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THE QUENCHING BEHAVIOR OF A PARABOLIC SYSTEM

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ABSTRACT. In this paper, we study the quenching behavior of solution of a parabolic system. We prove finite-time quenching for the solution. Further, we show that quenching occurs on the boundary under certain conditions. Furthermore, we show that the time derivative blows up at quenching time. Finally, we get a quenching criterion by using a comparison lemma and we also get a quenching rate.

1. INTRODUCTION

In this paper, we study the problem for the following parabolic system:

$$u_t = u_{xx} + (1-v)^{-p}, \ 0 < x < 1, \ 0 < t < T, \tag{1}$$

$$v_t = v_{xx} + (1-u)^{-q}, \ 0 < x < 1, \ 0 < t < T, \tag{2}$$

with boundary conditions

$$u_x(0,t) = 0 = u_x(1,t), \ 0 < t < T, \tag{3}$$

$$v_x(0,t) = 0 = v_x(1,t), \ 0 < t < T, \tag{4}$$

and initial conditions

$$u(x,0) = u_0(x) < 1, \ v(x,0) = v_0(x) < 1, \ 0 \le x \le 1, \tag{5}$$

where p, q are positive constants, and $u_0(x), v_0(x)$ are positive smooth functions satisfying the compatibility conditions

$$u_0'(0) = v_0'(0) = u_0'(1) = v_0'(1) = 0.$$

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Throughout this paper, we also assume that the initial functions (u_0, v_0) satisfies the inequalities

$$\begin{aligned} u_{xx}(x,0)+(1-v(x,0))^{-p} &> 0, \quad (6)\\ v_{xx}(x,0)+(1-u(x,0))^{-q} &> 0, \quad (7)\\ u_x(x,0) &\geq 0, \quad (8)\\ v_x(x,0) &\geq 0. \quad (9) \end{aligned}$$

Our main purpose is to examine the quenching behavior of the solutions of problem $(1)-(5)$. The solution of the problem $(1)-(5)$ is said to quench if there exists a finite time T such that

$$\lim_{t\to T^-} \max\{u(x,t), v(x,t) : 0 \leq x \leq 1\} \to 1^-.$$

From now on, we denote the quenching time of the problem $(1)-(5)$ with T.

Since 1975, quenching problems with various boundary conditions have been studied extensively (cf. the surveys by Chan [1, 2], Kirk and Roberts [13] and by the authors of [3], [4], [5], [6], [7], [8],[11], [12], [15], [18]). There are many papers about the quenching phenomenon for the solutions of nonlinear parabolic systems ([10], [14], [16], [19], [20]). In [9], Fu and Guo studied the blow-up phenomenon for the solution of a nonlinear parabolic system. In [19], Zheng and Wang considered the following problem

$$\begin{aligned} u_t &= \Delta u - v^{-p}, v_t = \Delta v - u^{-q}, (x,t) \in \Omega \times (0,T),\\ u &= v = 1, (x,t) \in \partial\Omega \times (0,T),\\ u(x,0) &= u_0(x), v(x,0) = v_0(x), x \in \bar{\Omega}, \end{aligned}$$

where $p, q > 0, \Omega \in R^N$ is a bounded domain with smooth boundary, and the initial data satisfy $u_0, v_0 \in C^2(\Omega) \cap C^1(\bar{\Omega}),\ u_0, v_0 = 1$ on $\partial\Omega,\ 0 < u_0, v_0 \leq 1$. They obtained the sufficient conditions for global existence and finite time quenching of solutions, and then determined the blow-up time-derivatives and the quenching set. Further, they obtained a simultaneous and non-simultaneous quenching criterion. In [16], de Pablo et al. considered the following problem

$$\begin{aligned} u_t &= u_{xx} - v^{-p}, v_t = v_{xx} - u^{-q}, (x,t) \in (0,1) \times (0,T),\\ u_x(0,t) &= v_x(0,t) = u_x(1,t) = v_x(1,t) = 0, t \in (0,T),\\ u(x,0) &= u_0(x), v(x,0) = v_0(x), x \in [0,1] \end{aligned}$$

where $p, q > 0$ and u_0, v_0 are positive, smooth and satisfy the compatibility conditions $u_0', v_0' \geq 0,\ u_0'' - v_0^{-p}, v_0'' - u_0^{-q} < 0$. They showed that $x = 0$ is the unique quenching point and (u_t, v_t) blows up at quenching time. In [20], Zhou et al. considered same problem. They show that the system exhibits simultaneous and non-simultaneous quenching. In addition, they gave a natural condition for this problem beyond quenching time T for the case of non-simultaneous quenching.

In above examples, the authors considered quenching problems with singular absorption terms $-v^{-p}$,$-u^{-q}$. Here, we would like to study a quenching problem due to a singular reaction terms $(1-v)^{-p}, (1-u)^{-q}$. This paper is organized as follows. In Section 2, we first show that quenching occurs in finite time under the conditions $(6)-(7)$. Then, we show that the only quenching point is $x=1$ under the condition $(8)-(9)$. Finally, we show that (u_t, v_t) blows up at quenching time. In Section 3, we give a quenching criterion by using a comparison lemma and we also get a quenching rate.

2. Quenching on the boundary and blow-up of (u_t, v_t)

In this section, we will investigate quenching set of the problem $(1)-(5)$. Later, we will prove that (u_t, v_t) blows up at quenching time.

Remark 1. If (u_0, v_0) satisfies $(6)-(9)$, then we get $u_x, v_x > 0$ and $u_t, v_t > 0$ in $(0,1)\times(0,T)$ by the maximum principle. Thus we get $u(1,t) = \max\limits_{0\le x\le 1} u(x,t)$ and $v(1,t) = \max\limits_{0\le x\le 1} v(x,t)$.

Theorem 1. If (u_0, v_0) satisfies $(6)-(7)$, then there exist a finite time T, such that the solution (u,v) of the problem $(1)-(5)$ quenches at time T.
Proof. Assume that (u_0, v_0) satisfies $(6)-(7)$. Then there exist

$$\begin{aligned} w_1 &= \int_0^1 (1-v(x,0))^{-p}\,dx > 0, \\ w_2 &= \int_0^1 (1-u(x,0))^{-q}\,dx > 0. \end{aligned}$$

Introduce a mass function; $m_1(t) = \int_0^1 (1-u(x,t))\,dx$ and $m_2(t) = \int_0^1 (1-v(x,t))\,dx$, $0<t<T$. Then

$$\begin{aligned} m_1'(t) &= -\int_0^1 (1-v(x,t))^{-p}\,dx \le -w_1, \\ m_2'(t) &= -\int_0^1 (1-u(x,t))^{-q}\,dx \le -w_2, \end{aligned}$$

by Remark 1. Thus, $m_1(t) \le m_1(0) - w_1 t$ and $m_2(t) \le m_2(0) - w_2 t$; which means that $m_1(T_0) = 0$ or $m_2(T_0) = 0$ for some $T_0 = \min(\frac{m_1(0)}{w_1}, \frac{m_2(0)}{w_2})$, $(0 < T \le T_0)$. Thus, (u,v) quenches in finite time. ■

Theorem 2. If (u_0, v_0) satisfies $(8)-(9)$, then $x=1$ is the only quenching point.
Proof. Define

$$J(x,t) = u_x - \varepsilon(1-x) \text{ in } [1-\eta, 1]\times[\tau, T),$$

where $\eta \in (0,1)$, $\tau \in (0,T)$ and ε is a positive constant to be specified later. Then,

$$J_t - J_{xx} = p(1-v)^{-p-1} v_x > 0 \text{ in } (1-\eta,1) \times (\tau,T),$$

since $v_x(x,t) > 0$ in $(0,1] \times (0,T)$. Thus, $J(x,t)$ cannot attain a negative interior minimum by the maximum principle. Further, if ε is small enough, $J(x,\tau) > 0$ since $u_x(x,t) > 0$ in $(0,1] \times (0,T)$. Furthermore, if ε is small enough,

$$\begin{aligned} J(1-\eta,t) &= u_x(1-\eta,t) - \varepsilon\eta > 0, \\ J(1,t) &= 0, \end{aligned}$$

for $t \in (\tau,T)$. By the maximum principle, we obtain that $J(x,t) \geq 0$, i.e. $u_x \geq \varepsilon(1-x)$ for $(x,t) \in [1-\eta,1] \times [\tau,T)$. Integrating last inequality with respect to x from x to 1, we have

$$u(x,t) \leq u(1,t) - \frac{\varepsilon(1-x)^2}{2} \leq 1 - \frac{\varepsilon(1-x)^2}{2},$$

for $x \in [1-\eta,1]$. So u does not quench in $[0,1)$. Similarly, we observe that v does not quench in $[0,1)$. The theorem is proved. ■

Theorem 3. (u_t, v_t) blows up at the quenching point $x = 1$.
Proof. Define

$$\begin{aligned} J_1(x,t) &= u_t - \varepsilon(1-v)^{-p} \text{ in } [0,1] \times [\tau,T), \\ J_2(x,t) &= v_t - \varepsilon(1-u)^{-q} \text{ in } [0,1] \times [\tau,T), \end{aligned}$$

where $\tau \in (0,T)$ and ε is a positive constant to be specified later. Then, $J_1(x,t)$ and $J_2(x,t)$ satisfy

$$(J_1)_t - (J_1)_{xx} - p(1-v)^{-p-1} J_2 = \varepsilon p(p+1)(1-v)^{-p-2} v_x^2 > 0$$

and

$$(J_2)_t - (J_2)_{xx} - q(1-u)^{-q-1} J_1 = \varepsilon q(q+1)(1-u)^{-q-2} u_x^2 > 0.$$

Thus, $J_1(x,t)$ and $J_2(x,t)$ cannot attain a negative interior minimum by the maximum principle for weakly coupled parabolic systems (cf. Theorem 15 of Chapter 3 in [17]). Further, if ε is small enough, $J_1(x,\tau) > 0$ and $J_2(x,\tau) > 0$. Furthermore,

$$\begin{aligned} (J_1)_x(0,t) &= 0, \ (J_1)_x(1,t) = 0, \\ (J_2)_x(0,t) &= 0, \ (J_2)_x(1,t) = 0, \end{aligned}$$

for $t \in (\tau,T)$. By the maximum principle, we obtain that $J_1(x,t) \geq 0$, i.e.

$$u_t \geq \varepsilon(1-v)^{-p}$$

for $(x,t) \in [0,1] \times [\tau,T)$ and $J_2(x,t) \geq 0$, i.e.

$$v_t \geq \varepsilon(1-u)^{-q}$$

for $(x,t) \in [0,1] \times [\tau,T)$. The theorem is proved. ■

3. A quenching criterion and a quenching rate

In this section, we will obtain a quenching criterion and a quenching rate. First, we give a comparison lemma.

Lemma 1. **a)** If $u_0(x) \geq v_0(x)$ for $x \in [0,1]$ and $p \geq q$, then $u(x,t) \geq v(x,t)$ in $[0,1] \times (0,T)$,
b) If $v_0(x) \geq u_0(x)$ for $x \in [0,1]$ and $q \geq p$, then $v(x,t) \geq u(x,t)$ in $[0,1] \times (0,T)$.
Proof. **a)** Define $M(x,t) = u - v$ in $[0,1] \times [0,T)$. Then, $M(x,t)$ satisfies

$$\begin{aligned} M_t - M_{xx} &= (1-v)^{-p} - (1-u)^{-q} \\ &= (1-v)^{-p} - (1-u)^{-p} + (1-u)^{-p} - (1-u)^{-q} \\ &\geq -p(1-\beta)^{-p-1}M \end{aligned}$$

where $\beta(x,t)$ lies between $u(x,t)$ and $v(x,t)$. Thus, $M(x,t)$ cannot attain a negative interior minimum by the maximum principle. Further, $M(x,0) \geq 0$ since $u_0 \geq v_0$ for $x \in (0,1)$. Furthermore,

$$M_x(0,t) = 0 = M_x(1,t)$$

for $t \in (0,T)$. By the maximum principle, we obtain that $M(x,t) \geq 0$ in $[0,1] \times (0,T)$, i.e. $u(x,t) \geq v(x,t)$ in $[0,1] \times (0,T)$.
b) Similarly, we get $v(x,t) \geq u(x,t)$ in $[0,1] \times (0,T)$ since $v_0(x) \geq u_0(x)$ for $x \in [0,1]$ and for $q \geq p$. ■

Corollary 1. From statement of the problem $(1) - (5)$, we show that

$$\begin{aligned} \text{if } \lim_{t \to T^-} v(1,t) &= 1, \text{ then } \lim_{t \to T^-} u_t(1,t) = \infty, \\ \text{if } \lim_{t \to T^-} u(1,t) &= 1, \text{ then } \lim_{t \to T^-} v_t(1,t) = \infty. \end{aligned}$$

Then, from Theorem 3 and Lemma 1, we get
a) if $v_0(x) \geq u_0(x)$ for $x \in [0,1]$ and $q \geq p$, then u_t blows up at the quenching point $x = 1$. Further, we get

$$u_t(1,t) \geq \varepsilon\,(1 - v(1,t))^{-p} \geq \varepsilon\,(1 - u(1,t))^{-p}.$$

So, integrating for t from t to T we get

$$u(1,t) \leq 1 - C_1(T-t)^{1/(p+1)}$$

where $C_1 = (\varepsilon(p+1))^{1/(p+1)}$.
b) if $u_0(x) \geq v_0(x)$ for $x \in [0,1]$ and $p \geq q$, then v_t blows up at the quenching point $x = 1$. Further, we get

$$v_t(1,t) \geq \varepsilon\,(1 - u(1,t))^{-q} \geq \varepsilon\,(1 - v(1,t))^{-q}.$$

So, integrating for t from t to T we get

$$v(1,t) \leq 1 - C_2(T-t)^{1/(q+1)}$$

where $C_2 = (\varepsilon(q+1))^{1/(q+1)}$.

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