MODIFIED $q$–BASKAKOV OPERATORS

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Abstract. In the present paper, a generalization of the sequences of $q$–Baskakov operators, which are based on a function $\tau$ having continuously differentiable on $[0, \infty)$ with $\tau(0) = 0$, $\inf \tau'(x) \geq 1$, has been considered. Uniform approximation of such a sequence has been studied and degree of approximation has been obtained. Moreover, monotonicity properties of the sequence of operators are investigated.

1. Introduction

In [7], Baskakov operator was introduced as

$$B_n(f)(x) = \frac{1}{(1+x)^n} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left( \frac{x}{1+x} \right)^k f \left( \frac{k}{n} \right)$$

for $n \in \mathbb{N}$, $x \in [0, \infty)$ and $f \in C[0, \infty)$ where $C[0, \infty)$ denote the space of all continuous and real valued functions defined on $[0, \infty)$. This operator and its various extentions have been intensively studied. Some are in [1], [6], [8], [15], [16].

Let us recall some notations on $q$–analysis ([10], [17]). The $q$–integer, $[n]$ and the $q$–factorial, $[n]!$ are defined by

$$[n]_{q} := \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1 \end{cases}$$

for $n \in \mathbb{N}$

$$[0]_{q} = 0,$$

and

$$[n]! := \begin{cases} [1]_{q}[2]_{q} \ldots [n]_{q}, & n = 1, 2, \ldots \\ 1, & n = 0 \end{cases},$$

for $n \in \mathbb{N}$ and $[0]! = 1$.
respectively where $q > 0$. For integers $n \geq r \geq 0$ the $q$–binomial coefficient is defined as

$$\binom{n}{r}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}.$$  

The $q$–derivative of $f(x)$ is denoted by $D_q f(x)$ and defined as

$$D_q f(x) := \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad D_q f(0) = f'(0),$$

also

$$D^0_q f := f, \quad D^n_q f := D_q(D_q^{n-1}f), \quad n = 1, 2, ...$$

$q$–Pochammer formula is given by

$$(x,q)_0 = 1,$$

$$(x,q)_n = \prod_{k=0}^{n-1} (1 - q^k x)$$

with $x \in \mathbb{R}, n \in \mathbb{N} \cup \{\infty\}$. The $q$–derivative of the product and quotient of two functions $f$ and $g$ are

$$D_q(f(x)g(x)) = g(x)D_q(f(x)) + f(qx)D_q(g(x))$$

and

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q(f(x)) - f(x)D_q(g(x))}{g(x)g(qx)},$$

respectively.

A generalization of the Baskakov operator based on $q$–integers is defined by Aral and Gupta [4]. The authors constructed the $q$–Baskakov operator as

$$B_{n,q}(f;x) = \sum_{k=0}^{\infty} \left[ \frac{n+k-1}{k} \right] q^{\frac{k(k-1)}{2}} x^k (-x,q)_{n+k}^{-1} f \left( \frac{[k]}{q^{k-1}[n]} \right), \quad n \in \mathbb{N}, \quad (1.1)$$

where $x \geq 0$, $q > 0$ and $f$ is a real valued continuous function on $[0, \infty)$. They established moments using $q$–derivatives, expressed the operator in terms of divided differences, studied the rate of convergence in a polynomial weighted norm and gave a theorem related to monotonic convergence of the sequence of operators with respect to $n$.

Finta and Gupta [11] obtained direct estimates for the operators (1.1), using the second order Ditzian-Totik modulus of smoothness. A Voronovskaja-type result for $q$–derivative of $q$–Baskakov operators is given in [2].

Yet, a different type of $q$–Baskakov operator has also been introduced by Aral and Gupta in [3].
Recently, Cárdenas-Morales, Garrancho and Raşa [9] introduced a new type generalization of Bernstein polynomials denoted by $B_n^r$ and defined as

$$B_n^r(f; x) = B_n \left( f \circ \tau^{-1}; \tau(x) \right) = \sum_{k=0}^{n} \binom{n}{k} \tau^k(x) (1 - \tau(x))^{n-k} \left( f \circ \tau^{-1} \right) \left( \frac{k}{n} \right),$$

where $B_n$ is the $n$-th Bernstein polynomial, $f \in C[0,1]$, $x \in [0,1]$ and $\tau$ is a continuously differentiable of infinite order on $[0,1]$ such that $\tau(0) = 0$, $\tau(1) = 1$ and $\tau'(x) > 0$ for $x \in [0,1]$. Also, the authors studied some shape preserving and convergence properties concerning the generalized Bernstein operators $B_n^r(f; x)$.

In [5], Aral, Inoan and Raşa constructed sequences of Szasz-Mirakyan operators which are based on a function $\rho$. They studied weighted approximation properties, Voronovskaja-type result for these operators. They also showed that the sequence of the generalized Szász-Mirakyan operators is monotonically nonincreasing under the $\rho$-convexity of the original function.

In the present paper, we consider a modification of the $q$-Baskakov operators (1.1) in the sense of [5], we study some approximation and shape preserving properties of the new operators.

Motivated from [5] and [9], we define a new generalization of $q$-Baskakov operators for $f \in C[0,1]$ by

$$B_{n,q}^\rho(f; x) = \sum_{k=0}^{\infty} \left( f \circ \rho^{-1} \right) \left( \frac{k}{q^{k-1}} \right) \left[ n + k - 1 \right] \rho^k(x) \left( -\rho(x), q \right) \left\{ n + k \right\} \left( \frac{k}{q^{k-1}} \right) \left( x \right).$$

$q > 0$ and $\rho$ is a continuously differentiable function on $[0, \infty)$ such that

$$\rho(0) = 0, \quad \inf_{x \in [0, \infty)} \rho'(x) \geq 1.$$ 

An example of such a function $\rho$ is given in [5]. Note that, in the setting (1.3) we have

$$B_{n,q}^\rho f := B_{n,q} \left( f \circ \rho^{-1} \right) \circ \rho,$$ 

where the operator $B_{n,q}$ is defined by (1.1). If $\rho = e_1$, then $B_{n,q}^e = B_{n,q}$. We can write the following equalities that are similar to the corresponding results for the $q$-Baskakov operators (1.1)

$$B_{n,q}^\rho(1; x) = 1,$$ 

$$B_{n,q}^\rho(\rho; x) = \rho(x)$$ 

$$B_{n,q}^\rho(\rho^2; x) = \rho^2(x) + \frac{\rho(x)}{n} \left( 1 + \frac{\rho(x)}{q} \right).$$
\[ B_{n,q}^\rho (\rho^3; x) = \rho^3(x) + \frac{1}{[n]} \left\{ \rho^2(x) \left( 1 + \frac{\rho(x)}{q} \right) \left( \frac{2q + 1}{q} \right) \right\} \]

The first purpose of the paper is to investigate uniform convergence of the operators (1.3) on weighted spaces which are defined using the function \( \rho \) and obtain the degree of weighted convergence, using weighted modulus of continuity. Next, we study the monotonic convergence under \( \rho \)-convexity of the function.

Throughout the paper we will consider the following class of functions. Let \( \varphi(x) = 1 + \rho^2(x) \)

\[ B_{\varphi} (\mathbb{R}^+) = \{ f : \mathbb{R}^+ \rightarrow \mathbb{R}, |f(x)| \leq M_f \varphi(x), x \geq 0 \} \]

where \( M_f \) is a constant depending on \( f \).

\[ C_{\varphi} (\mathbb{R}^+) = \{ f \in B_{\varphi} (\mathbb{R}^+) ; f \text{ is continuous on } \mathbb{R}^+ \} \]

\[ C_{\varphi}^k (\mathbb{R}^+) = \{ f \in C_{\varphi} (\mathbb{R}^+) ; \lim_{x \to \infty} \frac{f(x)}{\varphi(x)} = k_f \} \]

where \( k_f \) is a constant depending on \( f \).

\[ U_{\varphi} (\mathbb{R}^+) = \{ f \in C_{\varphi} (\mathbb{R}^+) ; \frac{f(x)}{\varphi(x)} \text{ is uniformly continuous on } \mathbb{R}^+ \} \]

These spaces are normed spaces with the norm

\[ \| f \|_{\varphi} = \sup_{x \in \mathbb{R}^+} \frac{|f(x)|}{\varphi(x)}. \]

Moreover, we shall use the following weighted modulus of continuity

\[ \omega_\rho (f; \delta) = \sup_{x,t \in \mathbb{R}^+} \frac{|f(t) - f(x)|}{\varphi(t) + \varphi(x)} \]

for each \( f \in C_{\varphi} (\mathbb{R}^+) \) and for every \( \delta > 0 \) [14]. We observe that \( \omega_\rho (f; 0) = 0 \) for every \( f \in C_{\varphi} (\mathbb{R}^+) \) and the function \( \omega_\rho (f; \delta) \) is nonnegative and nondecreasing with respect to \( \delta \) for \( f \in C_{\varphi} (\mathbb{R}^+) \).

**Definition 1.** A continuous, real valued function \( f \) is said to be convex in \( D \subseteq [0, \infty) \), if

\[ f \left( \sum_{i=1}^{m} \alpha_i x_i \right) \leq \sum_{i=1}^{m} \alpha_i f(x_i) \]

for every \( x_1, x_2, ..., x_m \in D \) and for every nonnegative numbers \( \alpha_1, \alpha_2, ..., \alpha_m \) such that \( \alpha_1 + \alpha_2 + ... + \alpha_m = 1 \).

In [9] Cárdenas-Morales, Garrancho and Raşa introduced the following definition of \( \rho \)-convexity of a continuous function.
Definition 2. A continuous, real valued function $f$ is said to be $\rho-$convex in $D$, if $f \circ \rho^{-1}$ is convex in the sense of Definition 1.

2. Approximation Properties

In this section, we obtain the weighted uniform convergence of $B_{n,q}^\rho$ to $f$ and the degree of approximation with the aid of weighted modulus of continuity. Let us recall the weighted form of the Korovkin Theorem ([12], [13]).

Lemma 1. [12] The positive linear operators $L_n$, $n \geq 1$, act from $C_\varphi (\mathbb{R}^+)$ to $B_\varphi (\mathbb{R}^+)$ if and only if the inequality

$$|L_n (\varphi; x)| \leq K_n \varphi (x), \quad x \geq 0$$

holds; where $K_n$ is a positive constant.

Theorem 1. [12] Let the sequence of linear positive operators $(L_n)_{n \geq 1}$ acting from $C_\varphi (\mathbb{R}^+)$ to $B_\varphi (\mathbb{R}^+)$ satisfy the three conditions

$$\lim_{n \to \infty} \|L_n \rho^v - \rho^v\|_\varphi = 0, \quad v = 0, 1, 2. \quad (2.1)$$

Then for any function $f \in C_\varphi^k (\mathbb{R}^+)$

$$\lim_{n \to \infty} \|L_n f - f\|_\varphi = 0.$$ 

Now, we are ready to give the following theorem.

Theorem 2. Let $B_{n,q}^\rho$ be the operator defined by (1.3). Then for any $f \in C_\varphi^k (\mathbb{R}^+)$ and $q > 1$, we have

$$\lim_{n \to \infty} \|B_{n,q}^\rho f - f\|_\varphi = 0.$$ 

Proof. By Lemma 1 $B_{n,q}^\rho$ are linear operators acting from $C_\varphi (\mathbb{R}^+)$ to $B_\varphi (\mathbb{R}^+).$

Indeed, from (1.4) and (1.6) we easily obtain that

$$|B_{n,q}^\rho (\varphi; x)| \leq (1 + \rho^2 (x)) \left(\frac{q [n] + q + 1}{q [n]}\right).$$

On the other hand, using (1.4), (1.5) and (1.6), one can write

$$\|B_{n,q}^\rho 1 - 1\|_\varphi = 0,$$

$$\|B_{n,q}^\rho (\rho) - \rho\|_\varphi = 0,$$

and

$$\|B_{n,q}^\rho (\rho^2) - \rho^2\|_\varphi = \sup_{x \in \mathbb{R}^+} \rho (x) \frac{(1 + \rho(x)) (1 + \rho(x))}{[n] 1 + \rho^2 (x)} \leq \frac{2}{[n]}. \quad (2.2)$$

Therefore, the conditions (2.1) are satisfied. By Theorem 1, the proof is completed. \qed
In [14], the following theorem is given.

**Theorem 3.** Let \( L_n : C_\varphi (\mathbb{R}^+) \to B_\varphi (\mathbb{R}^+) \) be a sequence of positive linear operators with

\[
\begin{align*}
\| L_n (1) - 1 \|_{\varphi^0} &= a_n, \\
\| L_n (\rho) - \rho \|_{\varphi^{\frac{1}{2}}} &= b_n, \\
\| L_n (\rho^2) - \rho^2 \|_{\varphi} &= c_n, \\
\| L_n (\rho^3) - \rho^3 \|_{\varphi^{\frac{3}{2}}} &= d_n,
\end{align*}
\]

where \( a_n, b_n, c_n \) and \( d_n \) tend to zero as \( n \to \infty \). Then

\[
\| L_n (f) - f \|_{\varphi^{\frac{3}{2}}} \leq (7 + 4a_n + 2c_n) \omega_\rho (f; \delta_n) + \| f \|_\varphi a_n
\]

for all \( f \in C_\varphi (\mathbb{R}^+) \), where

\[
\delta_n = 2\sqrt{(a_n + 2b_n + c_n) (1 + a_n) + (a_n + 3b_n + 3c_n + d_n)}.
\]

Applying the above theorem, we obtain the degree of approximation.

**Theorem 4.** For all \( f \in C_\varphi (\mathbb{R}^+) \) and \( q > 1 \), we have

\[
\| B_{n,q}^\rho (f) - f \|_{\varphi^{\frac{3}{2}}} \leq \left( 7 + \frac{4}{\lceil n \rceil} \right) \omega_\rho \left( f; \frac{2\sqrt{2}}{\sqrt{\lceil n \rceil}} + \frac{18}{\lceil n \rceil} \right).
\]

**Proof.** According to Theorem 3, we shall calculate the sequences \( a_n, b_n, c_n \) and \( d_n \).

From (1.4), (1.5), (2.2) and (1.7) we get

\[
a_n = \| B_{n,q}^\rho (1) - 1 \|_{\varphi^0} = 0, \\
b_n = \| B_{n,q}^\rho (\rho) - \rho \|_{\varphi^{\frac{1}{2}}} = 0,
\]

\[
c_n = \| B_{n,q}^\rho (\rho^2) - \rho^2 \|_{\varphi} = \sup_{x \in \mathbb{R}^+} \frac{\rho (x) \left( 1 + \frac{\rho(x)}{q} \right) \left( 2q + 1 \right)}{\lceil n \rceil (1 + \rho^2 (x))} \leq \frac{2}{\lceil n \rceil},
\]

\[
d_n = \| B_{n,q}^\rho (\rho^3) - \rho^3 \|_{\varphi^{\frac{3}{2}}} = \sup_{x \in \mathbb{R}^+} \left\{ \frac{1}{\lceil n \rceil} \left[ \rho^2 (x) \left( 1 + \frac{\rho(x)}{q} \right) \left( 2q + 1 \right) \right] \left( 1 + \rho^2 (x) \right)^{\frac{3}{2}} \right. \\
- \left. \frac{1}{\lceil n \rceil^2} \left[ \frac{\rho (x) \left( 1 + \frac{\rho(x)}{q} \right) \left( 1 + \frac{\rho(x)}{q} \right) \rho^2 (x) \left( 1 + \frac{\rho(x)}{q} \right)}{(1 + \rho^2 (x))^2} \right] \right\} \leq \frac{12}{\lceil n \rceil}.
\]

By Theorem 3, the proof is completed. \( \square \)
3. Monotonicity Properties of \( B_{n,q}^\rho \)

Here, we study the monotonic convergence of the operators (1.3) under the \( \rho \)-convexity.

**Theorem 5.** Let \( f \) be a \( \rho \)-convex function on \([0, \infty)\). Then we have

\[
B_{n,q}^\rho (f; x) \geq B_{n+1,q}^\rho (f; x)
\]

for \( n \in \mathbb{N} \).

**Proof.** From (1.3), one can write

\[
B_{n,q}^\rho (f; x) - B_{n+1,q}^\rho (f; x)
\]

\[
= \sum_{k=0}^{\infty} \left[ \binom{n+k-1}{k} q^{\frac{k(k-1)}{2}} \frac{\rho^k(x)}{(-\rho(x), q)^{n+k}} (f \circ \rho^{-1}) \left( \frac{[k]}{q^{k-1}[n]} \right) 
\]

\[-\sum_{k=0}^{\infty} \left[ \binom{n+k}{k} q^{\frac{k(k-1)}{2}} \frac{\rho^k(x)}{(-\rho(x), q)^{n+k+1}} (f \circ \rho^{-1}) \left( \frac{[k]}{q^{k-1}[n+1]} \right) 
\]

\[= \sum_{k=0}^{\infty} \left[ \binom{n+k-1}{k} q^{\frac{k(k-1)}{2}} \frac{\rho^k(x)}{(-\rho(x), q)^{n+k}} (f \circ \rho^{-1}) \left( \frac{[k]}{q^{k-1}[n]} \right) 
\]

\[-\sum_{k=0}^{\infty} \left[ \binom{n+k}{k} q^{\frac{k(k-1)}{2}} \frac{\rho^k(x)}{(-\rho(x), q)^{n+k+1}} (f \circ \rho^{-1}) \left( \frac{[k]}{q^{k-1}[n+1]} \right) 
\]

\[+ \sum_{k=0}^{\infty} \left[ \binom{n+k-1}{k} q^{\frac{k(k-1)}{2}} q^{n+k} \frac{\rho^{k+1}(x)}{(-\rho(x), q)^{n+k+1}} (f \circ \rho^{-1}) \left( \frac{[k]}{q^{k-1}[n+1]} \right) 
\]

\[= \sum_{k=1}^{\infty} \left[ \binom{n+k-1}{k} q^{\frac{k(k-1)}{2}} \frac{\rho^k(x)}{(-\rho(x), q)^{n+k}} (f \circ \rho^{-1}) \left( \frac{[k]}{q^{k-1}[n]} \right) 
\]

\[-\sum_{k=1}^{\infty} \left[ \binom{n+k}{k} q^{\frac{k(k-1)}{2}} \frac{\rho^k(x)}{(-\rho(x), q)^{n+k+1}} (f \circ \rho^{-1}) \left( \frac{[k]}{q^{k-1}[n+1]} \right) 
\]

\[+ \sum_{k=0}^{\infty} \left[ \binom{n+k-1}{k} q^{\frac{k(k-1)}{2}} q^{n+k} \frac{\rho^{k+1}(x)}{(-\rho(x), q)^{n+k+1}} (f \circ \rho^{-1}) \left( \frac{[k]}{q^{k-1}[n+1]} \right) 
\]

Rearranging the above equality, we have
\[
B_{n,q}^p (f; x) - B_{n+1,q}^p (f; x) = \sum_{k=0}^{\infty} \left[ \frac{n+k}{k+1} \right] q^{\frac{k(k-1)}{2}} q^k \left( -\rho \left( x \right), q \right)_{n+k+1} \left( f \circ \rho^{-1} \right) \left( \frac{[k+1]}{q^k [n]} \right) \\
- \sum_{k=0}^{\infty} \left[ \frac{n+k+1}{k+1} \right] q^{\frac{k(k-1)}{2}} q^k \left( -\rho \left( x \right), q \right)_{n+k+1} \left( f \circ \rho^{-1} \right) \left( \frac{[k+1]}{q^k [n+1]} \right) \\
+ \sum_{k=0}^{\infty} \left[ \frac{n+k}{k} \right] q^{\frac{k(k-1)}{2}} q^{n+k} \left( -\rho \left( x \right), q \right)_{n+k+1} \left( f \circ \rho^{-1} \right) \left( \frac{[k]}{q^{k-1} [n+1]} \right)
\]

By the following equalities
\[
\left[ \frac{n+k+1}{k+1} \right] = \left[ \frac{n+k}{k+1} \right] \left[ \frac{n+k}{k} \right] \\
\left[ \frac{n+k}{k+1} \right] = \left[ \frac{n+k}{k+1} \right] \left[ \frac{n+k}{k} \right]
\]
we get
\[
B_{n,q}^p (f; x) - B_{n+1,q}^p (f; x) = \sum_{k=0}^{\infty} \left[ \frac{n+k}{k} \right] q^{\frac{k(k-1)}{2}} q^k \left( -\rho \left( x \right), q \right)_{n+k+1} \left[ \frac{n+k+1}{k+1} \right] \\
\times \left\{ \left[ \frac{n}{n+k+1} \right] \left( f \circ \rho^{-1} \right) \left( \frac{[k+1]}{q^k [n]} \right) - \left( f \circ \rho^{-1} \right) \left( \frac{[k]}{q^{k-1} [n+1]} \right) \right\}.
\]
By taking, \( \lambda_1 = \frac{n}{n+k+1} \geq 0, \lambda_2 = q^n \frac{\left[ k+1 \right]}{\left[ n+k+1 \right]} \geq 0, \lambda_1 + \lambda_2 = 1 \) and \( x_1 = \frac{[k+1]}{q^k [n]}, x_2 = \frac{[k]}{q^{k-1} [n+1]} \) one has
\[
\lambda_1 x_1 + \lambda_2 x_2 = \frac{[k+1]}{q^k [n+1]}.
\]
Therefore, we obtain that
\[
B_{n,q}^p (f; x) - B_{n+1,q}^p (f; x) \geq 0
\]
by $\rho$–convexity of $f$ for $x \in [0, \infty)$ and $n \in \mathbb{N}$. This proves the theorem.

References


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