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POSITIVE SOLUTIONS FOR A FRACTIONAL BOUNDARY VALUE PROBLEM

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Abstract. We discuss the existence of positive solutions for a fractional boundary value problem by the help of some fixed point theorems and under suitable conditions on the nonlinear term. Two examples are also included to illustrate that the corresponding assumptions are satisfied.

1. Introduction

The purpose of the present work is to investigate sufficient conditions for the existence of three positive solutions for the following fractional boundary value problem (P):

$$^{c}D_{0^+}^{q}u(t)=a(t)f\left(u(t)\right), \quad 0<t<1,$$

$$u\left(0\right)=u'(0)=0, \quad u''\left(0\right)=\alpha u\left(1\right),$$

where $f:\mathbb{R}\to\mathbb{R}$ is a given function, $2<q<3$, $^{c}D_{0^+}^{q}$ denotes the Caputo's fractional derivative, $a\in C([0,1],\mathbb{R})$. We show that under certain growth conditions on the nonlinear term f, the fractional boundary value problem (P) has at least one or at least three positive solutions.

Fractional differential equations have recently proved to be valuable tools in the modelling of many phenomena in various fields of science and engineering, physics and economics. We can find numerous applications in viscoelasticity, electrochemistry, electrical networks, control theory, biosciences, electromagnetic, signal processes, mechanics and diffusion processes see [20, 21, 22, 23]. Significant developments in fractional differential equations can be find in the monographs of Kilbas et al. [20], Miller and Ross [22], Lakshmikantham et al. [21], Podlubny [23]. Ordinary differential equations and fractional differential equations have been

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studied by many authors by means of fixed point theory see [1, 2, 3, 4, 13, 14, 15, 16, 17, 18, 19, 24, 25].

Note that numerous works [6, 7, 8, 9, 10, 11, 12, 26] were dedicated to the research questions of local and non local boundary value problems for partial differential equations with boundary operators of high (integer and fractional) order. In [6], the initial boundary value problem for partial differential equations of higher order with the caputo fractional derivative was studied in the case when the order of the fractional derivative belongs to the interval (0,1).

In [16], El-Shahed consider the following nonlinear fractional boundary value problem

$$\begin{aligned}
&D_{0^+}^{q} u(t) + \lambda a(t) f\left(t, u(t)\right) = 0, \quad 0 < t < 1, \\
&u\left(0\right) = u'(0) = u'(1) = 0,
\end{aligned}$$

here $2 < q \leq 3$, and $D_{0^+}^{q}$ denotes the Riemann-Liouville fractional derivative. Using Krasnoselskii's fixed point theorem on cone, he proved the existence and nonexistence of positive solutions for the above fractional boundary value problem.

In [14], Bai and Lu investigated the existence and multiplicity of positive solutions for nonlinear fractional differential equation boundary value problem of type:

$$\begin{aligned}
D_{0^+}^{q} u(t) + f\left(t, u(t)\right) &= 0, \quad 0 < t < 1, \\
u\left(0\right) &= u(1) = 0,
\end{aligned}$$

where $1 < q \leq 2$, and $D_{0^+}^{q}$ denotes the Riemann-Liouville fractional derivative. Applying fixed point theorem on cone, they proved some existence and multiplicity results of positive solutions.

The organization of this paper is as follows. In Section 2, we introduce some definitions notations that will be used later. In the third Section, we discuss the existence of at least one positive solution of problem (P) by using Guo-Krasnosel'skii fixed point theorem in cone, then, under some sufficient conditions on the nonlinear source term, we apply Avery-Peterson theorem to prove the existence of at least three positive solutions. At the end of this section, we give two examples illustrating the previous results.

2. Preliminaries

In this section, we present some definitions and lemmas from fractional calculus theory, which will be needed later.

Definition 2.1. If $g \in C([a, b])$ and $\alpha > 0$, then the Riemann-Liouville fractional integral is defined by

$$I_{a^+}^{\alpha} g(t) = \frac{1}{\Gamma\left(\alpha\right)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2.2. Let $\alpha \geq 0, n = [\alpha] + 1$. If $f \in C^n[a, b]$ then the Caputo fractional derivative of order α of f defined by ${}^cD^{\alpha}_{a^+}g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^n(s)}{(t-s)^{\alpha-n+1}} ds$ exists almost everywhere on $[a, b]$ ($[\alpha]$ is the entire part of α).

Lemma 2.3. *For $\alpha > 0$, $g \in C([0, 1], \mathbb{R})$, the homogenous fractional differential equation ${}^cD^{\alpha}_{a^+}g(t) = 0$ has a solution $g(t) = c_1 + c_2t + c_3t^2 + \cdots + c_nt^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, \ldots, n$, and $n = [\alpha] + 1$.*

Define $E = C[0, 1]$ equipped with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Lemma 2.4. *Let $p, q \geq 0$, $f \in L_1[a, b]$. Then, $I^p_{0^+}I^q_{0^+}f(t) = I^{p+q}_{0^+}f(t) = I^q_{0^+}I^p_{0^+}f(t)$ and ${}^cD^q_{a^+}I^q_{0^+}f(t) = f(t)$, for all $t \in [a, b]$.*

Now we present the necessary definition from the theory of cone in Banach spaces.

Definition 2.5. A nonempty subset P of a Banach space E is called a cone if P is convex, closed and satisfies the following conditions:

(i) $\alpha x \in P$ for all $x \in P$ and $\alpha \in \mathbb{R}_+$,
(ii) $x, -x \in P$ implies $x = 0$.

Definition 2.6. A mapping is called completely continuous if it is continuous and maps bounded sets into relatively compact sets.

We start by solving an auxiliary problem which allows us to get the expression of the solution.

Lemma 2.7. *Assuming that $\alpha \neq 2$ and $y \in C([0, 1], \mathbb{R})$. Then, the problem (P_0)*

$$\begin{aligned} &{}^cD^q_{0^+}u(t) = y(t), \quad 0 < t < 1, \\ &u(0) = u'(0) = 0, \quad u''(0) = \alpha u(1), \end{aligned}$$

has a unique solution given by:

$$u(t) = \frac{1}{\Gamma(q)} \int_0^1 G(t, s)\, y(s) ds,$$

where

$$G(t, s) = \begin{cases} (t-s)^{q-1} + \frac{\alpha}{2-\alpha}t^2(1-s)^{q-1} & , \quad 0 \leq s \leq t, \\ \frac{\alpha}{2-\alpha}t^2(1-s)^{q-1} & , \quad 0 \leq t \leq s \leq 1. \end{cases}$$

Proof. Using Lemmas 2.3 and 2.4, we get

$$u(t) = I^q_{0^+}y(t) + a + bt + ct^2. \tag{2.1}$$

The boundary condition $u(0) = 0$ implies that $a = 0$. Differentiating both sides of (2.1) and using the initial condition $u'(0) = 0$, it yields $b = 0$. The condition

$u''(0) = \alpha u(1)$, $u''(0) = 2c = \alpha u(1)$, $2c = \alpha[I_{0+}^{q} y(1) + c]$, $2c - \alpha c = \alpha I_{0+}^{q} y(1)$, and $c = \dfrac{\alpha}{2-\alpha} I_{0+}^{q} y(1)$. Substituting a, b and c by their values in (2.1), we obtain

$$\begin{aligned} u(t) &= I_{0+}^{q} y(t) + \frac{\alpha}{2-\alpha} t^2 I_{0+}^{q} y(1) \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds + \frac{\alpha}{2-\alpha} \frac{1}{\Gamma(q)} t^2 \int_0^1 (t-s)^{q-1} y(s) ds \\ &= \frac{1}{\Gamma(q)} \int_0^1 G(t,s) y(s) ds. \end{aligned}$$

□

3. Existence of positive solutions

In this section we assume that $0 < \alpha < 2$ and :

(H_1) $a \in C([0,1], \mathbb{R}_+)$ and for all τ such that $0 < \tau < 1$ then $\int_\tau^1 (1-s)^{q-1} a(s) ds \neq 0$.

(H_2) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$.

Define the integral operator $T : E \to E$ by

$$T(u)(t) = \frac{1}{\Gamma(q)} \int_0^1 G(t,s) a(s) f(u(s)) ds, \tag{3.1}$$

that can be written as :

$$T(u)(t) = I_{0+}^{q} a(t) f(u(t) + \frac{\alpha}{2-\alpha} t^2 I_{0+}^{q} a(1) f(u(1). \tag{3.2}$$

Definition 3.1. A function u is called positive solution of problem (P) if $u(t) \geq 0, \forall t \in [0,1]$ and it satisfies the boundary condition in (P).

Let us introduce the following notation $A_0 = \lim\limits_{u \to 0} \dfrac{f(u)}{u}$, $A_\infty = \lim\limits_{u \to \infty} \dfrac{f(u)}{u}$. The case $A_0 = 0$ and $A_\infty = \infty$ is called superlinear case and the case $A_0 = \infty$ and $A_\infty = 0$ is called sublinear case.

Lemma 3.2. *If $0 < \alpha < 2$, then the function G has the following properties:*

1. *$G(t,s) \geq 0$, for all $t, s \in [0,1]$.*
2. *For all $t \in [\tau, 1]$ and $s \in [0,1]$, $\tau > 0$, $0 < \tau < 1$, we have*

$$0 \leq \alpha \tau^2 \gamma(s) \leq G(t,s) \leq 2\gamma(s), \tag{3.3}$$

and where $\gamma(s) = \frac{(1-s)^{q-1}}{2-\alpha}$.

Proof. Let $t \in [0,1]$, then we have

$$G(t,s) \leq (1-s)^{q-1} \left(\frac{2}{2-\alpha} \right) = 2\gamma(s),$$

and if $t \in [\tau, 1]$ then

$$G(t,s) \geq (1-s)^{q-1}\left(\frac{\alpha t^2}{2-\alpha}\right) \geq (1-s)^{q-1}\left(\frac{\alpha \tau^2}{2-\alpha}\right) = \alpha\tau^2\gamma(s). \tag{3.4}$$

□

Lemma 3.3. *The solution of fractional boundary value problem (P) satisfies*

$$\min u(t)_{t\in[\tau,1]} \geq \frac{\alpha\tau^2}{2}\|u\|. \tag{3.5}$$

Proof. The proof is easy, then we omit it. □

Theorem 3.4. *Assuming that $(H_1)-(H_2)$ holds, then the fractional boundary value problem (P) has at least one positive solution in the both cases superlinear as well as sublinear.*

To prove Theorem 3.4, we apply the well-known Guo-Krasnosel'skii fixed point theorem on cone.

Theorem 3.5. *[13] Let E be a Banach space, and let $K \subset E$, be a cone. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$\mathcal{A} : K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to K,$$

be a completely continuous operator such that

(i) $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_2$; *or*
(ii) $||\mathcal{A}u|| \geq ||u||$, $u \in K \cap \partial\Omega_1$, *and* $||\mathcal{A}u|| \leq ||u||$, $u \in K \cap \partial\Omega_2$.*Then $\mathcal{A}$ has a fixed point in $K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.*

Proof. Denote $E^+ = \{u \in E, u(t) \geq 0, \forall t \in [0,1]\}$ and define the cone K by

$$K = \left\{u \in E^+, \min_{t\in[\tau,1]} u(t) \geq \frac{\alpha\tau^2}{2}\|u\|\right\}, \tag{3.6}$$

It is easy to check that K is a nonempty closed and convex subset of E, hence it is a cone. One can check that $TK \subset K$. It is obvious that T is continuous since G, a and f are continuous. Let us prove that $T : K \to E$ is completely continuous mapping on K.

Claim 1. $T(B_r)$ is uniformly bounded, where $B_r = \{u \in K, \|u\| \leq r\}$.

Since the functions a and f are continuous, then there exists a constant c such that $\max_{t\in[0,1]} |a(t)f(u(t)| = c$ for any $u \in B_r$. By virtue of Lemma 3.2 we obtain

$$|Tu(t)| \leq \frac{2c}{(2-\alpha)\,\Gamma(q)}. \tag{3.7}$$

Hence T is uniformly bounded.

Claim 2. T is equicontinuous. We have for any $u \in B_r$

$$
\begin{aligned}
|T'u(t)| &= \left| \begin{array}{l} \frac{1}{\Gamma(q)}\int_0^1 (q-1)(t-s)^{q-2} a(s) f(u(s))\, ds \\ +\frac{1}{\Gamma(q)}\int_0^1 2t \frac{\alpha}{2-\alpha}(1-s)^{q-1} a(s) f(u(s))\, ds \end{array} \right| \\
&\leq \frac{c}{\Gamma(q-1)} \int_0^1 (1-s)^{q-2} ds + \frac{4c}{\Gamma q (2-\alpha)} \int_0^1 (1-s)^{q-1} ds \qquad (3.8) \\
&= \frac{c}{\Gamma q}\left(1 + \frac{4}{(2-\alpha)}\right) = \frac{c_1}{\Gamma q}.
\end{aligned}
$$

Therefore

$$
|Tu(t_2) - Tu(t_1)| = \left| \int_{t_1}^{t_2} T'u(t)\, dt \right| \leq \frac{c_1 (t_2 - t_1)}{\Gamma(q)}. \tag{3.9}
$$

Consequently T is equicontinuous. From Arzela-Ascoli theorem we deduce that T is completely continuous operator.

Let us consider the superlinear case. First, $A_0 = 0$, for any $\varepsilon > 0$, there exists $R_1 > 0$, such that if $0 < u \leq R_1$ then $f(u) \leq \varepsilon u$. Let $\Omega_1 = \{u \in E, \|u\| < R_1\}$, Letting $u \in K \cap \partial\Omega_1$, then we have

$$
\begin{aligned}
Tu(t) &= \frac{1}{\Gamma(q)} \int_0^1 G(t,s)\, a(s) f(u(s) ds, \\
&\leq \frac{2\varepsilon \|u\|}{\Gamma(q)} \int_0^1 \gamma(s) a(s) ds, \qquad (3.10)
\end{aligned}
$$

Then if we choose $\varepsilon = \Gamma(q)/2\int_0^1 \gamma(s)a(s)ds$, we get $||Tu|| \leq ||u||$, for any $u \in K \cap \partial\Omega_1$.

Second, since $A_\infty = \infty$, then for any $M > 0$, there exists $R_2 > 0$, such that $f(u) \geq Mu$ for $u \geq R_2$. Let $R = \max\left\{2R_1, \frac{2R_2}{\alpha\tau^2}\right\}$, and denote by $\Omega_2 = \{u \in E : ||u|| < R\}$. If $u \in K \cap \partial\Omega_2$ then

$$
\min_{t\in[\tau,1]} u(t) \geq \frac{\alpha\tau^2}{2}\|u\| = \frac{\alpha\tau^2}{2} R \geq R_2. \tag{3.11}
$$

Using the left-hand side of Lemmas 3.2 and 3.3, we obtain

$$
Tu(t) \geq \frac{\alpha\tau^2 M}{\Gamma(q)} \int_\tau^1 \gamma(s) a(s) u(s)\, ds, \tag{3.12}
$$

thus

$$
Tu(t) \geq \frac{\alpha^2\tau^4 M \|u\|}{2\Gamma(q)} \int_\tau^1 \gamma(s) a(s) ds, \tag{3.13}
$$

we can choose $M = 2\Gamma(q)/\alpha^2\tau^4 \int_\tau^1 \gamma(s)a(s)ds$, then we get $||Tu|| \geq ||u||$, $\forall u \in K \cap \partial\Omega_2$.The first statement of Theorem 3.5 implies that T has a fixed point in $K \cap \left(\overline{\Omega}_2 \diagdown \Omega_1\right)$ such that $R_2 \leq ||u|| \leq R$. Applying similar techniques as above, we prove the sublinear case. The proof of is complete. $\square$

Let us introduce the following functionals. Defining on K, the nonnegative, continuous, and concave functional Λ by $\Lambda(u) = \min_{t\in[\tau,1]} |u(t)|$, then $\Lambda(u) \leq \|u\|$. Defining the nonnegative, continuous, and convex functional φ and Φ on K by $\varphi(u) = \Phi(u) = \|u\|$ and the nonnegative continuous functional Ψ on K by $\Psi(u) = \|u\|$, then $\Psi(ku) \leq k\|u\|$ for $0 \leq k \leq 1$.

Theorem 3.6. *Assume that $(H_1) - (H_2)$ hold, and that there exists positive constants a, b, c, d, μ, β and ν such that $a < b$, $\mu > \frac{2}{(2-\alpha)\Gamma(q)} \int_0^1 (1-s)^{q-1} a(s)\, ds$, $\beta < \frac{\alpha\tau^2}{(2-\alpha)\Gamma(q)} \int_\tau^1 (1-s)^{q-1} a(s)\, ds$, and*

(i) $f(u) \leq \frac{d}{\mu}$ *for* $u \in [0, d]$.
(ii) $f(u) \leq \frac{b}{\beta}$ *for* $u \in [b, c]$.
(iii) $f(u) \leq \frac{a}{\mu}$ *for* $u \in [0, a]$.

Then the problem (P) has at least three positive solutions $u_1, u_2, u_3 \in \overline{K(\varphi, d)}$ such that

$\varphi(u_i) \leq d$ *for* $i = 1, 2, 3$, $b < \Lambda(u_1)$, $a < \Psi(u_2)$ *with* $\Lambda(u_2) < b$ *and* $\Psi(u_3) < a$.

To prove the existence of three positive solutions, we apply (Avery and Peterson fixed point Theorem).

Theorem 3.7. [5] *Let K be a cone in a real Banach space E. Let φ and Φ be nonnegative, continuous, and convex functional on K, let Λ be a continuous, nonnegative and concave functional on K, and let Ψ be a continuous and nonnegative functional on K satisfying $\Psi(ku) \leq k\|u\|$ for $0 \leq k \leq 1$. Define the sets, $K(\varphi, d)$, $K(\varphi, \Lambda, b, d)$, $K(\varphi, \Phi, \Lambda, b, c, d)$ and $R(\varphi, \Psi, a, d)$ by*

$$\begin{aligned} K(\varphi, d) &= \{u \in K, \varphi(u) < d\}, \\ K(\varphi, \Lambda, b, d) &= \{u \in K, b \leq \Lambda(u), \varphi(u) \leq d\}, \\ K(\varphi, \Phi, \Lambda, b, c, d) &= \{u \in K, b \leq \Lambda(u), \Phi(u) \leq c, \varphi(u) \leq d\}, \\ R(\varphi, \Psi, a, d) &= \{u \in K, a \leq \Psi(u), \varphi(u) \leq d\}. \end{aligned}$$

For M and d positive numbers we have $\Lambda(u) \leq \Psi(u)$ and $\|u\| \leq M\varphi(u)$ for any $u \in \overline{K(\varphi, d)}$. Assume $T : \overline{K(\varphi, d)} \longrightarrow \overline{K(\varphi, d)}$ is completely continuous and there exists positive numbers a, b and c with $a < b$ such that

$(S1)$ $\{u \in K(\varphi, \Phi, \Lambda, b, c, d), \Lambda(u) \succ b\} \neq \emptyset$ and $\Lambda(Tu) \succ b$ for $u \in K(\varphi, \Phi, \Lambda, b, c, d)$,

$(S2)$ $\Lambda(Tu) > b$ for $u \in K(\varphi, \Lambda, b, d)$ with $\Phi(Tu) > c$,

$(S3)$ $0 \notin R(\varphi, \Psi, a, d)$ and $\Psi(Tu) \prec a$ for $u \in R(\varphi, \Psi, a, d)$ with $\Psi(u) = a$.

Then T has at least three positive fixed points $u_1, u_2, u_3 \in \overline{K(\varphi, d)}$ such that

$\varphi(u_i) \leq d$ *for* $i = 1, 2, 3$, $b < \Lambda(u_1)$, $a < \Psi(u_2)$ *with* $\Lambda(u_2) < b$ *and* $\Psi(u_3) < a$.

Proof. Proceeding analogously as in the proof of Theorem 3.4, we prove that the mapping T is completely continuous on $\overline{K(\varphi, d)}$.

Claim 1. $T\left(\overline{K(\varphi,d)}\right) \subset \overline{K(\varphi,d)}$. Letting $u \in \overline{K(\varphi,d)}$, then $\|u\| \leq d$. Thus with the help of assumption (i) it yields

$$\begin{aligned}
\varphi(Tu) &= \|Tu\| = \max_{t\in[0,1]} \frac{1}{\Gamma(q)} \int_0^1 G(t,s)\, a(s) f(u(s)) ds \\
&\leq \frac{2}{(2-\alpha)\Gamma(q)} \int_0^1 (1-s)^{q-1} a(s) f(u(s))\, ds \\
&\leq \frac{d}{\mu} \frac{2}{(2-\alpha)\Gamma(q)} \int_0^1 (1-s)^{q-1} a(s)\, ds < d,
\end{aligned}$$

and hence $Tu \in \overline{K(\varphi,d)}$.

Claim 2. $(S1)$ holds, that is $\left\{u \in K\left(\varphi,\Phi,\Lambda,b,\frac{b}{1-\lambda},d\right), \Lambda(u) > b\right\} \neq \emptyset$ and $\Lambda(Tu) > b$ for $u \in K\left(\varphi,\Phi,\Lambda,b,\frac{b}{1-\lambda},d\right)$. Let $y(t) = b\left(\frac{\lambda}{1-\lambda}\right)$ with $\frac{1}{2} < \lambda < 1$, then

$$\Phi(y) = \varphi(y) = \|y\| = b\left(\frac{\lambda}{1-\lambda}\right) < \frac{b}{1-\lambda}.$$

Moreover we have

$$\Lambda(y) = \min_{t\in[\tau,1]} y(t) = b\left(\frac{\lambda}{1-\lambda}\right) > b > (1-\lambda)\|y\|.$$

Thus $y \in K\left(\varphi,\Phi,\Lambda,b,\frac{b}{1-\lambda},d\right)$, so $\left\{u \in K\left(\varphi,\Phi,\Lambda,b,\frac{b}{1-\lambda},d\right), \Lambda(u) > b\right\} \neq \emptyset$.

Letting $u \in K\left(\varphi,\Phi,\Lambda,b,\frac{b}{1-\lambda},d\right)$, then $b \leq u(t) \leq \frac{b}{1-\lambda}$, thus by virtue of (3.3) and assumption (ii), we obtain

$$\begin{aligned}
\Lambda(Tu) &= \min_{t\in[\tau,1]} |Tu(t)| \geq \frac{\alpha\tau^2}{(2-\alpha)\Gamma(q)} \int_\tau^1 (1-s)^{q-1} a(s) f(u(s))\, ds \\
&\geq \frac{\alpha\tau^2}{(2-\alpha)\Gamma(q)} \frac{b}{\beta} \int_\tau^1 (1-s)^{q-1} a(s)\, ds > b.
\end{aligned}$$

So condition (S1) is satisfied.

Claim 3. $(S2)$ holds. Letting $u \in K(\varphi,\Lambda,b,d)$ such that $\Phi(Tu) = \|Tu\| > c$, then

$$\Lambda(Tu) = \min_{t\in[\tau,1]} |Tu(t)| \geq b,$$

this implies that (S2) holds.

Claim 4. $(S3)$ holds. Letting $u \in R(\varphi,\Psi,a,d)$, then $0 < a \leq \|u\| \leq d$, and so $0 \notin R(\varphi,\Psi,a,d)$ with $\Psi(u) = \|u\| = a$, using Lemma 3.2 and assumption (iii) it yields

$$\begin{aligned}\Psi(Tu) &= \max_{t\in[0,1]} \frac{1}{\Gamma(q)} \int_0^1 G(t,s)\,a(s) f(u(s))\,ds \\ &\le \frac{2}{(2-\alpha)\Gamma(q)} \int_0^1 (1-s)^{q-1} a(s) f(u(s))\,ds \\ &\le \frac{a}{\mu} \frac{2}{(2-\alpha)\Gamma(q)} \int_0^1 (1-s)^{q-1} a(s)\,ds < a,\end{aligned}$$

Then (S3) is satisfied. □

Now we give two examples to illustrate Theorem 3.4 and 3.6

Example 3.8. Let us consider the following fractional boundary value problem

$${}^cD_{0^+}^{\frac{8}{3}} u(t) = a(t) f(u(t)), \quad 0 < t < 1,$$

where $q = \frac{8}{3}$, $\alpha = \frac{1}{2}$, $f(u) = \exp(-u)$, $a(t) = t$, $\tau = \frac{4}{5}$, by calculus we obtain $\int_0^{0,8} a(s)\,ds = \int_0^{0,8} s ds = 0,32 \neq 0$. The assumptions $(H_1) - (H_2)$ holds and that $A_0 = \infty$, $A_\infty = 0$, applying Theorem 3.4, we deduce that there exists at least one positive solution.

Example 3.9. Let us consider the following fractional boundary value problem

$${}^cD_{0^+}^{\frac{9}{4}} u(t) = a(t) f(u(t)), \quad 0 < t < 1,$$

where

$$q = \frac{9}{4},\ \alpha = 1,\ a(t) = \sqrt{1+t},\ \tau = \frac{9}{10},$$

$$f(u) = \begin{cases} \frac{u^3}{2} & , \quad 0 \le u \le 3, \\ \frac{7u^2}{2} - 18 & , \quad 3 \le u \le 4, \\ 38 & , \quad u \ge 4. \end{cases}$$

It is easy to see that (H_1) and (H_2) are satisfied. Let us check the assumptions of Theorem 3.6

$$\begin{aligned}\mu &> \frac{2(0,1)^{\frac{5}{4}}}{\Gamma\left(\frac{9}{4}\right)} \int_0^1 \sqrt{1+s} ds = 2.151\,7, \\ \beta &< \frac{(0,9)^2 (0,1)^{\frac{5}{4}}}{\Gamma\left(\frac{9}{4}\right)} \int_{0,9}^1 \sqrt{1+s} ds = 0,87149.\end{aligned}$$

If we choose $\mu = 2,30$, $\beta = 0,5$, $a = 2$, $b = 3$, $c = 0,1$, $d \ge 127,65$, then the assumptions of Theorem 3.6 are satisfied, consequently, there exists at least three positive solutions $u_1, u_2, u_3 \in \overline{K(\varphi, d)}$ such that

$\|u_i\| \le d = 128,3 < \min_{t\in\left[\frac{9}{10},1\right]} u_1(t)$, $2 < \|u_2\|$, with $\min_{t\in\left[\frac{9}{10},1\right]} u_2(t) < 3$ and $\|u_3\| < 2$.

Conclusion

In this paper, we have proved the existence of at least one positive solution of problem (P) by using Guo-Krasnosel'skii fixed point theorem in cone, then under some sufficient conditions on the nonlinear source term, we have applied Avery-Peterson theorem to prove the existence of at least three positive solutions. One can prove the existence of multiple positive solutions by using other fixed theorems.

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