Commun.Fac.Sci.Univ.Ank.Series A1
Volume 63, Number 2, Pages 1–11 (2014)
ISSN 1303–5991

ON THE BOUNDEDNESS OF THE MAXIMAL OPERATOR AND RIESZ POTENTIAL IN THE MODIFIED MORREY SPACES

CANAY AYKOL AND M. ESRA YILDIRIM

Abstract. In this paper we prove the boundedness of the maximal operator M and give the necessary and sufficient conditions for the boundedness of Riesz potential operator I_α in the modified Morrey spaces by using Guliyev method

1. Introduction

Morrey spaces $L_{p,\lambda}$ were introduced by Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations ([19]). Later, Morrey spaces found important applications to Navier Stokes ([18], [25]) and Schrödinger ([21, 22]) equations, elliptic problems with discontinuous coefficients ([5, 8]) and potential theory ([1, 2]). An exposition of the Morrey spaces can be found in the book [17]. Morrey spaces were widely studied during last decades, including the study of classical operators of harmonic analysis such as maximal, singular and potential operators ([1, 3, 4, 6, 7]). Modified Morrey spaces and the boundedness conditions of maximal operators and Riesz potential studied by some authors (see, for example [12, 13, 14, 15, 16]).

In [9] Guliyev considered the generalized Morrey spaces $\mathcal{M}_{p,\varphi}$ with a general function $\varphi(x,r)$ defining the Morrey-type norm. He found the conditions on the pair (φ_1, φ_2) without any assumption on monotonicity of φ_1, φ_2 which ensures the boundedness of the maximal operator in generalized Morrey spaces. He also proved the Spanne and Sobolev-Adams type theorems for the Riesz potential operator I_α.

In the present work, we prove the boundedness of the maximal operator M and Riesz potential operator I_α in modified Morrey spaces by using Guliyev methods given in [9].

Received by the editors July 09, 2015, Accepted: Aug. 04, 2014.
2000 *Mathematics Subject Classification.* 46E30, 42B20, 42B25.
Key words and phrases. Morrey space, modified Morrey space, maximal operator, fractional maximal operator, Riesz potential

M.E. Yildirim was partially supported by the Scientific and Technological Research Council of Turkey (TUBITAK Programme 2228-B).

2. Definitions and Preliminary Tools

Let $f \in L_{loc}^1(\mathbb{R}^n)$. As usual we define the Hardy-Littlewood maximal function of f, Mf, setting

$$Mf(x) := \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)| dy,$$

where $B(x,t)$ denotes the open ball centered at x of radius t for $x \in \mathbb{R}^n$ and $t > 0$. $|B(x,t)| = \omega_n t^n$ and ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

For $0 \le \alpha < n$, we define the fractional maximal function

$$M_\alpha f(x) := \sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}-1} \int_{B(x,t)} |f(y)| dy.$$

In the case $\alpha = 0$, we get $M_0 f = Mf$. The fractional maximal function $M_\alpha f$ is closely related to the Riesz potential operator

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y) dy}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n,$$

such that

$$M_\alpha f(x) \le \omega_n^{\frac{\alpha}{n}-1} (I_\alpha |f|)(x). \tag{2.1}$$

The operators M_α and I_α play important role in real and harmonic analysis (see, for example [1, 20, 23, 24]).

Definition 2.1. Let $1 \le p < \infty$, $0 \le \lambda \le n$, $[t]_1 = \min\{1, t\}$. We define the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$, and the modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ as the set of locally integrable functions f with the finite norms

$$\|f\|_{L_{p,\lambda}} := \sup_{x \in \mathbb{R}^n, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))}, \tag{2.2}$$

$$\|f\|_{\widetilde{L}_{p,\lambda}} := \sup_{x \in \mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,t))}, \tag{2.3}$$

respectively.

Note that

$$\widetilde{L}_{p,0}(\mathbb{R}^n) = L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$
$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \hookrightarrow L_{p,\lambda}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$$

and

$$\max\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}\} \le \|f\|_{\widetilde{L}_{p,\lambda}}$$

and if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Definition 2.2. [3, 4, 10, 11] Let $1 \le p < \infty$, $0 \le \lambda \le n$. We define the weak Morrey space $WL_{p,\lambda}(\mathbb{R}^n)$, and the modified weak Morrey space $W\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite norms

$$\|f\|_{WL_{p,\lambda}} := \sup_{x\in\mathbb{R}^n, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t))},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}} := \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,t))},$$

respectively.

Note that

$$WL_p(\mathbb{R}^n) = WL_{p,0}(\mathbb{R}^n) = W\widetilde{L}_{p,0}(\mathbb{R}^n),$$
$$L_{p,\lambda}(\mathbb{R}^n) \subset WL_{p,\lambda}(\mathbb{R}^n) \ and \ \|f\|_{WL_{p,\lambda}} \le \|f\|_{L_{p,\lambda}}$$

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \subset W\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \ and \ \|f\|_{W\widetilde{L}_{p,\lambda}} \le \|f\|_{\widetilde{L}_{p,\lambda}}.$$

The following lemmas give some embeddings between Morrey spaces which were proved in [14, 15].

Lemma 2.3. *Let* $0 \le \lambda < n$ *and* $0 \le \alpha < n-\lambda$. *Then for* $p = \frac{n-\lambda}{\alpha}$,

$$L_{p,\lambda}(\mathbb{R}^n) \hookrightarrow L_{1,n-\alpha}(\mathbb{R}^n), \ \ \|f\|_{L_{1,n-\alpha}} \le \omega_n^{1/p'} \|f\|_{L_{p,\lambda}}.$$

Lemma 2.4. *Let* $0 \le \lambda < n$ *and* $0 \le \alpha < n-\lambda$. *Then for* $\frac{n-\lambda}{\alpha} \le p < \frac{n}{\alpha}$

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \hookrightarrow L_{1,n-\alpha}(\mathbb{R}^n), \ \ \|f\|_{L_{1,n-\alpha}} \le \omega_n^{1/p'} \|f\|_{\widetilde{L}_{p,\lambda}}.$$

The following theorems proved by Guliyev in [9] will be our main tools to obtain the boundedness of maximal operator M and Riesz potential I_α in modified Morrey spaces, respectively.

Theorem A. Let $1 \le p < \infty$ and $f \in L_p^{loc}(\mathbb{R}^n)$. Then for $p > 1$

$$\|Mf\|_{L_{p(B(x,t))}} \le Ct^{\frac{n}{p}} \int_t^\infty r^{-\frac{n}{p}-1} \|f\|_{L_p(B(x,r))} dr, \tag{2.4}$$

and for $p = 1$

$$\|Mf\|_{WL_{1(B(x,t))}} \le Ct^{\frac{n}{p}} \int_t^\infty r^{-\frac{n}{p}-1} \|f\|_{L_1(B(x,r))} dr, \tag{2.5}$$

where C is a constant independent of f, $x \in \mathbb{R}^n$ and $t > 0$.

Theorem B. Let $1 \le p < \infty$, $0 < \alpha < \frac{n}{p}$ and $f \in L_p^{loc}(\mathbb{R}^n)$. Then

$$|I_\alpha f(x)| \le Ct^\alpha Mf(x) + C \int_t^\infty r^{\alpha-\frac{n}{p}-1} \|f\|_{L_p(B(x,r))} dr, \tag{2.6}$$

where C is a constant independent of f, x and t.

3. Boundedness of the maximal operator and the Riesz potential in the modified Morrey spaces

In this section we prove the boundedness of the maximal operator and the Riesz potential in modified Morrey spaces $\widetilde{L}_{p,\lambda}$. We prove Theorems 3.1 and 2 with the help of Theorems A and B, respectively.

Theorem 3.1. *Let $1 \le p < \infty$, $0 \le \lambda < n$ and $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$.*
(i) If $p > 1$, then the maximal operator M is bounded in $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$.

(ii)If $p = 1$, then M is bounded from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$.

Proof. (i) Let $1 < p < \infty$. From the inequality (2.4) we get

$$\begin{aligned}
\|Mf\|_{\widetilde{L}_{p,\lambda}} &= \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} \|Mf\|_{L_p(B(x,t))} \\
&\le C \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} t^{\frac{n}{p}} \int_t^\infty r^{-\frac{n}{p}-1} \|f\|_{L_p(B(x,r))} dr \\
&\le C \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} t^{\frac{n}{p}} \|f\|_{\widetilde{L}_{p,\lambda}} \min\{\int_t^\infty r^{-\frac{n}{p}-1} dr, \int_t^\infty r^{\frac{\lambda-n}{p}-1} dr\} \\
&= C \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} t^{\frac{n}{p}} \|f\|_{\widetilde{L}_{p,\lambda}} \min\{t^{-\frac{n}{p}}, t^{\frac{\lambda-n}{p}}\} \\
&= C \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} t^{\frac{n}{p}} \|f\|_{\widetilde{L}_{p,\lambda}} [t]_1^{\frac{\lambda}{p}} t^{-\frac{n}{p}} \\
&= C\|f\|_{\widetilde{L}_{p,\lambda}},
\end{aligned}$$

which implies that M is bounded in $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$.

(ii) Let $p=1$. From the inequality (2.5) we get

$$
\begin{aligned}
\|Mf\|_{W\widetilde{L}_{1,\lambda}} &= \sup_{x\in\mathbb{R}^n,t>0}[t]_1^{-\lambda}\|Mf\|_{WL_1(B(x,t))} \\
&\le C\sup_{x\in\mathbb{R}^n,t>0}[t]_1^{-\lambda}t^n\int_t^\infty r^{-n-1}\|f\|_{L_1(B(x,r))}dr \\
&\le C\sup_{x\in\mathbb{R}^n,t>0}[t]_1^{-\lambda}t^n\|f\|_{\widetilde{L}_{1,\lambda}}\min\{\int_t^\infty r^{-n-1}dr,\int_t^\infty r^{\lambda-n-1}dr\} \\
&= C\sup_{x\in\mathbb{R}^n,t>0}[t]_1^{-\lambda}t^n\|f\|_{\widetilde{L}_{1,\lambda}}\min\{t^{-n},t^{\lambda-n}\} \\
&= C\sup_{x\in\mathbb{R}^n,t>0}[t]_1^{-\lambda}t^n\|f\|_{\widetilde{L}_{1,\lambda}}[t]_1^{-\lambda}t^n \\
&= C\|f\|_{\widetilde{L}_{1,\lambda}},
\end{aligned}
$$

which implies that M is bounded from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$. □

In the following we give the necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces.

Theorem 3.2. *Let $0<\alpha<n$, $0\le\lambda<n-\alpha$ and $1\le p<\frac{n-\lambda}{\alpha}$.*

(i) If $1<p<\frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n}\le\frac{1}{p}-\frac{1}{q}\le\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

(ii) If $p=1<\frac{n-\lambda}{\alpha}$, then condition $\frac{\alpha}{n}\le 1-\frac{1}{q}\le\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator I_α from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

Proof. (i) *Sufficiency.* Let $1<p<\frac{n-\lambda}{\alpha}$, $\frac{\alpha}{n}\le\frac{1}{p}-\frac{1}{q}\le\frac{\alpha}{n-\lambda}$ and $f\in\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$. From the inequality (2.6) we get

$$
\begin{aligned}
&\|I_\alpha f\|_{\widetilde{L}_{q,\lambda}} \\
&= \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{q}} \|I_\alpha f\|_{L_q(B(x,t))} \\
&= \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda} \int_{B(x,t)} |I_\alpha f(y)|^q dy \right)^{\frac{1}{q}} \\
&\le C \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{q}} \left(\int_{B(x,t)} \left(r^\alpha Mf(y) + \int_r^\infty \tau^{\alpha-\frac{n}{p}-1} \|f\|_{L_p(B(x,\tau))} d\tau \right)^q dy \right)^{1/q} \\
&\le C \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{q}} \left(\int_{B(x,t)} \left(r^\alpha Mf(y) + \|f\|_{\widetilde{L}_{p,\lambda}} \min\{\int_r^\infty \tau^{\alpha-\frac{n}{p}-1} d\tau, \int_r^\infty \tau^{\alpha+\frac{\lambda-n}{p}-1} d\tau\} \right)^q dy \right)^{1/q} \\
&= C \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{q}} \left(\int_{B(x,t)} \left(r^\alpha Mf(y) + \|f\|_{\widetilde{L}_{p,\lambda}} \min\{r^{\alpha-\frac{n}{p}}, r^{\alpha+\frac{\lambda-n}{p}}\} \right)^q dy \right)^{1/q}.
\end{aligned}
$$

Minimizing with respect to r, with

$$
r = \left[\frac{\|f\|_{\widetilde{L}_{p,\lambda}}}{Mf(y)} \right]^{\frac{p}{n-\lambda}} \quad and \quad r = \left[\frac{\|f\|_{\widetilde{L}_{p,\lambda}}}{Mf(y)} \right]^{\frac{p}{n}}
$$

we have

$$
\begin{aligned}
&\|I_\alpha f\|_{\widetilde{L}_{q,\lambda}} \\
&\le C \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{q}} \left(\int_{B(x,t)} \left(\min\{ \left(\frac{Mf(y)}{\|f\|_{\widetilde{L}_{p,\lambda}}} \right)^{1-\frac{p\alpha}{n-\lambda}}, \left(\frac{Mf(y)}{\|f\|_{\widetilde{L}_{p,\lambda}}} \right)^{1-\frac{p\alpha}{n}} \} \|f\|_{\widetilde{L}_{p,\lambda}} \right)^q dy \right)^{1/q} \\
&\le C \|f\|_{\widetilde{L}_{p,\lambda}}^{1-\frac{p}{q}} \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{q}} \|Mf\|_{L_p(B(x,t))}^{\frac{p}{q}}.
\end{aligned}
$$

Hence by Theorem 3.1(i) we have

$$
\begin{aligned}
\|I_\alpha f\|_{\widetilde{L}_{q,\lambda}} &\le C \|f\|_{\widetilde{L}_{p,\lambda}}^{1-\frac{p}{q}} \|f\|_{\widetilde{L}_{p,\lambda}}^{\frac{p}{q}} \\
&= C \|f\|_{\widetilde{L}_{p,\lambda}},
\end{aligned}
$$

which implies that I_α is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

Necessity. Let $1 < p < \frac{n-\lambda}{\alpha}$, $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$. Suppose that I_α is bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$. Let us define $f_s(x) := f(sx)$, $[s]_{1,+} = \max\{1, s\}$. Then

$$\begin{aligned}
\|f_s\|_{\widetilde{L}_{p,\lambda}} &= \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{p}} \|f_s\|_{L_p(B(x,t))} \\
&= \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda} \int_{B(x,t)} |f_s(y)|^p dy \right)^{1/p} \\
&= s^{-\frac{n}{p}} \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda} \int_{B(x,st)} |f(y)|^p dy \right)^{1/p} \\
&= s^{-\frac{n}{p}} \sup_{t>0} \left(\frac{[st]_1}{[t]_1} \right)^{\frac{\lambda}{p}} \sup_{x\in\mathbb{R}^n, t>0} \left([st]_1^{-\lambda} \int_{B(x,st)} |f(y)|^p dy \right)^{1/p} \\
&= s^{-\frac{n}{p}} [s]_{1,+}^{\frac{\lambda}{p}} \|f\|_{\widetilde{L}_{p,\lambda}}, \qquad (3.1)
\end{aligned}$$

and

$$I_\alpha f_s(x) = s^{-\alpha} I_\alpha f(sx),$$

$$\begin{aligned}
\|I_\alpha f_s\|_{\widetilde{L}_{q,\lambda}} &= s^{-\alpha} \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda} \int_{B(x,t)} |I_\alpha f(sy)|^q dy \right)^{1/q} \\
&= s^{-\alpha-\frac{n}{q}} \sup_{t>0} \left(\frac{[st]_1}{[t]_1} \right)^{\frac{\lambda}{q}} \sup_{x\in\mathbb{R}^n, t>0} \left([st]_1^{-\lambda} \int_{B(sx,st)} |I_\alpha f(y)|^q dy \right)^{1/q} \\
&= s^{-\alpha-\frac{n}{q}} [s]_{1,+}^{\frac{\lambda}{q}} \|I_\alpha f\|_{\widetilde{L}_{q,\lambda}}.
\end{aligned}$$

By the boundedness of I_α from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ we get

$$\begin{aligned}
\|I_\alpha f\|_{\widetilde{L}_{q,\lambda}} &= s^{\alpha+\frac{n}{q}} [s]_{1,+}^{-\frac{\lambda}{q}} \|I_\alpha f_s\|_{\widetilde{L}_{q,\lambda}} \\
&\le s^{\alpha+\frac{n}{q}} [s]_{1,+}^{-\frac{\lambda}{q}} \|f_s\|_{\widetilde{L}_{p,\lambda}} \\
&\le C s^{\alpha+\frac{n}{q}-\frac{n}{p}} [s]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}} \|f\|_{\widetilde{L}_{p,\lambda}}.
\end{aligned}$$

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{n}$, then in the case $t \to 0$ we have $\|I_\alpha f\|_{\widetilde{L}_{q,\lambda}} = 0$ for all $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$.

If $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{n-\lambda}$, then in the case $t \to \infty$ we have $\|I_\alpha f\|_{\widetilde{L}_{q,\lambda}} = 0$ for all $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$.

Therefore we obtain $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$.

(*ii*) *Sufficiency.* Let $p = 1$ and $\frac{\alpha}{n} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. From the inequality (2.6) we have

$$\begin{aligned} |I_\alpha f(x)| &\le Ct^\alpha Mf(x) + C\int_t^\infty r^{\alpha-n-1}\|f\|_{L_1(B(x,r))}dr \\ &\le Ct^\alpha Mf(x) + C\|f\|_{\widetilde{L}_{1,\lambda}} \min\{t^{\alpha-n}, t^{\lambda+\alpha-n}\}. \end{aligned}$$

Minimizing with respect to t, with

$$t = \left[\frac{\|f\|_{\widetilde{L}_{1,\lambda}}}{Mf(x)}\right]^{\frac{1}{n-\lambda}} \quad \textit{and} \ \ t = \left[\frac{\|f\|_{\widetilde{L}_{1,\lambda}}}{Mf(x)}\right]^{\frac{1}{n}}$$

we have

$$|I_\alpha f(x)| \le C\min\left\{\left(\frac{Mf(x)}{\|f\|_{\widetilde{L}_{1,\lambda}}}\right)^{1-\frac{\alpha}{n-\lambda}}, \left(\frac{Mf(x)}{\|f\|_{\widetilde{L}_{1,\lambda}}}\right)^{1-\frac{\alpha}{n}}\right\}\|f\|_{\widetilde{L}_{1,\lambda}}.$$

Therefore we get

$$|I_\alpha f(x)| \le C(Mf(x))^{1/q}\|f\|_{\widetilde{L}_{1,\lambda}}^{1-1/q}. \tag{3.2}$$

Using the inequality (3.2) and from Theorem 3.1(*ii*) we get

$$\begin{aligned} \|I_\alpha f\|^q_{W\widetilde{L}_{q,\lambda}} &= \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\lambda}\|I_\alpha f\|^q_{WL_q(B(x,t))} \\ &= \sup_{r>0} r^q \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\lambda}|\{y\in B(x,t) : |I_\alpha f(y)| > r\}| \\ &\le \sup_{r>0} r^q \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\lambda}|\{y\in B(x,t) : C(Mf(y))^{1/q}\|f\|^{1-1/q}_{\widetilde{L}_{1,\lambda}} > r\}| \\ &= \sup_{r>0} r^q \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\lambda}\left|\left\{y\in B(x,t) : Mf(y) > \left(\frac{r}{C\|f\|^{1-1/q}_{\widetilde{L}_{1,\lambda}}}\right)^q\right\}\right| \\ &\le C\sup_{r>0} r^q\left(\frac{\|f\|^{1-\frac{1}{q}}_{\widetilde{L}_{1,\lambda}}}{r}\right)^q \|f\|_{\widetilde{L}_{1,\lambda}} \\ &= C\|f\|^q_{\widetilde{L}_{1,\lambda}}, \end{aligned}$$

which implies that I_α is bounded from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

Necessity. Let I_α is bounded from $\widetilde{L}_{1,\lambda}$ to $W\widetilde{L}_q(\mathbb{R}^n)$. We have

$$\begin{aligned}
\|I_\alpha f_s\|_{W\widetilde{L}_{q,\lambda}} &= \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\frac{\lambda}{q}} \|I_\alpha f_s\|_{W\widetilde{L}_{q,\lambda}(B(x,t))} \\
&= \sup_{r>0} r \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda} \int_{\{y\in B(x,t): |I_\alpha f_s(y)|>r\}} dy \right)^{1/q} \\
&= \sup_{r>0} r \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda} \int_{\{y\in B(xs,t): |I_\alpha f(sy)|>rs^\alpha\}} dy \right)^{1/q} \\
&= s^{-\alpha-\frac{n}{q}} \sup_{t>0} \left(\frac{[ts]_1}{[t]_1} \right)^{\frac{\lambda}{q}} \sup_{r>0} rs^\alpha \sup_{x\in\mathbb{R}^n, t>0} \left([ts]_1^{-\lambda} \int_{\{y\in B(x,ts): |I_\alpha f(y)|>rs^\alpha\}} \right)^{1/q} \\
&= s^{-\alpha-\frac{n}{q}} [s]_{1,+}^{\frac{\lambda}{q}} \|I_\alpha f\|_{W\widetilde{L}_{q,\lambda}}.
\end{aligned}$$

By using the boundedness of I_α from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$ we get

$$\begin{aligned}
\|I_\alpha f\|_{W\widetilde{L}_{q,\lambda}} &= s^{\alpha+\frac{n}{q}} [s]_{1,+}^{-\frac{\lambda}{q}} \|I_\alpha f_s\|_{W\widetilde{L}_{q,\lambda}} \\
&\le C s^{\alpha+\frac{n}{q}} [s]_{1,+}^{-\frac{\lambda}{q}} \|f_s\|_{\widetilde{L}_{1,\lambda}} \\
&= C s^{\alpha+\frac{n}{q}-n} [s]_{1,+}^{\lambda-\frac{\lambda}{q}} \|f\|_{\widetilde{L}_{1,\lambda}}.
\end{aligned}$$

If $1 < \frac{1}{q} + \frac{\alpha}{n}$, then in the case $t \to 0$ we have $\|I_\alpha f\|_{W\widetilde{L}_{q,\lambda}} = 0$ for all $f \in \widetilde{L}_{1,\lambda}(\mathbb{R}^n)$.

If $1 > \frac{1}{q} + \frac{\alpha}{n-\lambda}$, then in the case $t \to \infty$ we have $\|I_\alpha f\|_{W\widetilde{L}_{q,\lambda}} = 0$ for all $f \in \widetilde{L}_{1,\lambda}(\mathbb{R}^n)$.

Therefore $\frac{\alpha}{n} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. □

Corollary 1. *Let* $0 < \alpha < n$, $0 \le \lambda < n - \alpha$ *and* $1 \le p < \frac{n-\lambda}{\alpha}$.

*(*i*) If* $1 < p < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

*(*ii*) If* $p = 1 < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ *to* $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

Proof. Sufficiency of Corollary 1 is obtained from Theorem 3.2 and inequality (2.1).

Necessity. For the fractional maximal operator M_α the following equality

$$M_\alpha f_s(x) = s^{-\alpha} M_\alpha f(sx)$$

holds.

(i) Let $1 < p < \frac{n-\lambda}{\alpha}$ and M_α be bounded from $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$. Then we have

$$\|M_\alpha f_s\|_{\widetilde{L}_{q,\lambda}} = s^{-\alpha-\frac{n}{q}}[s]_{1,+}^{\frac{\lambda}{q}}\|M_\alpha f\|_{\widetilde{L}_{q,\lambda}}.$$

By similar methods in Theorem 3.2 we obtain $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$.

(i) Let M_α be bounded from $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

Then we have

$$\|M_\alpha f_s\|_{W\widetilde{L}_{q,\lambda}} = s^{-\alpha-\frac{n}{q}}[s]_{1,+}^{\frac{\lambda}{q}}\|M_\alpha f\|_{W\widetilde{L}_{q,\lambda}}.$$

Therefore we get $\frac{\alpha}{n} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$. □

References

[1] D.R. Adams, *A note on Riesz potentials*, Duke Math. **42** (1975), 765-778.

[2] D.R. Adams, *Choquet integrals in potential theory*, Publ. Mat. **42** (1998), 3-66.

[3] V.I. Burenkov and H.V. Guliyev, *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces*, Studia Mathematica **163**, 2 (2004), 157-176.

[4] V.I. Burenkov and V.S. Guliyev, *Necessary and sufficient conditions for the boundedness of the Riesz operator in local Morrey-type spaces*, Potential Analysis **30**, 3 (2009), 211-249.

[5] L. Caffarelli, *Elliptic second order equations*, Rend. Sem. Mat. Fis. Milano **58** (1990), 253-284.

[6] F. Chiarenza and M. Frasca *Morrey spaces and Hardy- Littlewood maximal function*, Rend. Math. **7** (1987), 273-279.

[7] G. Di Fazio and M. A. Ragusa *Commutators and Morrey spaces*, Bollettino U.M.I., **7** (5-A)(1991), 323-332.

[8] G. Di Fazio, D.K. Palagachev and M.A. Ragusa, *Glocal Morrey regularity of strong solutions to the Dirichlet problemfor elliptic equations with discontinuous coefficients*, J. Funct. Anal. **166** (1999), 179-196.

[9] V.S. Guliyev, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. 2009, Art ID 503948, 20 pp.

[10] V.S. Guliyev, *On maximal function and fractional integral, associated with the Bessl differentai operator*, Mathematical Inequalities and Applications **6**, 2(2003), 317-330.

[11] V.S. Guliyev and J. Hasanov, *Necessary and sufficient conditions for the boundedness of B-Riesz potential in the B-Morrey spaces*, Journal of Mathematical Analysis and Applications **347**, 1 (2008), 113-122.

[12] V.S. Guliyev, J. Hasanov and Y. Zeren, *Necessary and sufficient conditions for the boundedness of B-Riesz potential in modified Morrey spaces*, Journal of Mathematical Inequalities **5**, 4 (2011), 491-506.

[13] V.S. Guliyev and Y.Y. Mammadov, *Riesz potential on the Heisenberg group and modified Morrey spaces*, An. St. Univ. Ovidius Constanta **20**(1) (2012), 189-212.

[14] V.S. Guliyev and K. Rahimova, *Parabolic fractional maximal operator and modified parabolic Morrey spaces*, Journal of Function Spaces and Applications, 2012, Article ID 543475, 20 pages, 2012.

[15] V.S. Guliyev and K. Rahimova *Parabolic fractional integral operator in modified parabolic Morrey spaces*, Proc. Razmadze Mathematical Institute, 163 (2013), 85-106.

[16] V. Kokilashvili, A. Meskhi and H. Rafeiro, *Riesz type potential operators in generalized grand Morrey spaces*, Georgian Math. J. **20** (2013), no. 1, 43-64.

[17] A. Kufner, O. John and S. Fucik, *Function spaces*, Noordhoff, Leyden and Academia, Prague, 1977.

[18] A.L. Mazzucato, *Besov-Morrey spaces: function theory and applications to non-linear PDE.* Trans. Amer. Math. Soc. **355** (2003), 1297-1364.
[19] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations,* Trans. Amer. Math. Soc. **43** (1938), 126-166.
[20] B. Muckenhoupt and R. Whedeen, *Weighted norm inequalities for fractional integrals*, Trans. Amer.Math. Soc. **192** (1974), 261-274.
[21] A. Ruiz and L. Vega, *Unique continuation for Schrödinger operators with potential in Morrey spaces*, Publ. Mat. **35** (1991), 291-298.
[22] A. Ruiz and L. Vega, *On local regularity of Schrödinger equations*, Int. Math. Res. Notices **1993**, 1 (1993), 13-27.
[23] E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton Univ. Press (1971).
[24] E.M. Stein, *Singular integrals and differentiability properties of functions.* Princeton Math. Ser. 30. Princeton University Press, Princeton(1971).
[25] M.E. Taylor, *Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations*, Comm. Partial Differential Equations **17**(1992), 1407-1456.

Current address: Ankara University, Faculty of Sciences, Dept. of Mathematics, Ankara, TURKEY

E-mail address: Canay.Aykol@science.ankara.edu.tr, meyildirim@ankara.edu.tr

URL: http://communications.science.ankara.edu.tr/index.php?series=A1