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AN ANTI-PERIODIC CAPUTO $q-$FRACTIONAL BOUNDARY VALUE PROBLEM WITH A $p-$LAPLACIAN OPERATOR

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ABSTRACT. This paper studies the existence of solutions for an anti-periodic boundary value problem for the q-fractional p-Laplacian equation. The existence result depends on Leray-Schaefer's Fixed Point Theorem.

1. INTRODUCTION

To study the Turbulent flow in a porous medium, which is a fundamental mechanics problem, Leibenson introduced the $p-$Laplacian equation as follows:

$$(\phi_p(x'(t)))' = f(t, x(t), x'(t)), \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s, : p > 1$. Obviously, ϕ_p is invertible and its inverse operator is $\phi_r, : r > 1, : \frac{1}{p} + \frac{1}{r} = 1$. Then, many certain boundary value conditions have been associated to equation (1.1) during the past few decades. Chen and Liu in [1] considered the following BVP.

$${}^C D^{\beta}_{0^+} \phi_p({}^C D^{\alpha}_{0^+} x(t)) = f(t, x(t)), : t \in [0, 1], \tag{1.2}$$

with boundary conditions

$$x(0) = -x(1), \quad {}^C D^{\alpha}_{0^+} x(0) = -{}^C D^{\alpha}_{0^+} x(1),$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$ and $f : [0, 1] \times R \to R$ is continuous. When $p = 2$ we get the composition linear operator ${}^C D^{\beta}_{0^+} {}^C D^{\alpha}_{0^+}$. Depending on Schaefer's fixed point, the authors proved under certain nonlinear growth conditions of the nonlinearity, the following existence theorem:

Theorem 1.1. [1] *Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that there exist nonnegative functions $a, b \in C[0, 1]$ such that*

$$|f(t, u)| \leq a(t) + b(t)|u|^{p-1}, \ \forall t \in [0, 1], \ u \in \mathbb{R}. \tag{1.3}$$

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Then the Anti BVP (1.2) has at least one solution, provided that

$$\frac{3^r \|b\|_\infty^{r-1}}{2^r \Gamma(\alpha+1)\Gamma(\beta+1)^{r-1}} < 1. \tag{1.4}$$

The q-fractional calculus basics were initiated about the middle of the previous century [24, 14, 16, 19, 17, 18, 15, 4, 6, 7, 2, 3, 12]. In the last two decades this calculus started to attract many authors, when fractional calculus [20, 21, 22, 23, 37] applications started to appear in different branches of science and engineering and the discrete fractional calculus into nabla and delta started to be developed extensively [5, 8, 9, 10, 11, 26, 27, 28, 29, 30, 31, 31, 32, 33, 34, 35, 36, 13]. In this article we investigate the result stated in Theorem 1.1 in the sense of q-fractional calculus.

For $0 < q < 1$, let T_q be the time scale

$$T_q = \{q^n : n = 0, 1, ...\} \cup \{0\}$$

More generally, if α is a nonnegative real number then we define the time scale

$$T_q^\alpha = \{q^{n+\alpha} : n = 0, 1, ...\} \cup \{0\}$$

We write $T_q^0 = T_q$.
For a function $f : T_q \to R$, the nabla $q-$derivative of f is given by

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t},\ t \in T_q - \{0\} \tag{1.5}$$

The nabla $q-$integral of f is given by

$$\int_0^t f(s)\nabla_q s = (1-q)t\sum_{i=0}^{\infty} q^i f(tq^i) \tag{1.6}$$

and for $0 \le a \in T_q$

$$\int_a^t f(s)\nabla_q s = \int_0^t f(s)\nabla_q s - \int_0^a f(s)\nabla_q s$$

By the fundamental theorem in $q-$calculus we have

$$\nabla_q \int_0^t f(s)\nabla_q s = f(t) \tag{1.7}$$

and if f is continuous at 0, then

$$\int_0^t \nabla_q f(s)\nabla_q s = f(t) - f(0) \tag{1.8}$$

Also the following identity will be helpful

$$\nabla_q \int_a^t f(t,s)\nabla_q s = \int_a^t \nabla_q f(t,s)\nabla_q s + f(qt,t) \tag{1.9}$$

From the theory of $q-$calculus and the theory of time scale more generally, the following product rule is valid

$$\nabla_q(f(t)g(t)) = f(qt)\nabla_q g(t) + (\nabla_q f(t))g(t) \tag{1.10}$$

The $q-$binomial function for $n \in N$ is defined by

$$(t-s)_q^n = \prod_{i=0}^{n-1}(t - q^i s) \tag{1.11}$$

When α is a non positive integer, the $q-$binomial fractional function is defined by

$$(t-s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{1-\frac{s}{t}q^i}{1-\frac{s}{t}q^{i+\alpha}} \tag{1.12}$$

It has the following properties

- $(t-s)_q^{\beta+\gamma} = (t-s)_q^\beta (t-q^\beta s)_q^\gamma$
- $(at-as)_q^\beta = a^\beta (t-s)_q^\beta$
- The nabla $q-$derivative of the $q-$binomial function with respect to t is

$$\nabla_q (t-s)_q^\alpha = \frac{1-q^\alpha}{1-q}(t-s)_q^{\alpha-1}$$

- The nabla $q-$derivative of the $q-$binomial function with respect to s is

$$\nabla_q (t-s)_q^\alpha = -\frac{1-q^\alpha}{1-q}(t-qs)_q^{\alpha-1}$$

where $\alpha, \beta, \gamma \in R$.

Moreover, the $q-$fractional integral of order $\alpha \neq 0, -1, -2, ...$ is defined by

$${}_qI_0^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)_q^{\alpha-1} f(s)\nabla_q s. \tag{1.13}$$

Let $\alpha > 0$. If $\alpha \notin N$, then the $\alpha-$order Caputo (left) $q-$fractional derivative of a function f is defined by

$${}_qC_a^\alpha f(t) \triangleq :_q I_a^{(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-qs)_q^{n-\alpha-1}\nabla_q^n f(s)\nabla_q s \tag{1.14}$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the greatest integer less than or equal to α. If $\alpha \in N$, then ${}_qC_a^\alpha f(t) \triangleq \nabla_q^n f(t)$.

The following identity is useful to transform Caputo $q-$fractional difference equation into

$q-$fractional integrals.

Assume $\alpha > 0$ and f is defined in suitable domains. Then

$${}_qI_a^\alpha\, {}_qC_a^\alpha f(t) = f(t) - \sum_{k=0}^{n-1}\frac{(t-a)_q^k}{\Gamma_q(k+1)}\nabla_q^k f(a) \tag{1.15}$$

and if $0 < \alpha \leq 1$ then

$$ {}_qI_a^\alpha :_q C_a^\alpha f(t) = f(t) - f(a) \tag{1.16}$$

The following identity is essential to solve linear $q-$fractional equations

$$ {}_qI_a^\alpha (x-a)_q^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)}(x-a)_q^{\mu+\alpha} \quad (0 \leq a < x < b) \tag{1.17}$$

where $\alpha \in R^+$ and $\mu \in (-1, \infty)$.
For more about $q-$Gamma functions and other $q-$calculus concepts we refer, for example. to [14] The following theorem is of importance to be stated in the article.

Theorem 1.2. ***Leray-Schaefer's Fixed Point Theorem:***
let T be a continuous and compact mapping of a Banach space X into itself, such that the set

$$\Lambda = \{x \in X : x = \lambda T x, \textit{for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then T has a fixed point. For more general versions of Theorem 1.2 see [38].

2. An anti-Caputo $q-$fractional BVP

$$ {}_qC_0^\beta \phi_p \left({}_qC_0^\alpha x(t)\right) = f(t, x(t)), : t \in T_q \tag{2.1}$$

with anti-boundary conditions

$$x(0) = -x(1), \quad {}_qC_0^\alpha x(0) = -{}_qC_0^\alpha x(1), \tag{2.2}$$

where, $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$ and $f : T_q \times R \to R$ is continuous.

$C_q[0, 1]$ will denote the Banach space of all continuous real-valued functions defined on the time scale T_q with the supremum norm.

Lemma 2.1. *Given $h(t) = f(t, x(t))$ is continuous on T_q, then the solution of (2.1) and (2.2) is*

$$\begin{aligned} x(t) &= {}_qI_0^\alpha \phi_r({}_qI_0^\beta h(t) + Ah(t)) + Bh(t) \\ &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{\alpha-1} \phi_r \left(\frac{1}{\Gamma_q(\beta)} \int_0^s (s-q\tau)_q^{\beta-1} h(\tau) \nabla_q \tau + Ah(s) \right) \nabla_q s + Bh(s), \end{aligned}$$

$$\textit{where} \quad Ah(t) = \frac{-1}{2\Gamma_q(\beta)} \int_0^1 (1-qs)_q^{\beta-1} h(s) \nabla_q s \ \textit{ and}$$

$$Bh(t) = \frac{-1}{2\Gamma_q(\alpha)} \int_0^1 (1-qs)_q^{\alpha-1} \phi_r \left(\frac{1}{\Gamma_q(\beta)} \int_0^s (s-q\tau)_q^{\beta-1} h(\tau) \nabla_a \tau + Ah(s) \right) \nabla_q s$$

$\forall t \in [0, 1]$

Proof. Assume that $x(t)$ satisfies (2.1), then

$$ {}_qI_0^\beta \, {}_qC_0^\beta \phi_p({}_qC_0^\alpha x(t)) = {}_qI_0^\beta h(t)$$

Now using (1.16) we get

$$\phi_p({}_qC_0^\alpha x(t)) = {}_qI_0^\beta h(t) + \phi_p({}_qC_0^\alpha x(0))$$

Now using ${}_qC_0^\alpha x(0) = -{}_qC_0^\alpha x(1)$, we get

$$\phi_p({}_qC_0^\alpha x(0)) = -\frac{1}{2} {}_qI_0^\beta h(t)|_{t=1} = Ah(t)$$

Now we have

$$\phi_p({}_qC_0^\alpha x(t)) = {}_qI_0^\beta h(t) + Ah(t)$$

which is equivalent to ${}_qC_0^\alpha x(t) = \phi_r({}_qI_0^\beta h(t) + Ah(t))$. Using (2.1), we get

$$x(t) = x(0) + {}_qI_0^\alpha \phi_r({}_qI_0^\beta h(t) + Ah(t))$$

Using $x(0) = -x(1)$, we obtain

$$x(0) = -\frac{1}{2} :_q I_0^\alpha \phi_r({}_qI_0^\beta h(t) + Ah(t))|_{t=1} = Bh(t). : \square$$

□

Now, define the operator $F : C_q[0,1] \to C_q[0,1]$ by

$$\begin{aligned}
Fx(t) &= {}_qI_{0^+}^\alpha : \phi_r({}_qI_{0^+}^\beta Nx(t) + ANx(t)) + BNx(t) \\
&= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{\alpha-1} \phi_r \left(\frac{1}{\Gamma_q(\beta)} \int_0^s (s-q\tau)_q^{\beta-1} f(\tau, x(\tau)) \nabla_q \tau \right. \\
&- \left. \frac{1}{2\Gamma_q(\beta)} \int_0^1 (1-q\tau)_q^{\beta-1} f(\tau, x(\tau)) \nabla_q \tau \right) \nabla_q s \\
&- \frac{1}{2\Gamma_q(\alpha)} \int_0^1 (1-qs)_q^{\alpha-1} \phi_r \left(\frac{1}{\Gamma_q(\beta)} \int_0^s (s-q\tau)_q^{\beta-1} f(\tau, x(\tau)) \nabla_q \tau \right. \\
&- \left. \frac{1}{2\Gamma_q(\beta)} \int_0^1 (1-q\tau)_q^{\beta-1} f(\tau, x(\tau)) \nabla_q \tau \right) \nabla_q s,
\end{aligned}$$

$\forall t \in [0,1]$ where $N : C_q[0,1] \to C_q[0,1]$ is the Nemytskii operator defined by $Nx(t) = f(t, x(t))$, $\forall t \in [0,1]$. Then the fixed points of the operator F are solutions of (2.1) and (2.2). The next theorem is based on Schaefer's fixed point theorem.

Theorem 2.2. *let $f : T_q \times \mathbb{R} \to \mathbb{R}$ be continuous. Assume that there exist two functions $a, b \in C(T_q, \mathbb{R})$ such that*

$$|f(t,u)| \le a(t) + b(t)|u|^{p-1}, \ \forall t \in T_q, \ u \in \mathbb{R}. \tag{2.3}$$

Then the q-anti BVP (1-2) has at least one solution, provided that

$$\frac{3^r \|b\|_\infty^{r-1}}{2^r \Gamma_q(\alpha+1)\Gamma_q(\beta+1)^{r-1}} < 1. \tag{2.4}$$

Proof. Let us prove first that $F : C_q[0,1] \to C_q[0,1]$ is completely continuous. Let $\Omega \subset C_q[0,1]$ be an open bounded subset. By the continuity of f, we can get that F is continuous and $F(\overline{\Omega})$ is bounded. Moreover, there exists a constant $T > 0$ such that $|{}_qI_{0^+}^\beta Nx + ANx| \le T$, $\forall x \in \overline{\Omega}$, $t \in [0,1]$. Thus, in view of the Arzelá-Ascoli theorem, we need only to prove that $F(\overline{\Omega}) \subset C_q[0,1]$ is equicontinuous.

For $0 \leq t_1 < t_2 \leq 1$, $x \in \overline{\Omega}$, we have

$$\begin{aligned}
& |Fx(t_2) - Fx(t_1)| \\
= & \left| {}_qI^{\alpha}_{0^+}\phi_r({}_qI^{\beta}_{0^+}Nx(t) + ANx(t))|_{t=t_2} - {}_qI^{\alpha}_{0^+}\phi_r({}_qI^{\beta}_{0^+}Nx(t) + ANx(t))|_{t=t_1} \right| \\
= & \frac{1}{\Gamma_q(\alpha)} \left| \int_0^{t_2} (t_2 - qs)_q^{\alpha-1} \phi_r({}_qI^{\beta}_{0^+}Nx(s) + ANx(s)) \nabla_q s \right. \\
- & \left. \int_0^{t_1} (t_1 - qs)_q^{\alpha-1} \phi_r({}_qI^{\beta}_{0^+}Nx(s) + ANx(s)) \nabla_q s \right| \\
= & \frac{1}{\Gamma_q(\alpha)} \left| \int_0^{t_1} [(t_2 - qs)_q^{\alpha-1} - (t_1 - qs)_q^{\alpha-1}] \phi_r({}_qI^{\beta}_{0^+}Nx(s) + ANx(s)) \nabla_q s \right. \\
+ & \left. \int_{t_1}^{t_2} (t_2 - qs)_q^{\alpha-1} \phi_r({}_qI^{\beta}_{0^+}Nx(s) + ANx(s)) \nabla_q s \right| \\
\leq & \frac{T^{r-1}}{\Gamma_q(\alpha)} \left\{ \int_0^{t_1} [(t_1 - qs)_q^{\alpha-1} - (t_2 - qs)_q^{\alpha-1}] \nabla_q s + \int_{t_1}^{t_2} (t_2 - qs)_q^{\alpha-1} \nabla_q s \right\} \\
= & \frac{T^{r-1}}{\Gamma_q(\alpha+1)} [t_1^{\alpha} - t_2^{\alpha} + 2(t_1 - t_2)_q^{\alpha}].
\end{aligned}$$

Since t_q^{α} is uniformly continuous on $[0,1]$, we can obtain that $F(\overline{\Omega}) \subset C_q[0,1]$ is equicontinuous.
Now we need to prove that the set $\Omega = \left\{ x \in C_q[0,1] \,\middle|\, x = \lambda^{r-1} Fx, \lambda \in (0,1) \right\}$ is bounded.
By (2.3), we have

$$\begin{aligned}
|ANx(t)| & \leq \frac{1}{2\Gamma_q(\beta)} \int_0^1 (1 - qs)_q^{\beta-1} |f(s, x(s))| \nabla_q s \\
& \leq \frac{1}{2\Gamma_q(\beta)} \int_0^1 (1 - qs)_q^{\beta-1} (a(s) + b(s)|x(s)|^{p-1}) \nabla_q s \\
& \leq \frac{1}{2\Gamma_q(\beta)} (\|a\|_\infty + \|b\|_\infty \|x\|_\infty^{p-1}) \cdot \frac{1}{\beta} \\
& = \frac{1}{2\Gamma_q(\beta+1)} (\|a\|_\infty + \|b\|_\infty \|x\|_\infty^{p-1}), : \forall t \in [0,1],
\end{aligned}$$

which together with the monotonicity of s^{q-1}, implies that

$$\begin{aligned}
|BNx(t)| &\leq \frac{1}{2\Gamma_q(\alpha)}\int_0^1 (1-qs)_q^{\alpha-1}|{}_qI_{0^+}^{\beta}Nx(s)+ANx(s)|^{r-1}\nabla_q s \\
&\leq \frac{1}{2\Gamma_q(\alpha)}\int_0^1 (1-qs)_q^{\alpha-1}\left(\frac{t^{\beta}}{\Gamma_q(\beta+1)}(\|a\|_\infty+\|b\|_\infty\|x\|_\infty^{p-1})\right. \\
&+ \left.\frac{1}{2\Gamma_q(\beta+1)}(\|a\|_\infty+\|b\|_\infty\|x\|_\infty^{p-1})\right)^{r-1}\nabla_q s \\
&= \frac{1}{2\Gamma_q(\alpha)}\frac{(\|a\|_\infty+\|b\|_\infty\|x\|_\infty^{p-1})^{r-1}}{2^{r-1}(\Gamma_q(\beta+1))^{r-1}}(2t^{\beta}+1)^{r-1}\int_0^1 (1-qs)_q^{\alpha-1}\nabla_q s
\end{aligned}$$

Hence, we get

$$|BNx(t)| \leq \frac{3^{r-1}}{2^r\Gamma_q(\alpha+1)(\Gamma_q(\beta+1))^{r-1}}(\|a\|_\infty+\|b\|_\infty\|x\|_\infty^{p-1})^{r-1} \tag{2.5}$$

$\forall t\in[0,1]$ Similarly, we find that,

$$|{}_qI_{0^+}^{\alpha}\phi_r({}_qI_{0^+}^{\beta}Nx(t)+ANx(t))| \leq \frac{3^{r-1}(\|a\|_\infty+\|b\|_\infty\|x\|_\infty^{p-1})^{r-1}}{2^{r-1}\Gamma_q(\alpha+1)(\Gamma_q(\beta+1))^{r-1}} \tag{2.6}$$

For $x\in\Omega$, we get $x(t)=\lambda^{r-1}Fx(t)$. Thus, using (2.5)and (2.6), we obtain

$$\begin{aligned}
|x(t)| &\leq |{}_qI_{0^+}^{\alpha}\phi_r({}_qI_{0^+}^{\beta}Nx(t)+ANx(t))|+|BNx(t)| \\
&\leq \frac{3^r(\|a\|_\infty+\|b\|_\infty\|x\|_\infty^{p-1})^{r-1}}{2^r\Gamma_q(\alpha+1)(\Gamma_q(\beta+1))^{r-1}}\ \forall t\in[0,1].
\end{aligned}$$

There exist a constant $M>0$ such that $\|x\|_\infty\leq M$.
As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is the solution of ABVP (2.1) and (2.2). □

3. An example

Here is an example that illustrates the main result.

Consider the time scale T_q , and consider the following ABVP for the $q-$fractional $p-$Laplacian equation

$${}_qC_0^{\frac{1}{2}}\phi_3\left({}_qC_0^{\frac{3}{4}}x(t)\right)=\frac{1}{10}x^2(t)+\sin t,\ t\in T_q$$

with boundary conditions

$$x(0)=-x(1),\quad {}_qC_0^{\frac{3}{4}}x(0)=-{}_qC_0^{\frac{3}{4}}x(1).$$

Corresponding to ABVP (2.1) and (2.2), we have: $p=3, r=\frac{3}{2}, \alpha=\frac{3}{4}, \beta=\frac{1}{2}$ and $f(t,x(t))=\frac{1}{10}x^2(t)+\sin t$.

Choose $a(t) = 1, b(t) = \frac{1}{10}$, then after a straightforward calculations, we obtain that $\|b\|_\infty = \frac{1}{10}$ and

$$\frac{3^{\frac{3}{2}}(\frac{1}{10})^{\frac{1}{2}}}{2^{\frac{3}{2}}\Gamma_q(\frac{3}{4}+1)(\Gamma_q(\frac{1}{2}+1))^{\frac{1}{2}}} < 1.$$

The ABVP in this example satisfies all assumptions of the Theorem 2.2, hence it has at least one solution.

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