Commun.Fac.Sci.Univ.Ank.Series A1
Volume 64, Number 2, Pages 55–62 (2015)
DOI: 10.1501/Commua1_0000000733
ISSN 1303–5991

SOME PROPERTIES OF THE GENERALIZED BLEIMANN, BUTZER AND HAHN OPERATORS

DILEK SÖYLEMEZ-ÖZDEN

Abstract. In the present paper, we introduce sequences of Bleimann, Butzer and Hahn operators which are based on a function τ. This function is a continuously differentiable function on $\mathbb{R}^{+}$, such that $\tau(0)=0$, $\inf \tau'(x) \geq 1$. We give a Korovkin-type theorem and prove uniform approximation of the generalized Bleimann, Butzer and Hahn operator. We also investigate the monotonic convergence property of the sequence of the operators under $\tau-$convexity.

1. Introduction

Let, as usual, $C[0,\infty)$ denote the space of all continuous and real valued functions defined on $[0,\infty)$ and $C_B[0,\infty)$ denote the space of all bounded functions from $C[0,\infty)$. Obviously

$$\|f\|_{C_B} = \sup_{x \geq 0} |f(x)|$$

defines a norm on $C_B[0,\infty)$. In [6], Bleimann, Butzer and Hahn proposed a sequence of positive linear operators L_n defined by

$$L_n(f;x) = \frac{1}{(1+x)^n} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) \binom{n}{k} x^k, \ x \geq 0, \ n \in \mathbb{N} \tag{1.1}$$

for $f \in C[0,\infty)$. Here, the authors proved that $L_n(f;x) \to f(x)$ as $n \to \infty$ pointwise on $[0,\infty)$ when $f \in C_B[0,\infty)$. Moreover, the convergence being uniform on each compact subset of $[0,\infty)$. In [9], using test functions $\left(\frac{t}{1+t}\right)^v$ for $v = 0,1,2$, Gadjiev and Çakar stated a Korovkin-type theorem for the uniform convergence of functions belonging to some suitable function space by some linear positive operators. As an application of this result, they proved uniform approximation of Bleimann, Butzer and Hahn operators.

Received by the editors: June 06, 2015; Accepted: August 08, 2015.
2010 *Mathematics Subject Classification.* Primary ; 41A36; Secondary 41A20.
Key words and phrases. Generalized Bleimann, Butzer and Hahn operator; Korovkin-type theorem; $\tau-$convexity; monotonicity property.

Monotonicity properties of the Bleimann, Butzer and Hahn operators was investigated by Della Vecchia in [8]. The operator L_n and its generalizations have been studied by several authors some are in [1], [4], [11], [2].

In [7], Cárdenas-Moreles, Garrancho and Raşa introduce a new type generalization of Bernstein polynomials denoted by B_n^τ and defined as

$$\begin{aligned} B_n^\tau(f;x) \quad &: = B_n\left(f\circ\tau^{-1};\tau(x)\right) \\ &= \sum_{k=0}^{n}\binom{n}{k}\tau^k(x)\,(1-\tau(x))^{n-k}\,(f\circ\tau^{-1})(\frac{k}{n}), \end{aligned} \tag{1.2}$$

where B_n is the $n-$th Bernstein polynomial, $f\in C[0,1]$, $x\in[0,1]$ and τ being any function that is continuously differentiable of infinite order on $[0,1]$ such that $\tau(0)=0,\ \tau(1)=1$ and $\tau'(x)>0$ for $x\in[0,1]$. In this work, the authors studied some shape preserving and convergence properties concerning the generalized Bernstein operators $B_n^\tau(f;x)$. A similar generalization for Szász-Mirakyan operator was stated in [5] by Aral, Inoan and Raşa by taking ρ as continuously differentiable function on $[0,\infty)$, $\rho(0)=0$, $\inf_{x\in\mathbb{R}^+}\rho'(x)\geq 1$. Here, weighted approximation as well as the degree of the approximation were obtained. Among other results, they also showed that the sequence of the generalized Szász-Mirakyan operators is monotonically nonincreasing under the notion of $\rho-$convexity of the original function.

Now, accordingly, we consider the following generalized Bleimann, Butzer and Hahn operators for $f\in C[0,\infty)$:

$$L_n^\tau(f;x)=\frac{1}{(1+\tau(x))^n}\sum_{k=0}^{n}\left(f\circ\tau^{-1}\right)\left(\frac{k}{n-k+1}\right)\binom{n}{k}\tau(x)^k, \tag{1.3}$$

where τ is a continuously differentiable function defined on $[0,\infty)$ such that

$$\tau(0)=0,\ \inf_{x\in[0,\infty)}\tau'(x)\geq 1. \tag{1.4}$$

An example of such a function τ is given in [5]. Note that, in the setting of the operator (1.3) we have

$$L_n^\tau f:=L_n\left(f\circ\tau^{-1}\right)\circ\tau,$$

where L_n is the $n-$th Bleimann, Butzer and Hahn operator given by (1.1). If $\tau(x)=x$ then $L_n^\tau=L_n$. Obviously, we have

$$\begin{aligned} L_n^\tau(1;x) &= 1, \\ L_n^\tau\left(\frac{\tau}{1+\tau};x\right) &= \frac{n}{n+1}\frac{\tau(x)}{1+\tau(x)}, \\ L_n^\tau\left(\left(\frac{\tau}{1+\tau}\right)^2;x\right) &= \frac{n(n-1)}{(n+1)^2}\left(\frac{\tau(x)}{1+\tau(x)}\right)^2+\frac{n}{(n+1)^2}\frac{\tau(x)}{1+\tau(x)}. \end{aligned} \tag{1.5}$$

The first purpose of this paper is to extend the results of Gadjiev and Çakar in [9] to the generalized Bleimann, Butzer and Hahn operators (1.3). In this direction, we first give a generalized Korovkin-type theorem to obtain uniform convergence by $\{L_n^{\tau}(f;x)\}_{n\in\mathbb{N}}$ to $f(x)$ on $[0,\infty)$ for f belonging to some suitable subspace of continuous and bounded functions that we will denote it by H_{ω}^{τ}. Next, we study the monotonic convergence under the $\tau-$convexity of the function which is approximated.

For this purpose, we define the following class of functions

Let ω be a general functions of modulus of continuity, satisfying the following properties:
(a) ω is a continuous nonnegative increasing function on $[0,\infty)$,
(b) $\omega(\delta_1+\delta_2)\leq\omega(\delta_1)+\omega(\delta_2)$
(c) $\lim_{\delta\to 0}\omega(\delta)=0$.

Suppose that H_{ω}^{τ} denote the space of all real valued functions f defined on $[0,\infty)$ satisfying

$$|f(x)-f(y)|\leq\omega\left(\left|\frac{\tau(x)}{1+\tau(x)}-\frac{\tau(y)}{1+\tau(y)}\right|\right) \tag{1.6}$$

for all $x,y\in[0,\infty)$. It readily follows from (c) that if $f\in H_{\omega}^{\tau}$, then it is continuous on $[0,\infty)$. Moreover, if $f\in H_{\omega}^{\tau}$, then we have

$$\begin{aligned} |f(x)| &\leq |f(0)|+\omega\left(\frac{\tau(x)}{1+\tau(x)}\right) \\ &\leq |f(0)|+\omega(1), \quad (x\geq 0), \end{aligned}$$

by the assumption on τ, which clearly gives that f is bounded on $[0,\infty)$. Therefore we have the following inclusion:

$$H_{\omega}^{\tau}\subset C_B[0,\infty).$$

When $\omega(t)=Mt^{\alpha}$, $0<\alpha\leq 1$, the space of H_{ω}^{τ} will be denoted by H_{α}^{τ}. From (1.6) we get that

$$|f(x)-f(y)| \leq M\frac{|\tau(x)-\tau(y)|^{\alpha}}{(1+\tau(x))^{\alpha}(1+\tau(y))^{\alpha}}.$$

Hence we reach to

$$H_{\alpha}^{\tau}\subset Lip_M(\tau(x);\alpha),$$

where $Lip_M(\tau(x);\alpha)$, $0<\alpha\leq 1$, $M>0$, is the set of all functions $f\in C[0,\infty)$ satisfying the inequality

$$|f(t)-f(x)|\leq M|\tau(t)-\tau(x)|^{\alpha}, \ x, \ t\geq 0$$

(see [10]).

Definition 1. *A continuous, real valued function f is said to be convex in $D \subseteq [0,\infty)$, if*

$$f\left(\sum_{i=1}^{m}\alpha_i x_i\right) \leq \sum_{i=1}^{m}\alpha_i f\left(x_i\right)$$

for every $x_1, x_2, ..., x_m \in D$ and for every nonnegative numbers $\alpha_1, \alpha_2, ..., \alpha_m$ such that $\alpha_1 + \alpha_2 + ... + \alpha_m = 1$.

In [7] Cárdenas-Morales, Garrancho and Raşa introduced the following definition of $\tau-$convexity of a continuous function.

Definition 2. *A continuous, real valued function f is said to be $\tau-$convex in D, if $f \circ \tau^{-1}$ is convex in the sense of Definition 1.*

Here, we give a Korovkin-type theorem in the sense of Gadjiev and Çakar $([9])$.

2. Main Results

Here, we give a Korovkin-type theorem in the sense of Gadjiev and Çakar $([9])$.

Theorem 1. *Let $\{T_n^{\tau}(f)\}_{n\in\mathbb{N}}$ be a sequence of linear positive operators from H_{ω}^{τ} to $C_B[0,\infty)$. If*

$$\lim_{n\to\infty}\left\|T_n^{\tau}\left(\left(\frac{\tau(t)}{1+\tau(t)}\right)^{v};x\right)-\left(\frac{\tau(x)}{1+\tau(x)}\right)^{v}\right\|_{C_B}=0, \tag{2.1}$$

is satisfied for $v = 0, 1, 2$, then for $f \in H_{\omega}^{\tau}$ we have

$$\lim_{n\to\infty}\left\|T_n^{\tau}(f)-f\right\|_{C_B}=0.$$

Proof. Supposing that $f \in H_{\omega}^{\tau}$, we deduce from (1.6) that for any $\epsilon > 0$ there exist a positive $\delta > 0$ such that

$$|f(t)-f(x)|<\epsilon$$

whenever $\left|\frac{\tau(t)}{1+\tau(t)}-\frac{\tau(x)}{1+\tau(x)}\right|<\delta$. On the other hand, from the boundedness of f, we get

$$|f(t)-f(x)|<\frac{2M}{\delta^2}\left[\frac{\tau(t)-\tau(x)}{(1+\tau(t))(1+\tau(x))}\right]^2$$

when $\left|\frac{\tau(t)}{1+\tau(t)}-\frac{\tau(x)}{1+\tau(x)}\right|\geq\delta$. In this case, for all $t,\ x \in [0,\infty)$ we can write

$$|f(t)-f(x)|<\epsilon+\frac{2M}{\delta^2}\left[\frac{\tau(t)-\tau(x)}{(1+\tau(t))(1+\tau(x))}\right]^2. \tag{2.2}$$

Since T_n^{τ} is linear and positive, then applying the operator on $f(t)-f(x)$, we obtain

$$\begin{aligned}\left\|T_n^{\tau}(f;x)-f(x)\right\|_{C_B} &\leq \left\|T_n^{\tau}(|f(t)-f(x)|;x)\right\|_{C_B}+\left\|f(x)\right\|_{C_B}\left\|T_n^{\tau}(1;x)-1\right\|_{C_B}\\ &=: I_n^{\tau 1}+I_n^{\tau 2}.\end{aligned}$$

From (2.1) and the boundedness of f, $\|f\|_{C_B} \leq M$, we get

$$\lim_{n\to\infty} I_n^{\tau 2} = 0.$$

On the other hand, (2.1) gives that

$$\left\| T_n^\tau \left(\left(\frac{\tau(t) - \tau(x)}{(1+\tau(t))(1+\tau(x))} \right)^2 ; x \right) \right\|_{C_B} < C\epsilon_n \tag{2.3}$$

where $\epsilon_n \to 0$ as $n \to \infty$ and C is a positive constant independent of n.
Moreover, from (2.1), (2.2) and (2.3) it follows that

$$\begin{aligned} I_n^{\tau 1} &\leq \epsilon \left\| T_n^\tau (1;x) \right\|_{C_B} + \frac{2M}{\delta^2} \left\| T_n^\tau \left(\left(\frac{\tau(t) - \tau(x)}{(1+\tau(t))(1+\tau(x))} \right)^2 ; x \right) \right\|_{C_B} \\ &\leq \epsilon (1+\epsilon_n) + \frac{2M}{\delta^2} C\epsilon_n. \end{aligned}$$

Hence we deduce

$$\lim_{n\to\infty} I_n^{\tau 1} = 0,$$

which completes the proof. □

Theorem 2. *Let L_n^τ be the operator defined by (1.3). Then for any $f \in H_\omega^\tau$ we have*

$$\lim_{n\to\infty} \left\| L_n^\tau f - f \right\|_{C_B} = 0.$$

Proof. From Theorem 1 it suffices to show that (2.1) hold for L_n^τ. Indeed, from (1.5) we easily obtain that

$$L_n^\tau (1;x) = 1, \tag{2.4}$$

$$\begin{aligned} &\left\| L_n^\tau \left(\frac{\tau(t)}{1+\tau(t)} ; x \right) - \frac{\tau(x)}{1+\tau(x)} \right\|_{C_B} \\ &= \left\| \left(\frac{n}{n+1} - 1 \right) \frac{\tau(x)}{1+\tau(x)} \right\|_{C_B} \\ &\leq \frac{1}{n+1}, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &\left\| L_n^\tau \left(\left(\frac{\tau(t)}{1+\tau(t)} \right)^2 ; x \right) - \left(\frac{\tau(x)}{1+\tau(x)} \right)^2 \right\|_{C_B} \\ &\leq \left\| \frac{n(n-1)}{(n+1)^2} + \frac{n}{(n+1)^2} + 1 \right\|_{C_B} \\ &\leq \frac{4n+1}{(n+1)^2}. \end{aligned} \tag{2.6}$$

Therefore, using (2.4) – (2.6) we get that (2.1) holds. By Theorem 1, the proof is completed. □

3. Monotonicity Result for L_n^τ

Theorem 3. *If f is $\tau-$convex and non-increasing on $[0,\infty)$, then we have*

$$L_n^\tau(f;x) \geq L_{n+1}^\tau(f;x)$$

for $n \in \mathbb{N}$.

Proof. From (1.3),we can write

$$\begin{aligned}
& L_n^\tau(f;x) - L_{n+1}^\tau(f;x) \\
= \quad & \frac{1}{(1+\tau(x))^{n+1}} \sum_{k=0}^{n} \left(f\circ\tau^{-1}\right)\left(\frac{k}{n-k+1}\right)\binom{n}{k}\tau(x)^k \\
& + \frac{1}{(1+\tau(x))^{n+1}} \sum_{k=0}^{n} \left(f\circ\tau^{-1}\right)\left(\frac{k}{n-k+1}\right)\binom{n}{k}\tau(x)^{k+1} \qquad (3.1) \\
& - \frac{1}{(1+\tau(x))^{n+1}} \sum_{k=0}^{n+1} \left(f\circ\tau^{-1}\right)\left(\frac{k}{n-k+2}\right)\binom{n+1}{k}\tau(x)^k. \qquad (3.2)
\end{aligned}$$

removing, n. term and $n+1$.term from $(3.1),(3.2)$ respectively and taking into account of the fact $\frac{n+1}{n+1-k}\binom{n}{k} = \binom{n+1}{k}$, $\binom{n}{k-1} = \binom{n}{k}\frac{k}{n+1-k}$, we have

$$\begin{aligned}
& L_n^\tau(f;x) - L_{n+1}^\tau(f;x) \\
= \quad & \left(\frac{\tau(x)}{1+\tau(x)}\right)^{n+1}\left[\left(f\circ\tau^{-1}\right)(n) - \left(f\circ\tau^{-1}\right)(n+1)\right] \\
& + \frac{1}{(1+\tau(x))^{n+1}}\left[\left(f\circ\tau^{-1}\right)(0) - \left(f\circ\tau^{-1}\right)(0)\right] \\
& + \frac{1}{(1+\tau(x))^{n+1}} \sum_{k=1}^{n} \left(f\circ\tau^{-1}\right)\left(\frac{k}{n-k+1}\right)\binom{n}{k}\tau(x)^k \\
& + \frac{1}{(1+\tau(x))^{n+1}} \sum_{k=0}^{n-1} \left(f\circ\tau^{-1}\right)\left(\frac{k}{n-k+1}\right)\binom{n}{k}\tau(x)^{k+1} \\
& - \frac{1}{(1+\tau(x))^{n+1}} \sum_{k=1}^{n} \left(f\circ\tau^{-1}\right)\left(\frac{k}{n-k+2}\right)\frac{n+1}{n+1-k}\binom{n}{k}\tau(x)^k
\end{aligned}$$

$$
\begin{aligned}
= \quad & \left(\frac{\tau(x)}{1+\tau(x)}\right)^{n+1}\left[\left(f\circ\tau^{-1}\right)(n)-\left(f\circ\tau^{-1}\right)(n+1)\right] \\
& +\frac{1}{(1+\tau(x))^{n+1}}\sum_{k=1}^{n}\left(f\circ\tau^{-1}\right)\left(\frac{k}{n-k+1}\right)\binom{n}{k}\tau(x)^{k} \\
& +\frac{1}{(1+\tau(x))^{n+1}}\sum_{k=1}^{n}\left(f\circ\tau^{-1}\right)\left(\frac{k-1}{n-k+2}\right)\binom{n}{k-1}\tau(x)^{k} \\
& -\frac{1}{(1+\tau(x))^{n+1}}\sum_{k=1}^{n}\left(f\circ\tau^{-1}\right)\left(\frac{k}{n-k+2}\right)\frac{n+1}{n+1-k}\binom{n}{k}\tau(x)^{k} \\
= \quad & \left(\frac{\tau(x)}{1+\tau(x)}\right)^{n+1}\left[\left(f\circ\tau^{-1}\right)(n)-\left(f\circ\tau^{-1}\right)(n+1)\right] \\
& +\frac{1}{(1+\tau(x))^{n+1}}\sum_{k=1}^{n}\binom{n}{k}\tau(x)^{k} \\
& \times\left[\left(f\circ\tau^{-1}\right)\left(\frac{k}{n-k+1}\right)+\frac{k}{n+1-k}\left(f\circ\tau^{-1}\right)\left(\frac{k-1}{n-k+2}\right)\right. \\
& \left.-\frac{n+1}{n+1-k}\left(f\circ\tau^{-1}\right)\left(\frac{k}{n-k+2}\right)\right].
\end{aligned}
$$

By taking $\alpha_1=\frac{n-k+1}{n+1}\geq 0$, $\alpha_2=\frac{k}{n+1}\geq 0, \alpha_1+\alpha_2=1$, and $x_1=\frac{k}{n-k+1}$, $x_2=\frac{k-1}{n-k+2}$ one has

$$
\begin{aligned}
\alpha_1x_1+\alpha_2x_2 &= \frac{k}{n+1}+\frac{k}{n+1}\frac{k-1}{n-k+2} \\
&= \frac{k(n-k+2)+k^2-k}{(n+1)(n-k+2)} \\
&= \frac{k}{n-k+2}.
\end{aligned}
$$

Therefore, we obtain that

$$L_n^{\tau}(f;x)-L_{n+1}^{\tau}(f;x)\geq 0$$

by $\tau-$convexity and non-increasingness of f for $x\in[0,\infty)$.

$\square$

References

[1] Abel, U. and Ivan, M., Some identities for the operator of Bleimann, Butzer and Hahn involving divided differences, Calcolo **36**, (1999), 143-160.

[2] Adell, J.A., de la Cal, J. and San Miguel, M. Inverse Beta and Generalized Bleimann, Butzer and Hahn operators, Journ. Approx. Theory **76**, (1994), 54-64

[3] Altomare, F., Campiti, M., Korovkin-type approximation theory and its applications. Walter De Gruyter, Berlin, 1994.

[4] Aral, A. and Doğru, O., Bleimann, Butzer and Hahn Operators on the $q-$Integers, Hindawi Publishing Corporation Journal of Inequalities and Applications **2007**, (2007) Article I D 79410.
[5] Aral, A., Inoan, D. and Raşa, I., On the generalized Szász-Mirakyan operators. Result in Mathematics **65**, (2014), 441-452.
[6] Bleimann, G. , Butzer, P. L. and Hahn, L., A Bernstein Type operator approximating continuous functions on the semi-axis. Indag. Math. **42**, (1980), 255-262.
[7] Cárdenas-Moreles, D. Garrancho, P., Raşa, I., Berstein-type operators which preserve polynomials. Compt. Math. Appl. **62**, (2011), 158-163.
[8] Della Vecchia, B., Some Properties of a Rational Operator of Bernstein-Type, Progr. Approx. Theory (1991), 177-185.
[9] Gadjiev, A. D. and Çakar, Ö., On uniform approximation by Bleimann, Butzer and Hahn Operators on all positive semiaxis, Transactions of AS Azerbaijan. Series of Physical Technical and Mathematical Sciences **19** (5), (1999), 21-26.
[10] Gadjiev A. D., Aral A. The estimates of approximation by using a new type of weighted modulus of continuity, Comp. Math. Appl. **54** (2007), 127-135.
[11] Khan, R.A., A note on a Bernstein type operator of Bleimann, Butzer and Hahn, J. Approx. Theory. **53**, (1988), 295-303.

Address: Ankara University, Elmadag Vocational School, 06780, Elmadag, Ankara, Turkey
E-mail: `dsoylemez@ankara.edu.tr`

[0]Başlık: Genelleştirilmiş Bleimann, Butzer ve Hahn Operatörlerinin Bazı Özellikleri

Anahtar Kelimeler: Genelleştirilmiş Bleimann, Butzer ve Hahn Operatörleri, Korovkin-tipi teorem; $\tau-$konveksliği; monotonluk özelliği