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ON THE PROPERTIES OF QUASI-QUATERNION ALGEBRA

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Abstract. We study some fundamental properties of the quasi-quaternions and derive the De Moivre's and Euler's formulae for matrices associated with these quaternions. Furthermore, with the aid of the De-Moivre's formula, any powers of these matrices can be obtained.

1. Introduction

Quaternions are an efficient way understanding many aspects of physics and kinematics. Today, quaternions are used especially in the area of computer vision, computer graphics, animation, and to solve optimization problems involving the estimation of rigid body transformations [11]. The Euler's and De-Moivre's formulae for the complex numbers are generalized for quaternions [3]. These formulae are also investigated for the cases of split and dual quaternions in [7, 9].

Some algebraic properties of Hamilton operators are considered in [2] where real quaternions have been expressed in terms of 4×4 matrices by means of these operators. The theory of quaternion matrices has been applied in quaternionic mechanics and quantum fields [1]. Also, Yayli has considered homothetic motions with aid of the Hamilton operators in four-dimensional Euclidean space E^4 [13]. Eigenvalues, eigenvectors and the others algebraic properties of these matrices are studied by several authors [5, 15]. Recently, we have derived the De-Moivre's and Euler's formulae for matrices associated with real quaternion and every power of these matrices are obtained [6]. A brief introduction of the quasi-quaternions is provided in [10]. Special Galilean transformation in terms of the quasi-quaternions considered in [9, 15] and De Moivre's and Euler's formula for these quaternions are given in [4].

Here, we investigate some algebraic properties of quasi-quaternions. De-Moivre's and Euler's formulae for these quaternions are given. Also, we derive the nth root of quasi-quaternions. By the Hamilton operators, these quaternions have been expressed in terms of 4×4 matrices. With the aid of the De-Moivre's formula,

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we obtain any power of these matrices. Finally, we give some examples for more clarification.

2. Preliminaries

In this section, we give a brief summary of the real quaternions. For detailed information about these concepts, we refer the reader to [12].

Definition 2.1. A real quaternion is defined *as*

$$q = a_\circ + a_1 i + a_2 j + a_3 k$$

where $a_\circ, a_1, a_2$ and a_3 are real numbers and $1, i, j,\ k$ of q may be interpreted as the four basic vectors of Cartesian set of coordinates; and they satisfy the non-commutative multiplication rules

$$\begin{aligned} i^2 &= j^2 = k^2 = ijk = -1 \\ ij &= k = -ji, \quad jk = i = -kj \end{aligned}$$

and

$$ki = j = -ik.$$

A quaternion may be defined as a pair (S_q, V_q), where $S_q = a_\circ \in \mathbb{R}$ is scalar part and $V_q = a_1 i + a_2 j + a_3 k \in \mathbb{R}^3$ is the vector part of q. The quaternion product of two quaternions p and q is defined as

$$pq = S_p S_q - \langle V_p, V_q \rangle + S_p V_q + S_q V_p + V_p \wedge V_q$$

where"$\langle , \rangle$"and "$\wedge$" are the inner and vector products in $\mathbb{R}^3$, respectively. The norm of a quaternion is given by the sum of the squares of its components: $N_q = a_\circ^2 + a_1^2 + a_2^2 + a_3^2$, $N_q \in \mathbb{R}$. It can also be obtained by multiplying the quaternion by its conjugate, in either order since a quaternion and its conjugate commute: $N_q = \overline{q}q = q\overline{q}$. Every non-zero quaternion has a multiplicative inverse given by its conjugate divided by its norm: $q^{-1} = \frac{\overline{q}}{N_q}$. The quaternion algebra H is a normed division algebra, meaning that for any two quaternions p and q, $N_{pq} = N_p N_q$, and the norm of every non-zero quaternion is non-zero (and positive) and therefore the multiplicative inverse exists for any non-zero quaternion. Of course, as is well known, multiplication of quaternions is not commutative, so that in general for any two quaternions p and q, $pq \neq qp$. This can have subtle ramifications, for example: $(pq)^2 = pqpq \neq p^2 q^2$.

3. QUASI-QUATERNIONS

We introduce a type of quaternion, the quasi-quaternion, which is called $\frac{1}{4}$-quaternion in [10] and dual quaternion in [4, 8, 14].

Definition 3.1. A quasi-quaternion is defined *as*

$$q = a_\circ + a_1 i + a_2 j + a_3 k$$

where $a_\circ, a_1, a_2$ and a_3 are real numbers and $1, i, j, k$ of q may be interpreted as the four basic vectors of cartesian set of coordinates; and they satisfy the rules

$$\begin{aligned} i^2 &= j^2 = k^2 = 0 \\ ij &= ji = jk = kj = ki = ik = 0. \end{aligned}$$

The set of all quasi-quaternions are denoted by H°. A quasi-quaternion may be defined as a pair (S_q, V_q), where $S_q = a_\circ \in \mathbb{R}$ is scalar part and $V_q = a_1 i + a_2 j + a_3 k$ is the vector part of q.

The addition rule for quasi-quaternions is component-wise addition:

$$\begin{aligned} q + p &= (a_\circ + a_1 i + a_2 j + a_3 k) + (b_\circ + b_1 i + b_2 j + b_3 k) \\ &= (a_\circ + b_\circ) + (a_1 + b_1) i + (a_2 + b_2) j + (a_3 + b_3) k. \end{aligned}$$

This rule preserves the associativity and commutativity properties of addition. The product of scalar and a quasi-quaternion is defined in a straightforward manner. If c is a scaler and $q \in H^\circ$,

$$cq = cS_q + cV_q = (ca_\circ)1 + (ca_1)i + (ca_2)j + (ca_3)k.$$

The quasi-quaternion product of two quaternions q and p is defined as

$$qp = S_q S_p + S_q V_p + S_p V_q = pq.$$

Also, this can be written as

$$qp = \begin{bmatrix} a_\circ & 0 & 0 & 0 \\ a_1 & a_\circ & 0 & 0 \\ a_2 & 0 & a_\circ & 0 \\ a_3 & 0 & 0 & a_\circ \end{bmatrix} \begin{bmatrix} b_\circ \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Corollary 1. *In general case, quaternion multiplication is associative and distributive with respect to addition and subtraction, but the commutative law does not hold. For quasi-quaternion multiplication it hold.*

4. Some properties of quasi-quaternions

1) The conjugate of $q = a_\circ + a_1 i + a_2 j + a_3 k = S_q + V_q$ is

$$\overline{q} = a_\circ - (a_1 i + a_2 j + a_3 k) = S_q - V_q.$$

It is clear the scalar and vector part of q is denoted by $S_q = \frac{q+\overline{q}}{2}$ and $V_q = \frac{q-\overline{q}}{2}$.

2) The norm of q is defined as $N_q = q\overline{q} = \overline{q}q = a_\circ^2$. If $N_q = 1$, then q is called a unit quasi-quaternion.

Proposition 1. *Let $p, q \in H^\circ$ and $\lambda, \delta \in \mathbb{R}$. The conjugate and norm of quasi-quaternions satisfies the following properties;*

i) $\overline{\overline{q}} = q$, *ii)* $\overline{pq} = \overline{q}\,\overline{p}$, *iii)* $\overline{\lambda q + \delta p} = \lambda\overline{q} + \delta\overline{p}$,

iv) $N_{qp} = N_q N_p$, *v)* $N_{\lambda q} = \lambda^2 N_q$.

3) The inverse of q is defined as $q^{-1} = \frac{\overline{q}}{N_q}, N_q \neq 0$, with the following properties;

i) $(qp)^{-1} = p^{-1}q^{-1}$, *ii)* $(\lambda q)^{-1} = \frac{1}{\lambda}q^{-1}$, *iii)* $N_{q^{-1}} = \frac{1}{N_q}$.

4) To divide a semi-quaternion p by the semi-quaternion $q(\neq 0)$, one simply has to resolve the equation

$$xq = p \quad or \quad qy = p,$$

with the respective solutions

$$\begin{aligned} x &= pq^{-1} = p\frac{\overline{q}}{N_q}, \\ y &= q^{-1}p = \frac{\overline{q}}{N_q}p, \end{aligned}$$

and the relation $N_x = N_y = \frac{N_p}{N_q}$.

Theorem 4.1. *The algebra H° is isomorphic to the subalgebra of the algebra D_2 consisting of the (2×2)-matrices*

$$\tilde{A} = \begin{bmatrix} A & B \\ 0 & \overline{A} \end{bmatrix}.$$

Proof. The proof can be found in [10]. □

Theorem 4.2. *Let $q = 1 + a_1 i + a_2 j + a_3 k$ be a unit quasi-quaternion. Then q is a Galilean transformation in G_4.*

Proof. Since $q = 1 + a_1 i + a_2 j + a_3 k$, we have

$$\begin{aligned} qx &= (1 + a_1 i + a_2 j + a_3 k)(x_\circ + x_1 i + x_2 j + x_3 k) \\ &= x_\circ + (a_1 x_\circ + x_1)i + (b_1 x_\circ + x_2)j + (c_1 x_\circ + x_3)k, \end{aligned}$$

and

$$\|qx\| = \|x\| .$$

Thus q is a Galilean transformation [8]. □

5. De Moivre's formula for quasi-qauternions

Every nonzero quasi-quaternion $q = a_\circ + a_1 i + a_2 j + a_3 k$ can be written in the polar form

$$q = r(\cos\varphi + \overrightarrow{w}\sin\varphi),\ 0 \leq \varphi \leq 2\pi$$

where $r = \sqrt{N_q}$ and

$$\cos\varphi = \frac{a_\circ}{r},\ \ \sin\varphi = \varphi = \frac{\sqrt{a_1^2 + a_2^2 + a_3^2}}{r}.$$

and the unit vector $\overrightarrow{w}$ is given by

$$\overrightarrow{w} = \frac{a_1 i + a_2 j + a_3 j}{\sqrt{a_1^2 + a_2^2 + a_3^2}}.$$

Since $\overrightarrow{w}^2 = 0$, we have a natural generalization of Euler's formula for quasi-quaternions

$$\begin{aligned} e^{\overrightarrow{w}\varphi} &= 1 + \overrightarrow{w}\varphi + \frac{(\overrightarrow{w}\varphi)^2}{2!} + ... \\ &= 1 + \overrightarrow{w}\varphi = \cos\varphi + \overrightarrow{w}\sin\varphi. \end{aligned}$$

for any real number φ. For detalied information about Euler's formula, see [4].

Theorem 5.1. *(De-Moivre's formula) Let* $q = e^{\overrightarrow{w}\varphi} = \cos\varphi + \overrightarrow{w}\sin\varphi$ *be a unit quasi-quaternion. Then for every integer* n*;*

$$q^n = \cos n\varphi + \overrightarrow{w}\sin n\varphi.$$

Proof. The proof follows immediately from the induction (see [4]). □

The formula holds for all integer n since;

$$\begin{aligned} q^{-1} &= \cos\varphi - \overrightarrow{w}\sin\varphi, \\ q^{-n} &= \cos(-n\varphi) + \overrightarrow{w}\sin(-n\varphi) \\ &= \cos n\varphi - \overrightarrow{w}\sin n\varphi. \end{aligned}$$

Example 5.2. Let $q = 3 + 2i - 2j + k = 3(\cos\varphi + \overrightarrow{w}\sin\varphi)$ be a quasi-quaternion. Every powers of this quaternion are found to be with the aid of theorem 5.1, for example, 9-th power is

$$\begin{aligned} q^9 &= 3^9(\cos 9\varphi + \overrightarrow{w}\sin 9\varphi) \\ &= 3^9(1 + 9\overrightarrow{w}). \\ &= 3^9(1 + 6i - 6j + k). \end{aligned}$$

Corollary 2. *The equation* $q^n = 1$ *does not have solution for a general unit quasi-quaternion.*

Example 5.3. Let $q = 1 + (1, 1, 1)$ be a unit quasi-quaternion. There is no n $(n > 0)$ such that $q^n = 1$.

Theorem 5.4. *Let* $q = r(\cos\varphi + \overrightarrow{w}\sin\varphi)$ *be a quasi-quaternion. The equation* $x^n = q$ *has one root and this is*

$$x = \sqrt[n]{r}(\cos\frac{\varphi}{n} + \overrightarrow{w}\sin\frac{\varphi}{n}).$$

Proof. If $x^n = q$, q will have the same unit vector as x. So, assume that $x = N(\cos\varkappa + \overrightarrow{w}\sin\varkappa)$ is a root of the equation $x^n = q$. From theorem 5.1, we have

$$x^n = N^n(\cos n\varkappa + \overrightarrow{w}\sin n\varkappa).$$

Thus, $N^n = r$ and $\varkappa = \frac{\varphi}{n}$. Therefore, $x = \sqrt[n]{r}(\cos\frac{\varphi}{n} + \overrightarrow{w}\sin\frac{\varphi}{n})$ is a root of the equation $x^n = q$. □

Example 5.5. Let $q = 8 + i - 2j + 2k = 8(\cos\varphi + \overrightarrow{w}\sin\varphi)$ be a quasi-quaternion. The equation $x^3 = q$ has a root and this is

$$x = 2(1 + \frac{1}{8}\overrightarrow{w}).$$

6. De Moivre's Formula for Matrices of quasi-Qauternions

In this section, we introduce the $\mathbb{R}$-linear transformations representing left multiplication in H° and look for also the De-Moiver's formula for corresponding matrix representation. Let q be a quasi-quaternion, then $\varphi_l : H^\circ \to H^\circ$ defined as follows:

$$\varphi_l(x) = qx, \quad x \in H^\circ.$$

The Hamilton's operator φ_l, could be represented as the matrices;

$$A_{\varphi_l} = \begin{bmatrix} a_\circ & 0 & 0 & 0 \\ a_1 & a_\circ & 0 & 0 \\ a_2 & 0 & a_\circ & 0 \\ a_3 & 0 & 0 & a_\circ \end{bmatrix}.$$

We can express the matrix A_{φ_l} in polar form. Let q be a unit quasi-quaternion. Since

$$\begin{aligned} q &= a_\circ + a_1 i + a_2 j + a_3 k \\ &= \cos\varphi + \overrightarrow{w}\sin\varphi \\ &= \cos\varphi + (w_1, w_2, w_3)\sin\varphi \\ &= \cos\varphi + (w_1\sin\varphi, w_2\sin\varphi, w_3\sin\varphi) \end{aligned}$$

we have

$$\begin{bmatrix} a_\circ & 0 & 0 & 0 \\ a_1 & a_\circ & 0 & 0 \\ a_2 & 0 & a_\circ & 0 \\ a_3 & 0 & 0 & a_\circ \end{bmatrix} = \begin{bmatrix} \cos\varphi & 0 & 0 & 0 \\ w_1\sin\varphi & \cos\varphi & 0 & 0 \\ w_2\sin\varphi & 0 & \cos\varphi & 0 \\ w_3\sin\varphi & 0 & 0 & \cos\varphi \end{bmatrix}.$$

Theorem 6.1. *(De-Moivre's formula) Let $q = e^{\overrightarrow{w}\varphi} = \cos\varphi + \overrightarrow{w}\sin\varphi$ be a unit quasi-quaternion. For an integer n*

$$A = \begin{bmatrix} \cos\varphi & 0 & 0 & 0 \\ w_1\sin\varphi & \cos\varphi & 0 & 0 \\ w_2\sin\varphi & 0 & \cos\varphi & 0 \\ w_3\sin\varphi & 0 & 0 & \cos\varphi \end{bmatrix} \qquad (1.1)$$

the n-th power of the matrix A reads

$$A^n = \begin{bmatrix} \cos n\varphi & 0 & 0 & 0 \\ w_1\sin n\varphi & \cos n\varphi & 0 & 0 \\ w_2\sin n\varphi & 0 & \cos n\varphi & 0 \\ w_3\sin n\varphi & 0 & 0 & \cos n\varphi \end{bmatrix}.$$

Proof. The proof follows immediately from the induction. □

Example 6.2. Let $q = 3+2i-2j+k = 3(\cos\varphi + \overrightarrow{w}\sin\varphi)$ be a quasi-quaternion. The matrix corresponding to this quaternion is

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ -2 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix} = 3\begin{bmatrix} \cos\varphi & 0 & 0 & 0 \\ w_1\sin\varphi & \cos\varphi & 0 & 0 \\ w_2\sin\varphi & 0 & \cos\varphi & 0 \\ w_3\sin\varphi & 0 & 0 & \cos\varphi \end{bmatrix}$$

every powers of this matix are found to be with the aid of theorem 6.1, for example, 15-th power is

$$A^{15} = 3^{15}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 10 & 1 & 0 & 0 \\ -10 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}.$$

7. Euler's Formula for matrices accosiated quasi-quaternions

Let A be a matrix. We choose

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 \\ u_2 & 0 & 0 & 0 \\ u_3 & 0 & 0 & 0 \end{bmatrix}$$

then one immediately finds $A^2 = 0$. We have a netural generalization of Euler's formula for matrix A;

$$\begin{aligned} e^{A\theta} &= I_4 + A\varphi + \frac{(A\varphi)^2}{2!} + \frac{(A\varphi)^3}{3!} + \frac{(A\varphi)^4}{4!} + ... \\ &= I_4 + A\varphi \\ &= \cos\varphi + A\sin\varphi, \\ &= \begin{bmatrix} \cos\varphi & 0 & 0 & 0 \\ w_1\sin\varphi & \cos\varphi & 0 & 0 \\ w_2\sin\varphi & 0 & \cos\varphi & 0 \\ w_3\sin\varphi & 0 & 0 & \cos\varphi \end{bmatrix}. \end{aligned}$$

8. $n-th$ Root of Matrices of quasi-quaternions

The matrix accossiated with the quasi-quaternion q is of the form (1.1). The equation $x^n = A$ has one root. Thus

$$A^{\frac{1}{n}} = \begin{bmatrix} \cos\frac{\varphi}{n} & 0 & 0 & 0 \\ w_1\sin\frac{\varphi}{n} & \cos\frac{\varphi}{n} & 0 & 0 \\ w_2\sin\frac{\varphi}{n} & 0 & \cos\frac{\varphi}{n} & 0 \\ w_3\sin\frac{\varphi}{n} & 0 & 0 & \cos\frac{\varphi}{n} \end{bmatrix}.$$

Example 8.1. Let $q = 1 - i + 2j + 2k$ be a unit quasi-quaternion. The matrix corresponding to this quaternion is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

The cube roots of the matrix A can be achieved

$$A^{\frac{1}{3}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 & 0 \\ \frac{2}{3} & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & 1 \end{bmatrix}.$$

9. CONCLUSION

In this paper, we gave some of algebraic properties of the quasi-quaternions and investigated the Euler's and De Moivre's formulae for these quaternions and also for the matrices associated with quasi-quaternions. The $n-$th root of these matrices are obtained.

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