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PARALLEL SURFACES TO TRANSLATION SURFACES IN EUCLIDEAN 3-SPACE

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Abstract. In this paper, we investigate geometric properties of surfaces that are parallel to translation surfaces in 3-dimensional Euclidean space which are constructed by generator curves with constant curvatures and torsions are given.

1. Introduction

The theory of translation surfaces is always one of interesting topics in Euclidean space. Translation surfaces have been investigated by some differential geometers. Verstraelen et al. investigated the minimal translation surfaces in n-dimensional Euclidean spaces [5]. Liu gave the classification of the translation surfaces with constant mean curvature or constant Gauss curvature in 3-dimensional Euclidean space and 3-dimensional Minkowski space [3]. Yoon studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map satisfies the condition where denotes the Laplacian of the surface with respect to the induced metric and the set of real metrics [6]. Munteanu and Nistor studied the second fundamental form of translation surfaces in [4]. They gave a non-existence result for polynomial translation surfaces in with vanishing second Gauss curvature. They classified those translation surfaces for which and are proportional. Çetin et al. investigated the translation surfaces in 3-dimensional Euclidean space by using non-planar space curves and they gave the differential geometric properties for both translation surfaces and minimal translation surfaces [1].

In this paper, we investigate geometric properties of surfaces that are parallel to translation surfaces in 3-dimensional Euclidean space which are constructed by generator curves with constant curvatures and torsions are given.

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2. Parallel Surfaces to Translation Surfaces in Euclidean 3-Space

Let $\alpha(s)$ be a space curve with Frenet frame $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\kappa_\alpha,\ \tau_\alpha$ be curvature and torsion of the curve α respectively, and let $\beta(t)$ be a space curve with Frenet frame $\{T_\beta, N_\beta, B_\beta\}$ and $\kappa_\beta,\ \tau_\beta$ be curvature and torsion of the curve β respectively. Let $M(s,t) = \alpha(s) + \beta(t)$ be a translation surface in Euclidean 3-Space that is generated by two space curves with constant curvatures and torsions. Then we can write the equation of parallel surface to translation surface M as follow.

$$\overline{M}(s,t) = M(s,t) + \lambda U(s,t) \tag{2.1}$$

where $\lambda \in \mathbb{R}$ $(\lambda \neq 0)$ and $U(s,t)$ is the unit normal vector of M, and $U(s,t)$ can be written as follow.

$$U = \frac{T_\alpha \wedge T_\beta}{\sin\varphi} \tag{2.2}$$

where $\varphi(s)$ is the angle between tangent vector fields of α and β. Here, because of being easy the angle of φ has been constant.

Let α and β are space curves with non-zero curvature and torsion.

Differentiating (2.2) with respect to s, we get

$$U_s = \frac{\kappa_\alpha}{\sin\varphi}\left(N_\alpha \wedge T_\beta\right),$$

and

$$U_{ss} = -\frac{\kappa_\alpha^2}{\sin\varphi}\left(T_\alpha \wedge T_\beta\right) + \frac{\kappa_\alpha \tau_\alpha}{\sin\varphi}\left(B_\alpha \wedge T_\beta\right).$$

Similarly, differentiating (2.2) with respect to t, we get

$$U_t = \frac{\kappa_\beta}{\sin\varphi}\left(T_\alpha \wedge N_\beta\right),$$

and

$$U_{tt} = -\frac{\kappa_\beta^2}{\sin\varphi}\left(T_\alpha \wedge T_\beta\right) + \frac{\kappa_\beta \tau_\beta}{\sin\varphi}\left(T_\alpha \wedge B_\beta\right).$$

Also,

$$U_{st} = U_{ts} = \frac{\kappa_\alpha \kappa_\beta}{\sin\varphi}\left(N_\alpha \wedge N_\beta\right).$$

Differentiating (2.1) with respect to s, we get,

$$\overline{M_s} = T_\alpha + \frac{\lambda\kappa_\alpha}{\sin\varphi}\left(N_\alpha \wedge T_\beta\right)$$

$$\overline{M_{ss}} = \kappa_\alpha N_\alpha - \frac{\lambda\kappa_\alpha^2}{\sin\varphi}\left(T_\alpha \wedge T_\beta\right) + \frac{\lambda\kappa_\alpha\tau_\alpha}{\sin\varphi}\left(B_\alpha \wedge T_\beta\right).$$

And differentiating (2.1) with respect to t, we get

$$\overline{M_t} = T_\beta + \frac{\lambda\kappa_\beta}{\sin\varphi}\left(T_\alpha \wedge N_\beta\right)$$

$$\overline{M_{tt}} = \kappa_\beta N_\beta - \frac{\lambda \kappa_\beta^2}{\sin\varphi}\left(T_\alpha \wedge T_\beta\right) + \frac{\lambda \kappa_\beta \tau_\beta}{\sin\varphi}\left(T_\alpha \wedge B_\beta\right).$$

Also,

$$\overline{M_{st}} = \overline{M_{ts}} = \frac{\lambda \kappa_\alpha \kappa_\beta}{\sin\varphi}\left(N_\alpha \wedge N_\beta\right).$$

We can give following equalities.

$$\begin{array}{lll} \langle T_\alpha, T_\beta\rangle = A_1 & \langle T_\alpha, N_\beta\rangle = A_2 & \langle T_\alpha, B_\beta\rangle = A_3 \\ \langle N_\alpha, T_\beta\rangle = A_4 & \langle N_\alpha, N_\beta\rangle = A_5 & \langle N_\alpha, B_\beta\rangle = A_6 \\ \langle B_\alpha, T_\beta\rangle = A_7 & \langle B_\alpha, N_\beta\rangle = A_8 & \langle B_\alpha, B_\beta\rangle = A_9 \end{array}.$$

Also,

$$U = \cos\theta_\alpha N_\alpha + \sin\theta_\alpha B_\alpha \tag{2.3}$$

and

$$U = \cos\theta_\beta N_\beta + \sin\theta_\beta B_\beta. \tag{2.4}$$

can be written. From (2.4), we get

$$\cos\theta_\beta A_2 + \sin\theta_\beta A_3 = 0. \tag{2.5}$$

By using (2.3), we obtain

$$\cos\theta_\alpha A_4 + \sin\theta_\alpha A_7 = 0. \tag{2.6}$$

From (2.4), we get

$$U \wedge T_\beta = \sin\theta_\beta N_\beta - \cos\theta_\beta B_\beta. \tag{2.7}$$

From (2.3) and (2.7), we get

$$\cos\theta_\alpha \sin\theta_\beta A_5 - \cos\theta_\alpha \cos\theta_\beta A_6 + \sin\theta_\alpha \sin\theta_\beta A_8 - \sin\theta_\alpha \cos\theta_\beta A_9 = 0. \tag{2.8}$$

By using (2.3), we get

$$U \wedge T_\alpha = \sin\theta_\alpha N_\alpha - \cos\theta_\alpha B_\alpha. \tag{2.9}$$

From (2.4) and (2.9), we get

$$\sin\theta_\alpha \cos\theta_\beta A_5 + \sin\theta_\alpha \sin\theta_\beta A_6 - \cos\theta_\alpha \cos\theta_\beta A_8 - \cos\theta_\alpha \sin\theta_\beta A_9 = 0. \tag{2.10}$$

Taking the inner product of the U and U, we get

$$\cos\theta_\alpha \cos\theta_\beta A_5 + \cos\theta_\alpha \sin\theta_\beta A_6 + \sin\theta_\alpha \cos\theta_\beta A_8 + \sin\theta_\alpha \sin\theta_\beta A_9 = 1. \tag{2.11}$$

From (2.7), we obtain

$$\sin\theta_\beta A_2 - \cos\theta_\beta A_3 = -\sin\varphi. \tag{2.12}$$

By using (2.9), we get

$$\langle U \wedge T_\alpha, T_\beta\rangle = \sin\theta_\alpha A_4 - \cos\theta_\alpha A_7 = \sin\varphi. \tag{2.13}$$

From (2.7) and (2.9), we get

$$\sin\theta_\alpha \sin\theta_\beta A_5 - \sin\theta_\alpha \cos\theta_\beta A_6 - \cos\theta_\alpha \sin\theta_\beta A_8 + \cos\theta_\alpha \cos\theta_\beta A_9 = \cos\varphi. \tag{2.14}$$

Also, $\langle T_\alpha, T_\beta \rangle = \cos\varphi$, and thus we write

$$A_1 = \cos\varphi. \tag{2.15}$$

If the equation system consisting of (2.5), (2.6), (2.8), (2.10), (2.11), (2.12), (2.13), (2.14) and (2.15) is solved we will get following equalities.

$$\begin{aligned}
A_1 &= \cos\varphi, \quad A_2 = -\sin\theta_\beta \sin\varphi, \quad A_3 = \cos\theta_\beta \sin\varphi \\
A_4 &= \sin\theta_\alpha \sin\varphi, \quad A_5 = \cos\theta_\alpha \cos\theta_\beta + \sin\theta_\alpha \sin\theta_\beta \cos\varphi \\
A_6 &= \cos\theta_\alpha \sin\theta_\beta - \sin\theta_\alpha \cos\theta_\beta \cos\varphi, \quad A_7 = -\cos\theta_\alpha \sin\varphi \\
A_8 &= \sin\theta_\alpha \cos\theta_\beta - \cos\theta_\alpha \sin\theta_\beta \cos\varphi \\
A_9 &= \sin\theta_\alpha \sin\theta_\beta + \cos\theta_\alpha \cos\theta_\beta \cos\varphi
\end{aligned}$$

The first fundamental form $\overline{I}$ of the surface $\overline{M}$ is

$$\overline{I} = \overline{E} ds^2 + 2\overline{F} ds dt + \overline{G} dt^2$$

where $\overline{E}, \overline{F}$ and $\overline{G}$ are the coefficients of the $\overline{I}$ and

$$\begin{aligned}
\overline{E} &= 1 - 2\lambda\kappa_n^\alpha + \lambda^2 \left(\kappa_n^\alpha\right)^2 + \lambda^2 \left(k_1^\alpha\right)^2 \cot^2\varphi \\
\overline{F} &= \cos\varphi - \frac{\lambda^2 \kappa_n^\alpha \kappa_n^\beta \cot\varphi}{\sin\varphi} - \lambda^2 \kappa_g^\alpha \kappa_g^\beta \cot^2\varphi \\
\overline{G} &= 1 - 2\lambda\kappa_n^\beta + \lambda^2 \left(\kappa_n^\beta\right)^2 + \lambda^2 \left(k_1^\beta\right)^2 \cot^2\varphi
\end{aligned}$$

Let $\overline{U}$ be the unit normal of the surface $\overline{M}$. Then, it can be written as follow.

$$\overline{U} = \mu \begin{pmatrix} -\lambda\kappa_g^\beta T_\alpha + \left(\frac{\lambda^2 k_1^\alpha \kappa_n^\beta}{\sin\varphi} - \frac{\lambda k_1^\alpha}{\sin\varphi} \right) N_\alpha \\ + \left(\lambda\kappa_g^\alpha - \frac{\lambda^2 \kappa_g^\alpha \kappa_n^\beta}{\sin^2\varphi} + \frac{\lambda^2 \kappa_n^\alpha \kappa_g^\beta \cot\varphi}{\sin\varphi} \right) T_\beta \\ -\frac{\lambda k_1^\beta}{\sin\varphi} N_\beta + (T_\alpha \wedge T_\beta) \end{pmatrix}$$

where

$$\frac{1}{\mu} = \begin{pmatrix} -2\lambda\kappa_n^\alpha - 2\lambda\kappa_n^\beta - \lambda^2 \left(\kappa_g^\alpha\right)^2 - \lambda^2 \left(\kappa_g^\beta\right)^2 - 2\lambda^2 \kappa_g^\alpha \kappa_g^\beta \cos\varphi \\ -2\lambda^3 \kappa_n^\alpha \left(\kappa_g^\beta\right)^2 \cot^2\varphi + 2\lambda^3 \left(\kappa_g^\alpha\right)^2 \kappa_n^\beta + 2\lambda^2 \kappa_n^\alpha \kappa_n^\beta + \sin^2\varphi \\ + \frac{\lambda^4 \left(\kappa_g^\alpha\right)^2 \left(\kappa_n^\beta\right)^2}{\sin^4\varphi} - \frac{2\lambda^4 \kappa_n^\alpha \kappa_g^\alpha \kappa_n^\beta \kappa_g^\beta \cot^3\varphi}{\sin\varphi} + \frac{\lambda^4 (k_1^\alpha)^2 \left(\kappa_n^\beta\right)^2}{\sin^2\varphi} - \frac{2\lambda^3 (k_1^\alpha)^2 \kappa_n^\beta}{\sin^2\varphi} \\ + \frac{\lambda^2 (k_1^\alpha)^2}{\sin^2\varphi} - \frac{2\lambda^4 \left(\kappa_g^\alpha\right)^2 \left(\kappa_n^\beta\right)^2}{\sin^2\varphi} - \frac{2\lambda^3 \kappa_n^\alpha \left(\kappa_n^\beta\right)^2}{\sin^2\varphi} + \frac{2\lambda^2 \kappa_n^\alpha \kappa_n^\beta}{\sin^2\varphi} + \frac{\lambda^2 \left(k_1^\beta\right)^2}{\sin^2\varphi} \\ + \frac{\lambda^4 \left(\kappa_g^\alpha\right)^2 \left(\kappa_g^\beta\right)^2 \cot^2\varphi}{\sin^2\varphi} + \frac{2\lambda^2 \kappa_g^\alpha \kappa_g^\beta \cot\varphi}{\sin\varphi} \end{pmatrix}^{\frac{1}{2}}.$$

The second fundamental form $\overline{II}$ of the surface $\overline{M}$ is

$$\overline{II} = \bar{l} ds^2 + 2\overline{m} ds dt + \overline{n} dt^2$$

where $\bar{l}, \overline{m}$ and $\overline{n}$ are the coefficients of the $\overline{II}$ and

$$\bar{l} = \mu \begin{pmatrix} -\frac{\lambda^2 k_2^\alpha \kappa_g^\alpha \kappa_g^\beta}{\sin^2\varphi} - \frac{\lambda \kappa_n^\alpha \kappa_n^\beta \left(\lambda^2 (k_1^\alpha)^2 + \lambda k_2^\alpha \cot\varphi + 1\right)}{\sin\varphi} \\ + \frac{\lambda (k_1^\alpha)^2 \left(\lambda \kappa_n^\alpha + 2\lambda \kappa_n^\beta - 1\right)}{\sin\varphi} + \frac{\lambda^2 (k_1^\alpha)^2 k_2^\alpha \cot\varphi \left(\lambda \kappa_n^\beta - 1\right)}{\sin\varphi} \\ -\frac{\lambda^2 \left(\kappa_g^\alpha\right)^2 \kappa_n^\beta}{\sin\varphi} + \lambda \left(\kappa_g^\alpha\right)^2 \sin\varphi + \lambda k_g^\alpha \kappa_g^\beta \left(\lambda \kappa_n^\alpha \cot\varphi + \lambda k_2^\alpha - \cot\varphi\right) \\ + \kappa_n^\alpha \sin\varphi - \lambda \left(k_1^\alpha\right)^2 \sin\varphi + \lambda k_2^\alpha \kappa_n^\alpha \cos\varphi \end{pmatrix}$$

$$\overline{m} = \mu \begin{pmatrix} \frac{\lambda^3 \kappa_n^\alpha k_g^\alpha \kappa_n^\beta \kappa_g^\beta}{\sin^3\varphi} - \frac{\lambda^3 \cot\varphi \left(\left(\kappa_g^\alpha\right)^2 \left(\kappa_n^\beta\right)^2 + (\kappa_n^\alpha)^2 \left(\kappa_g^\beta\right)^2\right)}{\sin^2\varphi} \\ -\frac{\lambda^2 \kappa_g^\alpha \kappa_g^\beta \left(\kappa_n^\alpha + \kappa_n^\beta\right)}{\sin\varphi} + \frac{\lambda^3 \kappa_n^\alpha k_g^\alpha \kappa_n^\beta \kappa_g^\beta \cot^2\varphi}{\sin\varphi} \\ + \lambda^2 \kappa_n^\alpha \left(\kappa_g^\beta\right)^2 \cot\varphi + \lambda^2 \left(\kappa_g^\alpha\right)^2 \kappa_n^\beta \cot\varphi + \lambda \kappa_g^\alpha \kappa_g^\beta \sin\varphi \end{pmatrix}$$

and

$$\overline{n} = \mu \begin{pmatrix} \lambda \left(\kappa_g^\beta\right)^2 \sin\varphi + \left(\lambda \kappa_n^\beta - 1\right) \left(\frac{\lambda \kappa_n^\alpha \kappa_n^\beta}{\sin\varphi} + \lambda \kappa_g^\alpha \kappa_g^\beta \cot\varphi\right) \\ + \left(1 - \lambda \kappa_n^\beta\right) \left(\frac{\lambda^2 k_2^\beta \kappa_g^\alpha \kappa_g^\beta}{\sin^2\varphi} + \frac{\lambda^2 k_2^\beta \kappa_n^\alpha \kappa_n^\beta \cot\varphi}{\sin\varphi} + \frac{\lambda^2 \kappa_n^\alpha \left(k_1^\beta\right)^2}{\sin\varphi}\right) \\ + \lambda^2 k_2^\beta \kappa_g^\beta \left(\frac{\lambda \kappa_g^\alpha \kappa_n^\beta}{\sin^2\varphi} - \frac{\lambda \kappa_n^\alpha \kappa_g^\beta \cot\varphi}{\sin\varphi} - \kappa_g^\alpha\right) \\ + \frac{\lambda \left(k_1^\beta\right)^2}{\sin\varphi} \left(-1 + \kappa_n^\beta + k_2^\beta \cot\varphi\right) + \kappa_n^\beta \sin\varphi \\ -\lambda \left(k_1^\beta\right)^2 \sin\varphi - \lambda k_2^\beta \kappa_n^\beta \cos\varphi \end{pmatrix}.$$

3. Shape Operator Matrices of the Parallel Surfaces to Translation Surfaces

In [5], authors gave the shape operator matrix S, the Gauss curvature K and the mean curvature H of M are

$$S = \frac{1}{\sin^2\varphi} \begin{bmatrix} \kappa_n^\alpha & -\cos\varphi \kappa_n^\alpha \\ -\cos\varphi \kappa_n^\beta & \kappa_n^\beta \end{bmatrix} \tag{3.1}$$

$$K = \frac{\kappa_n^\alpha \kappa_n^\beta}{\sin^2\varphi} \tag{3.2}$$

and

$$H = \frac{\kappa_n^\alpha + \kappa_n^\beta}{2\sin^2\varphi} \tag{3.3}$$

respectively. The shape operator matrix $\overline{S}$ of the parallel surface $\overline{M}$ is expressed in the form

$$\overline{S} = S \left(I - \lambda S\right)^{-1}. \tag{3.4}$$

Then, by substituting (3.1) into (3.4), we obtain

$$\overline{S} = \frac{1}{\sin^2\varphi - \lambda \left(\kappa_n^\alpha + \kappa_n^\beta\right) + \lambda^2 \kappa_n^\alpha \kappa_n^\beta} \begin{bmatrix} \kappa_n^\alpha - \lambda \kappa_n^\alpha \kappa_n^\beta & -\kappa_n^\alpha \cos\varphi \\ -\kappa_n^\beta \cos\varphi & \kappa_n^\beta - \lambda \kappa_n^\alpha \kappa_n^\beta \end{bmatrix}. \tag{3.5}$$

From (3.5), we can find the Gauss curvature $\overline{K}$ of $\overline{M}$ as

$$\overline{K} = \frac{\kappa_n^\alpha + \kappa_n^\beta}{\sin^2\varphi - \lambda\left(\kappa_n^\alpha + \kappa_n^\beta\right) + \lambda^2\kappa_n^\alpha\kappa_n^\beta}, \tag{3.6}$$

and the mean curvature $\overline{H}$ of $\overline{M}$ as

$$\overline{H} = \frac{\kappa_n^\alpha + \kappa_n^\beta - 2\lambda\kappa_n^\alpha\kappa_n^\beta}{2\left(\sin^2\varphi - \lambda\left(\kappa_n^\alpha + \kappa_n^\beta\right) + \lambda^2\kappa_n^\alpha\kappa_n^\beta\right)}. \tag{3.7}$$

Also, the first and second principal curvature of $\overline{M}$ are

$$\overline{k_1} = \frac{\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi} + \sqrt{\left(\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi}\right)^2 - \frac{\kappa_n^\alpha\kappa_n^\beta}{\sin^2\varphi}}}{1 - \lambda\left(\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi} + \sqrt{\left(\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi}\right)^2 - \frac{\kappa_n^\alpha\kappa_n^\beta}{\sin^2\varphi}}\right)}$$

and

$$\overline{k_2} = \frac{\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi} - \sqrt{\left(\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi}\right)^2 - \frac{\kappa_n^\alpha\kappa_n^\beta}{\sin^2\varphi}}}{1 - \lambda\left(\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi} - \sqrt{\left(\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi}\right)^2 - \frac{\kappa_n^\alpha\kappa_n^\beta}{\sin^2\varphi}}\right)}.$$

respectively. In addition, we can find the harmonic mean curvature of $\overline{M}$ which is defined as $\overline{K}/\overline{H}$. Thus harmonic mean curvature is

$$\frac{\overline{K}}{\overline{H}} = \frac{\kappa_n^\alpha + \kappa_n^\beta}{\kappa_n^\alpha + \kappa_n^\beta - 2\lambda\kappa_n^\alpha\kappa_n^\beta}. \tag{3.8}$$

Corollary 1. ***i.*** *The parallel surface $\overline{M}$ of the translation surface M, which is constructed by generator curves α and β, is $\overline{K}$-flat if and only if the curve α and the curve β lying on the translation surface M are asymptotic line or $\kappa_n^\alpha = -\kappa_n^\beta$.*

ii. *The parallel surface $\overline{M}$ of the translation surface M has constant Gaussian curvature if and only if the surface M has constant normal curvatures along the generator curves.*

Corollary 2. ***i.*** *All parallel surfaces of M, which is constructed by generator curves, is minimal if and only if the curve α and the curve β lying on M are asymptotic line. If the generator curves lying on M aren't asymptotic line, then there is a minimal parallel surface in parallel surfaces for $\lambda = \frac{\kappa_n^\alpha+\kappa_n^\beta}{2\kappa_n^\alpha\kappa_n^\beta}$.*

ii. *The parallel surface $\overline{M}$ of the translation surface M has constant mean curvature if and only if the surface M has constant normal curvatures along the generator curves.*

iii. *The parallel surface* $\overline{M}$ *has the same Gauss and mean curvatures if and only if* $\lambda = \frac{-H}{K}$. *In this case, from (3.8) we get* $\left(\frac{\partial \kappa_n^\alpha}{\partial s}\right) \kappa_n^\beta$ *and* $\left(\frac{\partial \kappa_n^\alpha}{\partial t}\right) \kappa_n^\alpha$. *Thus, both of generator curves are asymptotic curves of* M.

Let K and H be Gauss curvature and mean curvature of M. Then, Gauss curvature and mean curvature of $\overline{M}$ given by (3.6) and (3.7) can be written in terms K and H by using (3.2) and (3.3) as follow.

$$\overline{K} = \frac{2H}{1 - 2\lambda H + \lambda^2 K} \tag{3.9}$$

and

$$\overline{H} = \frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}. \tag{3.10}$$

Both (3.9) and (3.10) give us important inequality $H^2 < K$ which is guarantee to us that λ is neither $\frac{H}{K}$ nor $\frac{H \mp \sqrt{H^2 - K}}{K}$. Thus, on these conditions, we can give the following important remarks.

Remark 1. ***i.*** *M is a minimal translation surface if and only if its parallel surface is a $K-$flat.*

ii. *Only translation surface whose Gauss and mean curvatures satisfies $H^2 < K$ has parallel surfaces, so it can be clearly state that if a translation surface has parallel surfaces then its Gauss curvature is positive.*

iii. *The harmonic mean curvature of M is equal to $\frac{-1}{\lambda} = const.$ if and only if $\overline{K} = \overline{H}$.*

iv. *Parallel surfaces of any translation surface can not be minimal surface.*

In addition, we can give following theorem.

Theorem 1. *Let K and H be Gauss curvature and mean curvature of the translation surface M, and let $\overline{K}$ and $\overline{H}$ be Gauss curvature and mean curvature of the parallel surface $\overline{M}$, respectively. Then, the following equality is satisfied.*

$$H\overline{K} - \lambda K\overline{K} + 2H\overline{H} = 0.$$

By using (3.9) and (3.10), we can write harmonic mean curvature of the parallel surface as

$$\frac{\overline{K}}{\overline{H}} = \frac{2H}{H - \lambda K}. \tag{3.11}$$

From (3.11), we obtain

$$\frac{\overline{H}}{\overline{K}} = \frac{1}{2} - \frac{\lambda}{2}\frac{H}{K}.$$

Then, we can give following theorem.

Theorem 2. *A translation surface has constant harmonic mean curvature if and only if its parallel surface has constant harmonic mean curvature.*

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