $i = 1,2$. It is clear that v and w are are solutions of (1) since for $i = 1,2$, v_{ni} and w_{ni} satisfy

$$\begin{aligned} v_{ni}' &= f_i(t, v_{n-1}) - M_i\left(v_{ni} - v_{(n-1)i}\right), \qquad v_{ni}(0) = g_i(v_{n-1}(T)) \\ w_{ni}' &= f_i(t, w_{n-1}) - M_i\left(w_{ni} - w_{(n-1)i}\right), \qquad w_{ni}(0) = g_i(w_{n-1}(T)). \end{aligned} \tag{3}$$

To prove that v and w are minimal and maximal solutions of (1), we have to show that if u is any solution of (1) such that $v_{0i} \leq u_i \leq w_{0i}$ on J, $i = 1,2$, then,

$$v_{0i} \leq v_i \leq u_i \leq w_i \leq w_{0i} \qquad on\ J\ for\ i = 1,2.$$

Suppose that for some n, $v_{ni} \leq u_i \leq w_{ni}$ on J and set $p_i = u_i - v_{(n+1)i}$, then we have

$$\begin{aligned} p_i' &= u_i' - v_{(n+1)i}' \\ &= f_i(t,u) - f_i(t,v_n) + M_i\left(v_{(n+1)i} - v_{ni}\right) \\ &\geq -M_i(u_i - v_{ni}) + M_i\left(v_{(n+1)i} - v_{ni}\right) = -M_i p_i \end{aligned}$$

and

$$p_i(0) = u_i(0) - v_{(n+1)i}(0) = g_i(u(T)) - g_i(v_n(T)) \geq 0.$$

These inequalities give us that $p_i\,(t) \geq 0$. So, $u_i \geq v_{(n+1)i}$ on J for $i = 1, 2$. Similarly it can be shown that $u_i \leq w_{(n+1)i}$ on J for $i = 1, 2$. Hence, $v_{(n+1)i} \leq u_i \leq w_{(n+1)i}$ on J for $i = 1, 2$. By using mathematical induction, this proves that for all n, $v_{ni} \leq u_i \leq w_{ni}$. Taking the limit as $n \to \infty$ gives us that $v_i \leq u_i \leq w_i$ on J for $i = 1, 2$. □

We note that, every element of the sequence $\{v_n\}$ is a lower solution and every element of the sequence $\{w_n\}$ is an upper solution for problem (1).

Theorem 2.3. *Let the conditions of Theorem 1 hold and moreover let*

$$f_i(t,x) - f_i(t,\bar{x}) \leq h_i(t)(x_i - \bar{x}_i) \quad for \quad v_i(t) \leq \bar{x}_i(t) \leq x_i(t) \leq w_i(t), \;\; t \in J, \; i = 1, 2$$

$$g_i(x\,(T)) - g_i(\bar{x}\,(T)) \leq L_i(T)(x_i\,(T) - \bar{x}_i\,(T)) \; for \; v_i(T) \leq \bar{x}_i(T) \leq x_i(T) \leq w_i(T),$$

where $h_i : J \to \mathbb{R}$ *are integrable functions on* J *and* $L_i : J \to \mathbb{R}^+$ *are nonnegative functions for* $i = 1,\ 2$ *such that*

$$L_i(T) \exp\left(\int_0^T h_i(s)ds \right) < 1. \tag{4}$$

Then the problem (1) *has a unique solution in the set* Δ.

Proof. The existence of the solution of the problem (1) follows from Theorem 1. So, we need to prove the uniqueness of the solution. Let $y, z \in \Delta$ be two arbitrary solutions of (1). Without loss of generality we can assume y and z satisfy the conditions

$$y_i(t) > z_i(t) \quad for \quad t \in J = [0, T]\,, \quad i = 1, 2.$$

Set $p_i = y_i - z_i$, $i = 1,\ 2$. Hence,

$$\begin{aligned} p_i'\,(t) &= y_i'\,(t) - z_i'\,(t) = f_i\,(t, y\,(t)) - f_i\,(t, z\,(t)) \\ &\leq h_i\,(t)\,(y_i\,(t) - z_i\,(t)) \\ &= h_i\,(t)\,p_i\,(t) \end{aligned}$$

and

$$\begin{aligned} p_i\,(0) &= y_i\,(0) - z_i\,(0) = g_i\,(y\,(T)) - g_i\,(z\,(T)) \\ &\leq L_i\,(T)\,(y_i\,(T) - z_i\,(T)) \\ &= L_i\,(T)\,p_i\,(T)\,. \end{aligned} \tag{5}$$

So, we can write

$$\frac{p_i'\,(t)}{p_i\,(t)} \leq h_i\,(t)\,, \qquad i = 1, 2. \tag{6}$$

If we integrate (6) on the interval $[0, T]$, we get

$$p_i\,(T) \leq p_i\,(0) \exp\left(\int_0^T h_i\,(s)\,ds \right), \qquad i = 1, 2. \tag{7}$$

Using (5) and (7), we have

$$0 < p_i(0) \le L_i(T)\, p_i(T) \le L_i(T)\, p_i(0) \exp\left(\int_0^T h_i(s)\, ds\right), \qquad i = 1, 2.$$

By the condition given in (4), this inequality yields $p_i(0) = 0$ which contradicts with the assumption

$$p_i(t) = y_i(t) - z_i(t) > 0, \quad t \in J, \quad i = 1, 2.$$

So, there exists a $t_0 \in J$ such that $y_i(t_0) = z_i(t_0)$, $i = 1, 2$. If $t_0 = T$ or $t_0 = 0$, then

$$y_i(0) = g_i(y(T)) = g_i(z(T)) = z_i(0), \qquad i = 1,\ 2.$$

This and (6) yields

$$p_i(t) = y_i(t) - z_i(t) = 0, \quad t \in J, \quad i = 1, 2$$

which is a contradiction. Let $t_0 \in (0, T)$. Then, $y_i(t) = z_i(t)$ on $[t_0, T]$, since $y_i(T) = z_i(T)$ and $y_i(0) = z_i(0)$. So, $y_i(t) = z_i(t)$ on J for $i = 1, 2$. This is also a contradiction. Consequently, (1) has a unique solution in the set Δ. □

Note that if f and g satisfy the conditions of Theorem 2 besides the conditions of Theorem 3, the sequences $\{v_n\}$ and $\{w_n\}$ converge to the unique solution u, uniformly as $n \to \infty$.

Functions $v, w \in C^1\left(J, \mathbb{R}^2\right)$ are called weakly coupled lower and upper solutions of (1), if

$$\begin{aligned}
v_i'(t) &\le f_i(t, v(t)), && t \in J,\ i = 1, 2 && (8)\\
v_i(0) &\le g_i(w(T)) && &&\\
w_i'(t) &\ge f_i(t, w(t)), && t \in J,\ i = 1, 2 && (9)\\
w_i(0) &\ge g_i(v(T)). && &&
\end{aligned}$$

If the inequalities are converted to equalities in (8) and (9), v and w are called coupled quasisolutions of (1).

Theorem 2.4. *Let $f \in C(J \times \mathbb{R}^2, \mathbb{R}^2)$, $g \in C(\mathbb{R}^2, \mathbb{R}^2)$, y_0, z_0 be weakly coupled lower and upper solutions of* (1) *such that $y_{0i}(t) \le z_{0i}(t)$ on J for $i = 1, 2$ and let g be nonincreasing. Suppose further that*

$$-M_i[w_i - v_i] \le f_i(t, w) - f_i(t, v) \quad \text{for } y_{0i}(t) \le v_i(t) \le w_i(t) \le z_{0i}(t),\ t \in J,\ i = 1, 2.$$

Then there exists monotone sequences $\{y_n\}$ and $\{z_n\}$ such that $y_n \to y$, $z_n \to z$ monotonically and uniformly on J. Moreover, y and z are coupled quasisolutions of (1).

Proof. For any $\eta, \mu \in C(J, \mathbb{R}^2)$ such that $y_{0i} \le \eta_i \le z_{0i}$, $y_{0i} \le \mu_i \le z_{0i}$ $i = 1, 2$, we consider the following problems:

$$u_i' = f_i(t, \eta) - M_i(u_i - \eta_i), \qquad u_i(0) = g_i(\mu(T)) \tag{10}$$

For every such η, μ, problem (10) has unique solution u on J. Define a mapping A as $A_i[\eta,\mu]=u_i$, $i=1,2$. This mapping will be used to define the sequences $\{y_n\}$ and $\{z_n\}$. First, we will prove that
(a) $y_{0i}\le A_i[y_0,z_0]$, $z_{0i}\ge A_i[z_0,y_0]$, $i=1,2$.
(b) A_i are monotone operators on Δ, $i=1,2$.
To prove (a), set $A_i[y_0,z_0]=y_{1i}$ where y_1 is the unique solution of (10), for $\eta_i=y_{0i}$, $\mu_i=z_{0i}$, $i=1,2$. Setting $p_i=y_{1i}-y_{0i}$, we have

$$p_i'=y_{1i}'-y_{0i}'\ge f_i(t,y_0)-M_i(y_{1i}-y_{0i})-f_i(t,y_0)=-M_ip_i \quad , \quad i=1,2,$$

and

$$p_i(0)=y_{1i}(0)-y_{0i}(0)\ge g_i(z_0(T))-g_i(z_0(T))=0.$$

This gives us that $p_i(t)\ge 0$, so $y_{0i}\le A_i[y_0,z_0]$, $i=1,2$. Similarly, it can be proven that $z_{oi}\ge A_i[z_0,y_0]$, $i=1,2$.
To prove (b), let $\bar\eta_i,\ \tilde\eta_i,\ \mu_i\in[y_{0i},z_{0i}]$ such that $\tilde\eta_i\le\bar\eta_i$, $i=1,2$. Suppose that

$$u_1=A[\tilde\eta,\mu] \quad and \quad u_2=A[\bar\eta,\mu].$$

Here, u_{1i} and u_{2i} are the unique solutions of (10) for $[\tilde\eta,\mu]$ and $[\bar\eta,\mu]$, respectively. Set $p_i=u_{2i}-u_{1i}$, $i=1,2$, then,

$$\begin{aligned} p_i' &= u_{2i}'-u_{1i}'=f_i(t,\bar\eta)-M_i(u_{2i}-\bar\eta_i)-f_i(t,\tilde\eta)+M_i(u_{1i}-\tilde\eta_i)\\ &\ge -M_i(\bar\eta_i-\tilde\eta_i)-M_i(u_{2i}-u_{1i}-\bar\eta_i+\tilde\eta_i)=-M_ip_i \end{aligned}$$

and

$$p_i(0)=u_{2i}(0)-u_{1i}(0)=g_i(\mu(T))-g_i(\mu(T))=0.$$

So, $p_i(t)\ge 0$. Let $\eta_i,\ \tilde\mu_i,\ \bar\mu_i\in[y_{0i},z_{0i}]$ such that $\tilde\mu_i\le\bar\mu_i$, $i=1,2$. Suppose that

$$u_1=A[\eta,\tilde\mu] \quad and \quad u_2=A[\eta,\bar\mu].$$

Set $p_i=u_{1i}-u_{2i}$, $i=1,2$, then,

$$\begin{aligned} p_i' &= u_{1i}'-u_{2i}'=f_i(t,\eta)-M_i(u_{1i}-\eta_i)-f_i(t,\eta)+M_i(u_{2i}-\eta_i)\\ &= -M_ip_i \end{aligned}$$

and

$$p_i(0)=u_{1i}(0)-u_{2i}(0)=g_i(\tilde\mu(T))-g_i(\bar\mu(T))\ge 0.$$

So, $p_i(t)\ge 0$ on J for $i=1,2$. As a result of (a) and (b), the sequences $y_n=A[y_{n-1},z_{n-1}]$ and $z_n=A[z_{n-1},y_{n-1}]$ can be defined. We can also show, by using mathematical induction, that

$$y_{0i}\le y_{1i}\le ...\le y_{ni}\le z_{ni}\le ...\le z_{2i}\le z_{1i}\le z_{0i} \quad on\ J\ for\ i=1,2.$$

It then follows

$$\lim_{n\to\infty}y_{ni}=y_i \quad and \quad \lim_{n\to\infty}z_{ni}=z_i$$

monotonically and uniformly on J for $i = 1, 2$. It is clear that y and z are coupled quasisolutions of (1), since y_{ni} and z_{ni} satisfy

$$\begin{aligned} {y_{ni}}'(t) &= f_i\left(t, y_{(n-1)i}(t)\right) - M_i\left[y_{ni}(t) - y_{(n-1)i}(t)\right], \\ y_{ni}(0) &= g_i\left(z_{(n-1)i}(T)\right), \qquad t \in J,\ i = 1, 2, \\ z'_{ni}(t) &= f_i\left(t, z_{(n-1)i}(t)\right) - M_i\left[z_{ni}(t) - z_{(n-1)i}(t)\right], \\ z_{ni}(0) &= g_i\left(y_{(n-1)i}(T)\right), \qquad t \in J,\ i = 1, 2. \end{aligned} \tag{11}$$

□

Theorem 2.5. *Assume that $f \in C(J \times \mathbb{R}^2, \mathbb{R}^2)$, $g \in C(\mathbb{R}^2, \mathbb{R}^2)$. Let the functions $y_0, z_0 \in C^1\left(J, \mathbb{R}^2\right)$ be weakly coupled lower and upper solutions of (1) satisfying $y_{0i}(t) \leq z_{0i}(t)$, $t \in J$, $i = 1, 2$. Moreover, assume that there exists nonnegative constants M_i, integrable functions $K_i : J \to \mathbb{R}$ and nonnegative functions $N_i : J \to \mathbb{R}^+$ for $i = 1, 2$ such that*

$$\begin{aligned} &-M_i\left[w_i - v_i\right] \leq f_i(t, w) - f_i(t, v) \leq K_i(t)\left[w_i - v_i\right] \\ &\qquad \text{for } y_{0i}(t) \leq v_i(t) \leq w_i(t) \leq z_{0i}(t),\ t \in J,\ i = 1, 2, \\ &0 \leq g_i(v(T)) - g_i(w(T)) \leq N_i(T)\left[w_i(T) - v_i(T)\right] \\ &\qquad \text{for } y_{0i}(T) \leq v_i(T) \leq w_i(T) \leq z_{0i}(T),\ i = 1, 2 \end{aligned}$$

and

$$N_i(T)\exp\left(\int_0^T K_i(s)\,ds\right) < 1. \tag{12}$$

Then problem (1) has a unique solution $u \in \Delta$,

$$y_{0i} \leq y_{1i} \leq ... \leq y_{ni} \leq z_{ni} \leq ... \leq z_{2i} \leq z_{1i} \leq z_{0i} \quad \text{on } J \text{ for } i = 1, 2. \tag{13}$$

and $\{y_n\}$ and $\{z_n\}$ converge to u uniformly, for $n \to \infty$, on J.

Proof. Since the assumptions of Theorem 3 are satisfied, problem (1) has a unique solution, $u \in \Delta$. It is known from Theorem 4 that $\{y_n\}$ and $\{z_n\}$ converge to their limit functions uniformly and monotonically as $n \to \infty$. Let

$$\begin{aligned} \lim_{n\to\infty}(y^n)(t) &= y(t), \\ \lim_{n\to\infty}(z^n)(t) &= z(t). \end{aligned}$$

Then, y and z satisfy

$$\begin{aligned} y'_i(t) &= f_i(t, y(t)) \\ y_i(0) &= g_i(z(T)), \qquad t \in J,\ i = 1, 2, \\ z'_i(t) &= f_i(t, z(t)) \\ z_i(0) &= g_i(y(T)), \qquad t \in J,\ i = 1, 2 \end{aligned}$$

and

$$\left(y^0\right)_i(t) \leq y_i(t) \leq z_i(t) \leq \left(z^0\right)_i(t), \qquad t \in J,\ i = 1, 2.$$

Set $p_i = z_i - y_i$ so, $p_i(t) \geq 0,\ t \in J,\ i = 1, 2$. So,

$$p_i'(t) = z_i'(t) - y_i'(t) = f_i(t, z(t)) - f_i(t, y(t)) \leq K_i(t)\, p_i(t) \tag{14}$$

and

$$p_i(0) = z_i(0) - y_i(0) = g_i(y(T)) - g_i(z(T)) \leq N_i(T)\, p_i(T), \qquad i = 1, 2. \tag{15}$$

Using (14), we have

$$p_i(T) \leq p_i(0) \exp \int_0^T K_i(s)\, ds. \tag{16}$$

(15) and (16) give us

$$0 \leq p_i(0) \leq A_i(T)\, p_i(T) \leq N_i(T)\, p_i(0) \exp\left(\int_0^T K_i(s)\, ds\right).$$

Condition (12) yields to $p_i(0) = 0,\ i = 1, 2$. So, $p_i(t) = 0$, $t \in J,\ i = 1, 2$. Since the problem (1) has a unique solution on Δ, $u_i = y_i = z_i,\ i = 1, 2$. □

Example 2.6. Consider

$$\begin{aligned} x_1' &= x_1(10 - 2x_1 - 5x_2) \\ x_2' &= x_2(-3 + 5x_1) \\ x_1(0) &= \frac{1}{2e^{20}} x_1(2) \geq 0 \\ x_2(0) &= x_1(2) + \frac{1}{6} x_2(2) \geq 0, \qquad t \in [0, 2]. \end{aligned} \tag{17}$$

$v_i(t) \equiv 0$ is a lower solution of (17). The solution of the system

$$\begin{aligned} w_1' &= w_1(10 - 2w_1) \\ w_2' &= w_2(-3 + 5w_1) \\ w_1(0) &= \frac{1}{2e^{20}} w_1(2) = g_1(w(2)) \geq 0 \\ w_2(0) &= w_1(2) + \frac{1}{6} w_2(2) = g_2(w(2)) \geq 0, \qquad t \in [0, 2] \end{aligned} \tag{18}$$

is an upper solution of (17). We note that, the functions on the right-hand side of the system (18) are obtained by

$$\begin{aligned} &\sup_{0 \leq \varphi \leq x_2} x_1(10 - 2x_1 - 5\varphi) \\ &\sup_{0 \leq \varphi \leq x_1} x_2(-3 + 5\varphi). \end{aligned}$$

From (18),

$$
\begin{aligned}
w_1(t) &= \left(\frac{(2-2e^{20})e^{-10t}}{5}\right)^{-1} \\
w_2(t) &= \frac{15e^4\left(-1+2e^{20-10t}-e^{-10t}\right)^{\frac{1}{10}}\left(2e^{20}-1\right)^{\frac{1}{10}}}{e^{2t}\left(2e^{20}-1\right)^{\frac{1}{10}}\left(e^{-20}-1\right)\left(6e^4\left(2e^{20}-1\right)^{\frac{1}{10}}-\left(1-e^{-2}\right)^{\frac{1}{10}}\right)}
\end{aligned}
$$

is an upper solution of (17). From Theorem 1, (1) has a solution u satisfying the condition $v_i(t) \leq u_i(t) \leq w_i(t)$, $t \in J$, $i = 1, 2$. Since the functions g_i are nondecreasing for the i^{th} component, Theorem 2 gives us that (17) has a minimal and a maximal solution on Δ. Moreover, L_1, L_2, M_1 and M_2 constants for g_1, g_2, f_1 and f_2 are $\frac{1}{2e^{20}}$, $\frac{1}{6}$, 10 and 1, respectively. Since

$$\frac{1}{2e^{20}} \exp \int_0^2 10dt < 1$$

and

$$\frac{1}{6} \exp \int_0^2 dt < 1,$$

(17) has a unique solution.

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Current address: Ankara University, Faculty of Sciences, Dept. of Mathematics, Ankara, TURKEY

E-mail address: edemirci@ankara.edu.tr - nozalp@science.ankara.edu.tr

URL: http://communications.science.ankara.edu.tr/index.php?series=A1