STABILITY AND SUPER STABILITY OF FUZZY APPROXIMATELY *-HOMOMORPHISMS

N. EGHBALI

Abstract. In this paper we introduce the concept of fuzzy Banach *-algebra. Then we study the stability and super stability of approximately *-homomorphisms in the fuzzy sense.

1. Introduction

It seems that the stability problem of functional equations had been first raised by Ulam [12]. In 1941, Hyers [3] showed that if \( \delta > 0 \) and if \( f : E_1 \to E_2 \) is a mapping between Banach spaces \( E_1 \) and \( E_2 \) with \( \|f(x+y)-f(x)-f(y)\| \leq \delta \) for all \( x, y \in E_1 \), then there exists a unique \( T : E_1 \to E_2 \) such that \( T(x+y) = T(x)+T(y) \) with \( \|f(x)-T(x)\| \leq \delta \) for all \( x, y \in E_1 \). In 1978, a generalized solution to Ulam’s problem for approximately linear mappings was given by Th. M. Rassias [10]. Suppose \( E_1 \) and \( E_2 \) are two real Banach spaces and \( f : E_1 \to E_2 \) is a mapping. If there exist \( \delta \geq 0 \) and \( 0 < p < 1 \) such that \( \|f(x+y)-f(x)-f(y)\| \leq \delta (\|x\|^p+\|y\|^p) \) for all \( x, y \in E_1 \), then there is a unique additive mapping \( T : E_1 \to E_2 \) such that \( \|f(x)-T(x)\| \leq 2\delta \|x\|^p/2-2^p \) for every \( x \in E_1 \). In 1991, Gajda [1] gave a solution to this question for \( p > 1 \). For the case \( p = 1 \), Th. M. Rassias and Šemrl [11] showed that there exists a continuous real-valued function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f \) cannot be approximated with an additive map.

Găvruta [2] generalized Rassias’s result: Let \( G \) be an abelian group and \( X \) a Banach space. Denote by \( \varphi : G \times G \to [0, \infty) \) a function such that \( \tilde{\varphi}(x,y) = \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x, 2^k y) < \infty \) for all \( x, y \in G \). Suppose that \( f : G \to X \) is a mapping satisfying
\[
\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y)
\]
for all \( x, y \in G \). Then there exists a unique additive mapping \( T : G \to X \) such that
\[
\|f(x) - T(x)\| \leq 1/2 \tilde{\varphi}(x,x)
\]

Received by the editors: October 10, 2013; Accepted: April 04, 2015.
2010 Mathematics Subject Classification. Primary: 46S40; Secondary: 39B52, 39B82, 26E50, 46S50.
Key words and phrases. Fuzzy normed space; approximately *-homomorphism; stability.
for all \( x \in G \). Recently, Park [9] applied Găvruta’s result to linear functional equations in Banach modules over a C*-algebra.

B. E. Johnson [4] also investigated almost algebra *-homomorphisms between Banach *-algebras.

Fuzzy notion introduced firstly by Zadeh [13] that has been widely involved in different subjects of mathematics. Zadeh’s definition of a fuzzy set characterized by a function from a nonempty set \( X \) to \([0, 1]\).

Later, in 1984 Katsaras [7] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Defining the class of approximately solutions of a given functional equation one can ask whether every mapping from this class can be somehow approximated by an exact solution of the considered equation in the fuzzy Banach *-algebra. To answer this question, we use here the definition of fuzzy normed spaces given in [7] to exhibit some reasonable notions of fuzzy approximately *-homomorphism in fuzzy normed algebras and we will prove that if \( A \) is a Banach *-algebra, then under some suitable conditions a fuzzy approximately *-homomorphism \( f : A \to A \) can be approximated in a fuzzy sense by a *-homomorphism \( H : A \to A \). This is applied to show that for a fuzzy approximately map \( f : A \to A \) on a C*-algebra \( A \), there exists a unique *-homomorphism \( H : A \to A \) such that \( f = H \).

2. Preliminaries

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions.

**Definition 2.1.** Let \( X \) be a real linear space. A function \( N : X \times \mathbb{R} \to [0, 1] \) is said to be a fuzzy norm on \( X \) if for all \( x, y \in X \) and all \( t, s \in \mathbb{R} \),

1. \( N(x, c) = 0 \) for \( c \leq 0 \);
2. \( x = 0 \) if and only if \( N(x, c) = 1 \) for all \( c > 0 \);
3. \( N(cx, t) = N(x, \frac{t}{|c|}) \) if \( c \neq 0 \);
4. \( N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\} \);
5. \( N(x, \cdot) \) is a non-decreasing function on \( \mathbb{R} \) and \( \lim_{t \to \infty} N(x, t) = 1 \);
6. for \( x \neq 0 \), \( N(x, \cdot) \) is (upper semi) continuous on \( \mathbb{R} \).

The pair \((X, N)\) is called a fuzzy normed linear space.

**Example 2.2.** Let \((X, ||\cdot||)\) be a normed linear space. Then

\[
N(x, t) = \begin{cases} 
0, & t \leq 0; \\
\frac{t}{||x||}, & 0 < t \leq ||x||; \\
1, & t > ||x||.
\end{cases}
\]

is a fuzzy norm on \( X \).

**Definition 2.3.** Let \((X, N)\) be a fuzzy normed linear space and \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) is said to be convergent if there exists \( x \in X \) such that
\( \lim_{n \to \infty} N(x_n - x, t) = 1 \) for all \( t > 0 \). In that case, \( x \) is called the limit of the sequence \( \{x_n\} \) and we denote it by \( N \lim_{n \to \infty} x_n = x \).

**Definition 2.4.** A sequence \( \{x_n\} \) in \( X \) is called Cauchy if for each \( \varepsilon > 0 \) and each \( t > 0 \) there exists \( n_0 \) such that for all \( n \geq n_0 \) and all \( p > 0 \), we have \( N(x_{n+p} - x_n, t) > 1 - \varepsilon \).

It is known that every convergent sequence in a fuzzy normed space is Cauchy and if each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and furthermore the fuzzy normed space is called a complete fuzzy normed space.

Let \( X \) be an algebra and \( (X,N) \) be complete fuzzy normed space. The pair \( (X,N) \) is said to be a fuzzy Banach algebra if for every \( x,y \in X \) and \( s,t \in \mathbb{R} \) we have \( N(xy,st) \geq \min\{N(x,s),N(y,t)\} \).

**Definition 2.5.** Let \( X \) be a linear space and \( \varphi : X \times X \to [0, \infty) \). We say that \( \varphi \) is control function if we have
\[
\varphi(x,y) = \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n x, 2^n y) < \infty,
\]
for all \( x,y \in X \).

We give the following results proved in [8].

**Theorem 2.6.** Let \( X \) be a linear space and \( (Y,N) \) be a fuzzy Banach space. Suppose that \( \varphi : X \times X \to [0, \infty) \) is a control function and \( f : X \to Y \) is a uniformly approximately additive function with respect to \( \varphi \) in the sense that
\[
\lim_{t \to \infty} N(f(x + y) - f(x) - f(y), t\varphi(x,y)) = 1
\]
uniformly on \( X \times X \). Then \( T(x) = N - \lim_{t \to \infty} \frac{f(2^n x)}{2^n} \) for all \( x \in X \) exists and defines an additive mapping \( T : X \to Y \) such that for some \( \delta > 0, \alpha > 0 \)
\[
N(f(x + y) - f(x) - f(y), \delta \varphi(x,y)) > \alpha,
\]
for all \( x,y \in X \), then
\[
N(T(x) - f(x), \delta/2 \varphi(x,x)) > \alpha,
\]
for every \( x \in X \).

**Corollary 2.7.** Let \( X \) be a linear space and \( (Y,N) \) be a fuzzy Banach space. Let \( \varphi : X \times X \to [0, \infty) \) be a control function and \( f : X \to Y \) be a uniformly approximately additive function with respect to \( \varphi \) in the sense that
\[
\lim_{t \to \infty} N(f(x + y) - f(x) - f(y), t\varphi(x,y)) = 1
\]
uniformly on \( X \times X \). Then there is a unique additive mapping \( T : X \to Y \) such that
\[
\lim_{t \to \infty} N(T(x) - f(x), t\varphi(x,x)) = 1,
\]
uniformly on \( X \).

**Theorem 2.8.** Let \( X \) be a linear space and let \( (Z,N') \) be a fuzzy normed space. Let \( \psi : X \times X \to Z \) be a function such that for some \( 0 < \alpha < 2 \),
\[
N'(\psi(2x, 2y), t) \geq N'(\alpha \psi(x,y), t)
\]
for all \( x,y \in X \) and \( t > 0 \). Let \( (Y,N) \) be a fuzzy Banach space and let \( f : X \to Y \) be a mapping in the sense that
N(f(x + y) − f(x) − f(y), t) ≥ N(ψ(x, y), t)
for each t > 0 and x, y ∈ X. Then there exists a unique additive mapping T : X → Y such that
\[ N(f(x) − T(x), t) \geq N(\frac{2\psi(x,x)}{2-\alpha}, t), \]
where x ∈ X and t > 0.

3. Stability and super stability of fuzzy approximately *-homomorphisms on a fuzzy Banach *-algebra in uniform version

We start our work with definition of fuzzy Banach *-algebra.

**Definition 3.1.** A fuzzy Banach *-algebra \( A \) is a *-algebra \( A \) with a fuzzy complete \( N \)-norm \( N \) such that \( N(a, t) = N(a^*, t) \) for all \( a ∈ A \).

Throughout this paper, let \( A_{sa} \) be the set of self-adjoint elements of \( A \) and \( U(A) \) the set of unitary elements in \( A \).

**Lemma 3.2.** Let \( X \) be a fuzzy normed *-algebra and \( N - \lim_{n→∞} x_n = x \). Then \( N - \lim_{n→∞} x_n^* = x^* \).

**Proof.** By Definition 2.3 we have \( \lim_{t→∞} N(x_n − x, t) = 1 \). So \( \lim_{t→∞} N(x_n^* − x^*, t) = \lim_{t→∞} N((x_n − x)^*, t) = 1 \). It means that \( N - \lim_{n→∞} x_n^* = x^* \). \( \square \)

**Theorem 3.3.** Let \( A \) be a fuzzy Banach *-algebra and let \( ϕ : A × A → [0, ∞) \) be a control function and suppose that \( f : A → A \) is a function such that
\[ \lim_{t→∞} N(f(μx + μy) − μf(x) − μf(y), tϕ(x, y)) = 1, \quad (3.1) \]
uniformly on \( A × A \),
\[ \lim_{t→∞} N(f(x^*) − f(x)^*, tϕ(x, x)) = 1, \quad (3.2) \]
uniformly on \( A \), and
\[ \lim_{t→∞} N(f(zw) − f(z)f(w), tϕ(z, w)) = 1, \quad (3.3) \]
uniformly on \( A × A \) for all \( μ ∈ T^1 = \{ λ ∈ ℂ : |λ| = 1 \} \), all \( z, w ∈ A_{sa} \), and all \( x, y ∈ A \). Then there exists a unique algebra *-homomorphism \( H : A → A \) such that
\[ \lim_{t→∞} N(H(x) − f(x), tϕ(x, x)) = 1 \quad (3.4) \]
uniformly on \( A \).

**Proof.** Put \( μ = 1 ∈ T^1 \). It follows from Theorem 2.6 and Corollary 2.7 that, there exists a unique additive mapping \( H : A → A \) such that the equality (3.4) holds. The additive mapping \( H : A → A \) is given by \( H(x) = N - \lim_{n→∞} \frac{1}{2^n} f(2^n x) \) for all \( x ∈ A \).

By the assumption we have,
\[ \lim_{t→∞} N(f(2^n μx) − 2μf(2^{n−1}x), tϕ(2^{n−1}x, 2^{n−1}x)) = 1, \]
for all $\mu \in T^1$ and all $x \in A$. We have
\[
N(\mu f(2^n x) - 2\mu f(2^{n-1} x), t\varphi(2^{n-1} x, 2^n x))
= N(f(2^n x) - 2f(2^{n-1} x), |\mu|^{-1} t\varphi(2^{n-1} x, 2^n x))
= N(f(2^n x) - 2f(2^{n-1} x), t\varphi(2^{n-1} x, 2^n x)),
\]
for all $\mu \in T^1$ and all $x \in A$. On the other hand
\[
N(f(2^n \mu x) - \mu f(2^n x), t\varphi(2^{n-1} x, 2^n x)) 
\geq \min \{N(f(2^n \mu x) - 2\mu f(2^{n-1} x), t/2\varphi(2^{n-1} x, 2^n x)),
N(2\mu f(2^{n-1} x) - \mu f(2^n x), t/2\varphi(2^{n-1} x, 2^n x))\},
\]
for all $\mu \in T^1$ and $x \in A$. Thus
\[
limit_{n \to \infty} N(f(2^n \mu x) - \mu f(2^n x), t\varphi(2^{n-1} x, 2^n x)) = 1.
\]
So
\[
limit_{n \to \infty} N(2^{-n} f(2^n \mu x) - 2^{-n} \mu f(2^n x), 2^{-n} t\varphi(2^{n-1} x, 2^n x)) = 1.
\]
Since $\lim_{n \to \infty} 2^{-n} t\varphi(2^{n-1} x, 2^n x) = 0$, there is some $n_0 > 0$ such that
\[
2^{-n} t\varphi(2^{n-1} x, 2^n x) < t,
\]
for all $n \geq n_0$ and $t > 0$. Hence
\[
N(2^{-n} f(2^n \mu x) - 2^{-n} \mu f(2^n x), t) \geq N(2^{-n} f(2^n \mu x) - 2^{-n} \mu f(2^n x), 2^{-n} t\varphi(2^{n-1} x, 2^n x)).
\]
Given $\varepsilon > 0$ we can find some $t_0 > 0$ such that
\[
N(2^{-n} f(2^n \mu x) - 2^{-n} \mu f(2^n x), 2^{-n} t\varphi(2^{n-1} x, 2^n x)) \geq 1 - \varepsilon,
\]
for all $x \in A$ and all $t \geq t_0$. So $N(2^{-n} f(2^n \mu x) - 2^{-n} \mu f(2^n x), t) = 1$ for all $t > 0$.
Hence by items (N5) and (N2) of definition 2.1 we have
\[
N - \lim_{n \to \infty} 2^{-n} f(2^n \mu x) = N - \lim_{n \to \infty} 2^{-n} \mu f(2^n x),
\]
for all $\mu \in T^1$ and all $x \in A$. Hence
\[
H(\mu x) = N - \lim_{n \to \infty} f(2^n \mu x) = N - \lim_{n \to \infty} \frac{\mu f(2^n x)}{2^n} = \mu H(x),
\]
for all $\mu \in T^1$ and all $x \in A$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and let $M$ be an integer greater than $4|\lambda|$. Then
\[
|\frac{\lambda}{2^n}| < 1/4 < 1/3. \text{ By (5), Theorem 1), there exist three elements } \mu_1, \mu_2, \mu_3 \in T^1
\]
such that $3\frac{\lambda}{2^n} = \mu_1 + \mu_2 + \mu_3$. We have $H(x) = H(3.1/3x) = 3H(1/3x)$ for all $x \in A$. So $H(1/3x) = 1/3H(x)$ for all $x \in A$. Thus
\[
H(\lambda x) = H(\frac{3\lambda}{3} \frac{1}{3} x) = MH(1/3.3 \frac{1}{3} x) = M/3H(\mu_1 x + \mu_2 x + \mu_3 x)
= M/3H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)) = M/3(\mu_1 + \mu_2 + \mu_3)H(x) = \frac{M}{3} \lambda H(x) = \lambda H(x),
\]
for all $x \in A$. Hence
\[
H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y),
\]
for all $\zeta, \eta \in \mathbb{C}$ ($\zeta, \eta \neq 0$) and all $x, y \in A$, and $H(0x) = 0H(x)$ for all $x \in A$.
So the unique additive mapping $H : A \to A$ is a $\mathbb{C}$-linear mapping.

By using (3.2) we have
\[
\lim_{n \to \infty} N(2^{-n} f(2^n x^*) - 2^{-n} f(2^n x^*), 2^{-n} t\varphi(x, x)) = 1.
\]
Since $\lim_{n \to \infty} 2^{-n} t\varphi(x, x) = 0$, there is some $n_0 > 0$ such that $2^{-n} t\varphi(x, x) < t$
for all $n \geq n_0$ and $t > 0$. Hence
\[
N(2^{-n} f(2^n x^*) - 2^{-n} f(2^n x^*), t) \geq N(2^{-n} f(2^n x^*) - 2^{-n} f(2^n x^*), 2^{-n} t\varphi(x, x)).
\]
Given $\varepsilon > 0$ we can find some $t_0 > 0$ such that
\[ N(2^{-n}f(2^n x^*) - 2^{-n}f(2^n x^*), 2^{-n}t\varphi(x, x)) \geq 1 - \varepsilon, \]
for all $x \in A$ and all $t \geq t_0$. So $N(2^{-n}f(2^n x^*), 2^{-n}f(2^n x^*), t) = 1$ for all $t > 0$.

Hence by items (N5) and (N2) of Definition 2.1 we have
\[ N - \lim_{n \to \infty}(2^{-n}f(2^n x^*)) = N - \lim_{n \to \infty}2^{-n}f(2^n x^*). \quad (3.5) \]

By (3.5) and Lemma 3.2, we get
\[ H(x^*) = N - \lim_{n \to \infty}f(2^n x^*) = N - \lim_{n \to \infty}(f(2^n x)^*) = (N - \lim_{n \to \infty}f(2^n x))^* = H(x)^*, \]
for all $x \in A$.

Now it follows from (3.3) that
\[ \lim_{n \to \infty}N(4^{-n}f(2^{-n}z2^{-n}w) - 4^{-n}f(2^{-n}z)f(2^{-n}w), 4^{-n}t\varphi(2^{-n}z, 2^{-n}w)) = 1. \]

Since $\lim_{n \to \infty}4^{-n}t\varphi(2^{-n}z, 2^{-n}w) = 0$, there is some $n_0 > 0$ such that
\[ 4^{-n}t\varphi(2^{-n}z, 2^{-n}w) < t, \]
for all $n \geq n_0$ and $t > 0$. Hence
\[ N(4^{-n}f(2^{-n}z2^{-n}w) - 4^{-n}f(2^{-n}z)f(2^{-n}w), 4^{-n}t\varphi(2^{-n}z, 2^{-n}w)). \]

Given $\varepsilon > 0$ we can find some $t_0 > 0$ such that
\[ N(4^{-n}f(2^{-n}z2^{-n}w) - 4^{-n}f(2^{-n}z)f(2^{-n}w), 4^{-n}t\varphi(2^{-n}z, 2^{-n}w)) \geq 1 - \varepsilon, \]
for all $x \in A$ and all $t \geq t_0$. So $N(4^{-n}f(2^{-n}z2^{-n}w) - 4^{-n}f(2^{-n}z)f(2^{-n}w), t) = 1$
for all $t > 0$. Hence by items (N5) and (N2) of definition 2.1 we have
\[ N - \lim_{n \to \infty}4^{-n}f(2^{-n}z2^{-n}w) = N - \lim_{n \to \infty}4^{-n}f(2^{-n}z)f(2^{-n}w), \]
for all $z, w \in A_{sa}$; but $\sum_{j=0}^{\infty}4^{-j}\varphi(2^j z, 2^j w) \leq \sum_{j=0}^{\infty}2^{-j}\varphi(2^j z, 2^j w)$ for all $z, w \in A_{sa}$. So
\[ H(zw) = N - \lim_{n \to \infty}f(2^n zw) = N - \lim_{n \to \infty}f(2^n z)f(2^n w) = N - \lim_{n \to \infty}f(2^n z), \]
for all $z, w \in A_{sa}$.

For elements $x, y \in A, x = \frac{x + x^* + i\bar{x} - x^*}{2}$ and $y = \frac{y + y^* + i\bar{y} - y^*}{2}$, where $x_1 = \frac{x + x^*}{2}$, $x_2 = \frac{x - x^*}{2}, y_1 = \frac{y + y^*}{2}$ and $y_2 = \frac{-y + y^*}{2}$ are self-adjoint. Since $H$ is $C$-linear, $H(xy) = H(x_1y_1 - x_2y_2 + i(x_1y_2 + x_2y_1)) = H(x_1y_1) - H(x_2y_2) + iH(x_1y_2) + iH(x_2y_1)$, for all $x, y \in A$. Hence the additive mapping $H$ is an algebra $*$-homomorphism satisfying the inequality (3.4), as desired.

The proof of the uniqueness property of $H$ is similar to the proof of Corollary 2.7.
Corollary 3.4. Let $A$ be a fuzzy Banach $^*$-algebra, $\theta \geq 0$ and $q > 0$, $q \neq 1$. Suppose that $f : A \to A$ is a function such that

$$\lim_{t \to \infty} N(f(\mu x + \mu y) - \mu f(x) - \mu f(y), t\theta(||x||^q + ||y||^q)) = 1,$$

(3.6)

uniformly on $A \times A$,

$$\lim_{t \to \infty} N(f(x^*) - f(x)^*, 2t\theta||x||^q) = 1,$$

(3.7)

uniformly on $A$, and

$$\lim_{t \to \infty} N(f(zw) - f(z)f(w), t\theta(||z||^q + ||w||^q)) = 1,$$

(3.8)

uniformly on $A \times A$ for all $\mu \in T^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, all $z, w \in A_{sa}$, and all $x, y \in A$. Then there exists a unique algebra $^*$-homomorphism $H : A \to A$ such that

$$\lim_{t \to \infty} N(H(x) - f(x), \frac{2\theta t||x||^q}{1 - 2^{q-1}}) = 1,$$

(3.9)

uniformly on $A$.

Proof. Considering the control function $\varphi(x, y) = \theta(||x||^q + ||y||^q)$ for some $\theta > 0$, we obtain this corollary. □

In the following example we will show that Corollary 3.4 does not necessarily hold for $q = 1$.

Example 3.5. Let $X$ be a Banach $^*$-algebra, $x_0 \in X$ and $\alpha, \beta$ are real numbers such that $|\alpha| \geq 1 - (||x|| + ||y||)$ and $|\beta| \leq ||x|| + ||y||$ for every $x, y \in X$. Put

$f(x) = \alpha x + \beta x_0||x||$, $(x \in X)$.

Moreover for each fuzzy norm $N$ on $X$, we have

$N(f(x + y) - f(x) - f(y), t(||x|| + ||y||))$

$= N(\beta x_0(||x + y|| - ||x|| - ||y||), t(||x|| + ||y||))$

$= N(\beta x_0, \frac{t(||x|| + ||y||)}{||x + y|| - ||x|| - ||y||}) \geq N(\beta x_0, t) (x, y \in X, t \in \mathbb{R}).$

Therefore by the item (N5) of the Definition 2.1, we get

$$\lim_{t \to \infty} N(f(x + y) - f(x) - f(y), t(||x|| + ||y||)) = 1,$$

uniformly on $X \times X$. 
Also
\[ N(f(xy) - f(x)f(y), t(||x|| + ||y||)) = N(\alpha xy + \beta x_0||xy|| - (\alpha x + \beta x_0||x||)(\alpha y + \beta x_0||y||), t(||x|| + ||y||)) \]
\[ = N(\alpha xy + \beta x_0||xy|| - \alpha^2 xy - \alpha \beta x_0||y|| - \alpha \beta x_0 y||x|| - \beta^2 x_0^2||x||y||, t(||x|| + ||y||)) \]
\[ \geq \min\{N((1 - \alpha)\alpha xy, t(||x|| + ||y||)/5), N(||xy||\beta x_0, t(||x|| + ||y||)/5), N(\beta^2 x_0^2||x||y||, t(||x|| + ||y||)/5), N(\alpha \beta x_0 y||x||, t(||x|| + ||y||)/5)\} \]

where \( x \in X \) and \( t \in \mathbb{R} \).

Taking into account the following inequalities

\[ N((1 - \alpha)\alpha xy, t(||x|| + ||y||)/5) = N(\alpha xy, t(||x|| + ||y||)/5(1 - \alpha)) \geq N(\alpha xy, t/5), \quad (3.10) \]
\[ N(||xy||\beta x_0, t(||x|| + ||y||)/5) = N(||xy||x_0, t(||x|| + ||y||)/5|\beta|) \geq N(||xy||x_0, t/5), \quad (3.11) \]
\[ N(\beta^2 x_0^2||x||y||, t(||x|| + ||y||)/5) = N(\beta||x||y||x^2_0, t/5|\beta|) \geq N(\beta||x||y||x^2_0, t/5), \quad (3.12) \]
\[ N(\alpha \beta x_0 y||x||, t(||x|| + ||y||)/5) = N(\alpha xx_0||y||, t(||x|| + ||y||)/5|\beta|) \geq N(\alpha xx_0||y||, t/5), \quad (3.13) \]
\[ N(\alpha \beta x_0 y||x||, t(||x|| + ||y||)/5) = N(\alpha x_0 y||x||, t(||x|| + ||y||)/5|\beta|) \geq N(\alpha x_0 y||x||, t/5), \quad (3.14) \]
it can be easily seen that \( \lim_{t\to\infty} N(f(xy) - f(x)f(y), t(||x|| + ||y||)) = 1 \) uniformly on \( X \times X \).

Also we have

\[ N(f(x^*) - f(x)^*, 2t||x||) = N(\alpha x^* - \alpha x^* + \beta x_0 ||x^*|| - \beta x_0^* ||x||, 2t||x||) \]
\[ \geq \min\{N(\beta x_0, 2t||x||/||x^*||), N(\beta x_0^*, 2t||x||/||x||)\}. \]
So \( \lim_{t \to \infty} N(f(x^*) - f(x)^*, 2t|x||) = 1 \) uniformly on \( X \) and therefore the conditions of Corollary 3.4 are fulfilled.

Now we suppose that there exists a unique \(*\)-homomorphism \( H \) satisfying the conditions of Corollary 3.4. By the equation

\[
\lim_{t \to \infty} N(f(x + y) - f(x) - f(y), t(||x|| + ||y||)) = 1, 
\]

for given \( \varepsilon > 0 \), we can find some \( t_0 > 0 \) such that

\[
N(f(x + y) - f(x) - f(y), t(||x|| + ||y||)) \geq 1 - \varepsilon, 
\]

for all \( x, y \in X \) and all \( t \geq t_0 \). By using the simple induction on \( n \), we shall show that

\[
N(f(2^n x) - 2^n f(x), tn2^n||x||) \geq 1 - \varepsilon. 
\]

Putting \( y = x \) in (3.15), we get (3.16) for \( n = 1 \). Let (3.16) holds for some positive integer \( n \). Then

\[
N(f(2^{n+1} x) - 2^{n+1} f(x), t(n + 1)2^{n+1}||x||) \\
\geq \min\{N(f(2^{n+1} x) - 2f(2^n x), t(||2^n x|| + ||2^n x||)), \\
N(2f(2^n x) - 2^{n+1} f(x), 2tn(||2^{n-1} x|| + ||2^{n-1} x||)) \\
\geq 1 - \varepsilon. 
\]

This completes the induction argument. We observe that

\[
\lim_{n \to \infty} N(H(x) - f(x), nt||x||) \geq 1 - \varepsilon. 
\]

Hence

\[
\lim_{n \to \infty} N(H(x) - f(x), nt||x||) = 1. 
\]

One may regard \( N(x, t) \) as the truth value of the statement 'the norm of \( x \) is less than or equal to the real number \( t \). So (3.17) is a contradiction with the non-fuzzy sense. This means that there is no such the \( H \).

**Theorem 3.6.** Let \( A \) be a \( C^* \)-algebra and let \( f : A \to A \) be a bijective mapping satisfying \( f(xy) = f(x)f(y) \) and \( f(0) = 0 \) for which there exists function \( \varphi : A \times A \to [0, \infty) \) satisfying (3.1) and (3.3) such that

\[
\lim_{t \to \infty} N(f(u^*) - f(u)^*, t\varphi(u, u)) = 1, 
\]

for all \( u \in U(A) \). Assume that \( N - \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \) is invertible, where \( e \) is the identity of \( A \). Then the bijective mapping \( f \) is a bijective \(*\)-homomorphism.

**Proof.** By the same reasoning as in the proof of Theorem 3.3 there exists a unique \( \mathbb{C} \)-linear mapping \( H : A \to A \) such that

\[
\lim_{t \to \infty} N(H(x) - f(x), t\varphi(x, x)) = 1, 
\]

for all \( x \in A \). The \( \mathbb{C} \)-linear mapping \( H : A \to A \) is given by

\[
H(x) = N - \lim_{n \to \infty} \frac{f(2^n x)}{2^n}, 
\]
for all \( x \in A \).

By using (3.18) we have
\[
\lim_{t \to \infty} N(2^{-n}f(2^n u^*) - 2^{-n} f(2^n u^*), 2^{-n}t \varphi(u, u)) = 1.
\]
Since \( \lim_{n \to \infty} 2^{-n}t \varphi(u, u) = 0 \), there is some \( n_0 > 0 \) such that \( 2^{-n}t \varphi(u, u) < t \) for all \( n \geq n_0 \) and \( t > 0 \). Hence
\[
N(2^{-n}f(2^n u^*) - 2^{-n} f(2^n u^*), t) \geq N(2^{-n}f(2^n u^*) - 2^{-n} f(2^n u^*), 2^{-n}t \varphi(u, u)).
\]

Given \( \varepsilon > 0 \) we can find some \( t_0 > 0 \) such that
\[
N(2^{-n}f(2^n u^*) - 2^{-n} f(2^n u^*), 2^{-n}t \varphi(u, u)) \geq 1 - \varepsilon,
\]
for all \( x \in A \) and all \( t \geq t_0 \). So \( N(2^{-n} f(2^n u^*) - 2^{-n} f(2^n u^*), t) = 1 \) for all \( t > 0 \).

Hence by items (N5) and (N2) of definition 2.1 we have
\[
N - \lim_{n \to \infty} (2^{-n} f(2^n u^*)) = N - \lim_{n \to \infty} 2^{-n} f(2^n u^*). \tag{3.20}
\]

By (3.20) and Lemma 3.2, we get
\[
H(u^*) = N - \lim_{n \to \infty} \frac{(f(2^n u^*))}{2^n} = N - \lim_{n \to \infty} \frac{(f(2^n u^*))^*}{2^n} = H(u^*),
\]
for all \( u \in U(A) \).

Since \( H \) is \( \mathbb{C} \)-linear and each \( x \in A \) is a finite linear combination of unitary elements [6],
\[
H(x^*) = H(\sum_{j=1}^{m} \lambda_j u_j^*) = \sum_{j=1}^{m} \lambda_j H(u_j^*) = \sum_{j=1}^{m} \lambda_j H(u_j)^* = (\sum_{j=1}^{m} \lambda_j H(u_j))^* = H(\sum_{j=1}^{m} \lambda_j u_j)^* = H(x)^*,
\]
for all \( x \in A \).

Since \( f(xy) = f(x)f(y) \) for all \( x, y \in A \),
\[
H(xy) = N - \lim_{n \to \infty} f(2^n xy) = N - \lim_{n \to \infty} \frac{f(2^n x)f(y)}{2^n} = H(x)f(y), \tag{3.21}
\]
for all \( x, y \in A \). By the additivity of \( H \) and (3.21),
\[
2^n H(xy) = H(2^n xy) = H(x(2^n y)) = H(x)f(2^n y),
\]
for all \( x, y \in A \). Hence
\[
H(xy) = \frac{H(x)f(2^n y)}{2^n} = H(x)\frac{f(2^n y)}{2^n}, \tag{3.22}
\]
for all \( x, y \in A \). Taking the \( N \)-limit in (3.22) as \( n \to \infty \), we obtain
\[
H(xy) = H(x)H(y),
\]
for all \( x, y \in A \). By (3.21) we have,
\[
H(x) = H(ex) = H(e)f(x), \tag{3.23}
\]
for all \( x \in A \). Since \( H(e) = N - \lim_{n \to \infty} \frac{2^n}{2^n} \) is invertible and the mapping \( f \) is bijective, the \( \mathbb{C} \)-linear mapping \( H \) is a bijective *-homomorphism.

Now we have,
\[
H(e)H(x) = H(ex) = H(x) = H(e)f(x),
\]
for all \( x \in A \). Since \( H(e) \) is invertible, \( H(x) = f(x) \) for all \( x \in A \). Hence the bijective mapping \( f \) is a bijective *-homomorphism. \( \square \)
4. Non-uniform type of Stability and super stability of fuzzy approximately *-homomorphisms

We are in a position to give non-uniform type of Theorems 3.3 and 3.6.

**Theorem 4.1.** Let $(B, N')$ be a fuzzy normed algebra, $A$ a fuzzy Banach *-algebra and let $\varphi : A \times A \to B$ be a function such that for some $0 < \alpha < 2$,

$$N'(\varphi(2x, 2y), t) \geq N'(\varphi(x, y), t)$$

for all $x, y \in A$ and $t > 0$. Let $f : A \to A$ be a function such that

$$N(f(\mu x + \mu y) - \mu f(x) - \mu f(y), t) \geq N'(\varphi(x, y), t),$$

for all $x, y \in A$,

$$N(f(x^*) - f(x)^*, t) \geq N'(\varphi(x, x), t), \quad (4.1)$$

for all $x \in A$ and

$$N(f(zw) - f(z)f(w), t) \geq N'(\varphi(z, w), t), \quad (4.2)$$

for all $t > 0$, all $\mu \in T^1 = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$, and all $z, w \in A_{sa}$. Then there exists a unique algebra *-homomorphism $H : A \to A$ such that

$$N(H(x) - f(x), t) \geq N'(\frac{2\varphi(x, x)}{2 - \alpha}, t)$$

for all $x \in A$ and all $t > 0$.

**Proof.** Theorem 2.8 shows that there exists an additive function $H : A \to A$ such that

$$N(f(x) - T(x), t) \geq N'(\frac{2\varphi(x, x)}{2 - \alpha}, t),$$

where $x \in A$ and $t > 0$.

Put $\mu = 1 \in T^1$. The additive mapping $H : A \to A$ is given by $H(x) = N - \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$ for all $x \in A$.

By assumption for each $\mu \in T^1$,

$$N(f(2^n \mu x) - 2\mu f(2^{n-1} x), t) \geq N^{n-1} x, 2^{n-1} x, t),$$

for all $x \in A$. We have

$$N(\mu f(2^n x) - 2\mu f(2^{n-1} x), t) = N(f(2^n x) - 2f(2^{n-1} x), |\mu|^{-1} t) = N(f(2^n x) - 2f(2^{n-1} x), t) \geq N^{n-1} x, 2^{n-1} x, t),$$

for all $\mu \in T^1$ and all $x \in A$. So

$$N(f(2^n \mu x) - \mu f(2^n x), t) \geq \min\{N(f(2^n \mu x) - 2\mu f(2^{n-1} x), t/2),$$

$$N(2\mu f(2^{n-1} x) - \mu f(2^n x), t/2) \} \geq N^n x, 2^n x, t/2),$$

for all $\mu \in T^1$ and all $x \in A$. Taking $n$ to infinity in (4.3) and using the items (N2) and (N5) of Definition 2.1, we see that

$$N - \lim_{n \to \infty} 2^{-n} f(2^n x) = N - \lim_{n \to \infty} 2^{-n} \mu f(2^n x),$$

for all $\mu \in T^1$ and all $x \in A$.

Now by using the similar proof of the Theorem 3.3 the unique additive mapping $H : A \to A$ is a C-linear mapping.

By using (4.1) we have
As same as the proof of the Theorems 3.6 and 4.1, we can prove this Theo-
Proof.

\begin{equation}
N(2^{-n} f(2^n x) - 2^{-n} f(2^n x)^*, t) \geq N''(x, 2^n x, 2^n t),
\end{equation}
for all \( x \in A \). Taking \( n \) to infinity in (4.4) and using the items (N2) and (N5) of Definition 2.1, we see that
\( N - \lim_{n \to \infty} 2^{-n} f(2^n x^*) = N - \lim_{n \to \infty} 2^{-n} f(2^n x)^* \).

Again by using the similar proof of the Theorem 3.3 we have \( H(x^*) = H(x)^* \).

Now it follows from (4.2) that
\begin{equation}
N(4^{-n} f(2^{-n} z 2^{-n} w) - 4^{-n} f(2^{-n} z) f(2^{-n} w), 4^n t).
\end{equation}
for all \( z, w \in A_{sa} \). Taking \( n \) to infinity in (4.5) and using the items (N2) and (N5) of Definition 2.1, we see that
\( N - \lim_{n \to \infty} 4^{-n} f(2^{-n} z 2^{-n} w) = N - \lim_{n \to \infty} 4^{-n} f(2^{-n} z) f(2^{-n} w), \)
for all \( z, w \in A_{sa} \). By the proof of Theorem 3.3, \( H \) is a *-homomorphism as desired.

To prove the uniqueness property of \( H \), assume that \( H^* \) is another *-homomorphism satisfying \( N(f(x) - H^*(x), t) \geq N'(\frac{2\varphi(x,x)}{2-a}, t) \). Since both \( H \) and \( H^* \) are additive we deduce that
\( N(H(a) - H^*(a), t) \geq \min\{N(H(a)-n^{-1} f(na), t/2), N(n^{-1} f(na) - H^*(a), t/2)\} \geq \)
\( N'(\frac{2\varphi(na,na)}{2-a}, nt/2) \)
for all \( a \in A \) and all \( t > 0 \). Letting \( n \) tend to infinity we get that \( H(a) = H^*(a) \)
for all \( a \in A \).

\[ \Box \]

**Theorem 4.2.** Let \( A \) be a C*-algebra, \((B, N')\) a fuzzy normed algebra and let \( \varphi : A \times A \to B \) be a function such that for some \( 0 < \alpha < 2 \),
\( N'(\varphi(2x,2y), t) \geq N'(\varphi(x,y), t) \)
for all \( x, y \in A \) and \( t > 0 \). Let \( f : A \to A \) be a bijective mapping satisfying \( f(xy) = f(x)f(y) \) and \( f(0) = 0 \) such that
\( N(f(\mu x + \mu y) - \mu f(x) - \mu f(y), t) \geq N'(\varphi(x,y), t) \),
and
\( \lim_{t \to \infty} N(f(u^*) - f(u)^*, t \varphi(u, u)) = 1, \)
for all \( x, y \in A \) and \( u \in U(A) \). Assume that \( N - \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \) is invertible, where \( e \) is the identity of \( A \). Then the bijective mapping \( f \) is a bijective *-homomorphism.

**Proof.** As same as the proof of the Theorems 3.6 and 4.1, we can prove this Theorem.

\[ \Box \]

**References**


Address: Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran.
E-mail: nasrineghbali@gmail.com, eghbali@uma.ac.ir

---

Başlık: Bulanık yaklaşıklık *-homomorfizmin kararlılığı ve süper kararlılığı
Anahtar Kelimeler: Bulanık normlu uzay, yaklaşıklık *-homomorfizm, kararlılık