Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.
Volume 65, Number 1, Pages 53–70 (2016)
DOI: 10.1501/Commua1_0000000744
ISSN 1303–5991

COLIMITS IN THE CATEGORY OF QUADRATIC MODULES

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ABSTRACT. To use of colimits to put structures together is not uncommon in mathematics; coproducts particularly of structures such as groups and vector spaces have been known for a long time. Colimits have also been used in computer science for example to put together labeled graphs, in system theory etc. The importance of category for theoretical computer scientists is everyday increasing. In order to contribute usage of colimits, we show existence of finite (co)limits in the category of quadratic modules of groups by careful construction of (co)product object and (co)equilaser of morphisms of quadratic modules. Moreover, we give some examples of coproduct.

INTRODUCTION

Crossed modules were initially defined by Whitehead [12] as an algebraic model for homotopy connected 2-types. Corresponding definitions were given for some varieties of algebras in [8, 11, 10]. Brown, Higgins and Siviera [5] gave a construction of a coproduct object in the category of crossed modules over a group R. Then they obtained the colimits for the category of crossed R-modules of groups by using the equivalence of categories of Cat^1-groups and that of crossed modules. In his thesis, Nizar [9] has shown the existence of finite limits and colimits in the category of crossed R-modules of commutative algebras. The Lie algebra case of the similar work has been done by Ladra and Casas in [7].

Recently, Baues [2] defined the notion of a quadratic module of groups as an algebraic model for homotopy connected 3-types and gave a relation between quadratic modules and simplicial groups. In [1] Arvasi and Ulualan have explored relations among some algebraic models for homotopy 3-types such as quadratic modules, 2-crossed module,crossed square and simplicial groups.

Received by the editors: Feb. 25, 2015, Accepted: Nov. 18, 2015.
2010 *Mathematics Subject Classification.* 18D35, 18G30, 18G50, 18G55.
Key words and phrases. Colimit, coproduct, quadratic module.

To understand the working category of quadratic modules we need the description of its very remarkable constructions. One of the important categorical constructions is colimit. In this paper, we show the existence of finite colimits of category of quadratic modules. The interest in computing colimits of quadratic modules for algebraic topology is the 2-dimensional Van Kampen theorem due to Brown and Higgins [4, 3, 6, 7]. The following proposition [5] leads us to find finite colimit in the category of quadratic modules.

Proposition : *If functors $S \to C$ admit (co)products and (co)equilasers, then they admit (co)limits.*

Therefore, we construct the notions of products and coproducts for quadratic R-modules and we explore equilaser and coequilaser of morphisms of quadratic modules with the same domain and codomain. Hence we show the existence of finite limits and colimits in the category of quadratic R-modules.

1. Quadratic Modules

Recall that *a pre-crossed module* is a group homomorphism $\partial : M \to Q$ together with an action of Q on M, written m^q for $q \in Q$ and $m \in M$, satisfying the condition $\partial(m^q) = q^{-1}\partial(m)q$ for all $m \in M$ and $q \in Q$.

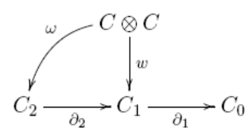
Let $\partial : M \to Q$ be a pre-crossed module. The Peiffer commutator is defined as $\langle m, m' \rangle = m^{-1}m'^{-1}mm'^{\partial(m)}$. The pre-crossed modules in which all Peiffer commutators are trivial are precisely the crossed modules. Namely, a crossed module is a pre-crossed module $\partial : M \to Q$ satisfying the extra condition: $m'^{\partial(m)} = m^{-1}m'm$ for all $m, m' \in M$. We will denote the category of crossed modules by **XMod**.

Since the Peiffer commutators are always defined in a pre-crossed module, it is a natural idea to factor out by the normal subgroup that they generate and consider the induced map from the quotient. The Peiffer subgroup $P_2(\partial) = \langle M, M \rangle$ of M is the subgroup of M generated by all Peiffer commutators. For any pre-crossed module $\partial : M \to Q$, the Peiffer subgroup $P_2(\partial)$ of M is an N-invariant normal subgroup. Let $M^{cr} = M/P_2(\partial)$. This quotient group is a Q-group. Then we can say that for the pre-crossed module $\partial : M \to Q$, the induced map gives a crossed module $\partial^{cr} : M^{cr} \to Q$ which is called the *crossed module associated to* ∂.

A nil(2)-module [2] is a pre-crossed module $\partial : M \to Q$ with an additional "nilpotency"condition. This condition is $P_3(\partial) = 1$, where $P_3(\partial)$ is the subgroup of M generated by Peiffer commutator $\langle m, m', m'' \rangle$ of length 3 for $m, m', m'' \in M$.

We shall denote the category of *nil(2)-modules* by $\mathbf{Nil(2)}$. Now we give the following definition from [2].

Definition 1.1. *A quadratic module* $(\omega, \partial_2, \partial_1)$ *is a diagram*



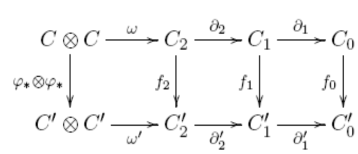
of homomorphisms between groups such that the following axioms are satisfied.
QM1) *The homomorphism* $\partial_1 : C_1 \to C_0$ *is a nil(2)-module with Peiffer commutator map* w *defined above. The quotient map* $C_1 \twoheadrightarrow C = (C_1^{cr})^{ab}$ *is given by* $x \mapsto \{x\}$*, where* $\{x\} \in C$ *denotes the class represented by* $x \in C_1$ *and* $C = (C_1^{cr})^{ab}$ *is the abelianization of the associated crossed module* $C_1^{cr} \to C_0$*.*
QM2) *The boundary homomorphisms* ∂_2 *and* ∂_1 *satisfy* $\partial_1\partial_2 = 1$ *and the quadratic map* ω *is a lift of the Peiffer commutator map* w*, that is* $\partial_2\omega = w$*.*
QM3) C_2 *is a* C_0*-group and all homomorphisms of the diagram are equivariant with respect to the action of* C_0*. Moreover, the action of* C_0 *on* C_2 *satisfies the formula* $(a \in C_2, x \in C_1)$

$$a^{\partial_1 x} = \omega((\{x\} \otimes \{\partial_2 a\})(\{\partial_2 a\} \otimes \{x\}))a.$$

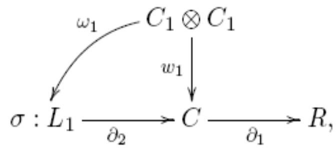
QM4) *Commutators in* C_2 *satisfy the formula* $(a, b \in C_2)$

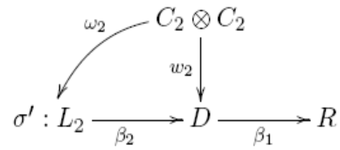
$$\omega(\{\partial_2 a\} \otimes \{\partial_2 b\}) = [b, a].$$

A morphism $\varphi : (\omega, \partial_2, \partial_1) \to (\omega', \partial_2', \partial_1')$ between quadratic modules is given by a commutative diagram, $\varphi = (f_2, f_1, f_0)$

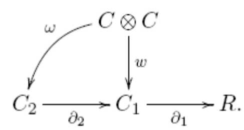


where (f_1, f_0) is a morphism between nil(2)-modules which induces $\varphi_* : C \to C'$ and where f_2 is an f_0-equivariant homomorphism. We denote the category of quadratic





modules by **Quad**. With a fixed group R, consider the category of quadratic R-modules



We will denote the category of such quadratic modules by **Quad/R**. A morphism between quadratic R-modules is a quadratic module morphism $\varphi = (f_2, f_1, f_0)$ as defined above in which f_0 is the identity homomorphism on the group R.

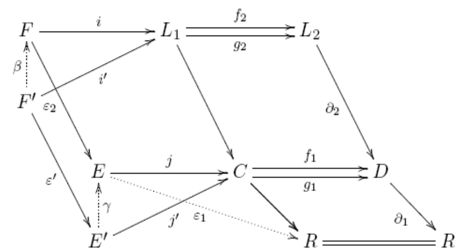
2. Finite Limits in **Quad/R**

In this section, we will show that the category of quadratic R-modules has finite limits by constructing the product of two quadratic modules and equaliser of two morphisms in the category of quadratic R-modules.

Proposition 2.1. *In* **Quad/R** *every pair of morphisms with common domain and codomain has an equaliser.*

Proof: Let $(f, g) : (L_1, C, R) \to (L_2, D, R)$ be two morphisms of quadratic modules where

$f = (f_2, f_1)$ and $g = (g_2, g_1)$, $\quad f_2, g_2 : L_1 \to L_2, \quad f_1, g_1 : C \to D$. Let E and F be sets as follows $E = \{c \in C : f_1(c) = g_1(c)\}$ and $F = \{l_1 \in L_1 : f_2(l_1) = g_2(l_1)\}$. It is clear that $(F, E, \varepsilon_2, \varepsilon_1)$ is a subquadratic module of $(L_1, C, \partial_2, \partial_1)$ in which ε_2 and ε_1 are induced from ∂_2 and ∂_1 respectively. The inclusion $(i, j) : (F, E, R) \to (L_1, C, R)$ is a morphism of quadratic R-modules. Suppose that there exist a quadratic module $(F', E', \varepsilon_2', \varepsilon_1')$ and a morphism $(i', j') : (F', E', R) \to (L_1, C, R)$ of quadratic R-modules such that $f_2 i'(y) = g_2 i'(y)$, $f_1 j'(x) = g_1 j'(x)$ for all $x \in E', y \in F'$. Hence $i'(y) \in F, j'(x) \in E$. Thus we define $\gamma : E' \to E$ as $\gamma(x) = j'(x)$ and $\beta : F' \to F$ as $\beta(y) = i'(y)$. The fact (i', j') being a quadratic module morphism immediately gives that (γ, β) is one as well. This morphism is unique for the commutative diagram



Namely, $j\gamma = j', i\beta = i'$. Thus the morphism (i, j) is an equalizer of (f, g). $\square$

Proposition 2.2. *The category* **Quad/R** *has pullbacks.*

Proof: Let $(f_2, f_1) : (L_1, C, R) \to (L_2, B, R)$ and $(g_2, g_1) : (L_3, D, R) \to (L_2, B, R)$ be two morphisms of quadratic modules where

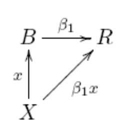
$$\sigma_1 : C_1 \otimes C_1 \xrightarrow[\omega_1]{} L_1 \xrightarrow[\partial_2]{} C \xrightarrow[\partial_1]{} R$$

$$\sigma_2 : C_2 \otimes C_2 \xrightarrow[\omega_2]{} L_2 \xrightarrow[\beta_2]{} B \xrightarrow[\beta_1]{} R$$

$$\sigma_3 : C_3 \otimes C_3 \xrightarrow[\omega_3]{} L_3 \xrightarrow[\delta_2]{} D \xrightarrow[\delta_1]{} R$$

are quadratic R-modules. We form the groups X, Y such that $X = \{(c, d) : f_1(c) = g_1(d)\} \subset C \times D$ and $Y = \{(l_1, l_3) : f_2(l_1) = g_2(l_3)\} \subset L_1 \times L_3$ and morphisms x, y such that $x : X \longrightarrow B$ is given by $(c, d) \mapsto f_1(c) = g_1(d)$ and $y : Y \longrightarrow X$ is given by $(l_1, l_3) \mapsto (\partial_2(l_1), \delta_2(l_3))$.

Thus we obtain the following commutative diagram.



Since

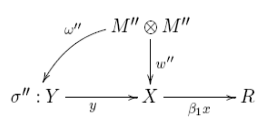
$$\begin{aligned} \beta_1 x((c,d)^r) &= \\ \beta_1 x(c^r, d^r) &= \beta_1(f_1(c^r)) = \beta_1(f_1(c)^r) = r^{-1}\beta_1(f_1(c))r = r^{-1}(\beta_1 x(c,d))r \end{aligned}$$

for all $r \in R$, $(c,d) \in X$, the map $\beta_1 x$ is a pre-crossed module. For (c,d), (c',d'), $(c'',d'') \in X$ we obtain

$$\begin{aligned}
& \langle\langle (c,d),(c',d')\rangle,(c'',d''^{-1}(c',d'^{-1}(c,d)(c',d'^{\beta_1 x(c,d)},(c'',d'')\rangle \\
= \; & \langle (c^{-1}c'^{-1}c, d^{-1}d'^{-1}d)(c',d'^{\beta_1 f_1 c},(c'',d'')\rangle \\
= \; & \langle (c^{-1}c'^{-1}cc'^{\beta_1 f_1 c}, d^{-1}d'^{-1}dd'^{\beta_1 f_1 c}),(c'',d'')\rangle \\
= \; & \langle (c^{-1}c'^{-1}cc'^{\partial_1 c}, d^{-1}d'^{-1}dd'^{\beta_1 g_1 d}),(c'',d'')\rangle \\
= \; & \langle (\langle c,c'\rangle,\langle d,d'\rangle),(c'',d'')\rangle \\
= \; & (\langle c,c'^{-1},\langle d,d'^{-1})(c''^{-1},d''^{-1})(\langle c,c'\rangle,\langle d,d'\rangle)(c'',d''^{\beta_1 x(\langle c,c'\rangle,\langle d,d'\rangle)} \\
= \; & (\langle c,c'^{-1}c''^{-1}\langle c,c'\rangle c''^{\beta_1 f_1(\langle c,c'\rangle)},\langle d,d'^{-1}d''^{-1}\langle d,d'\rangle d''^{\beta_1 g_1(\langle d,d'\rangle)}) \\
= \; & (\langle\langle c,c'\rangle,c''\rangle,\langle\langle d,d'\rangle,d''\rangle) \\
= \; & (1,1).
\end{aligned}$$

Therefore, $\beta_1 x : X \to R$ is a nil(2)-module.

Now, we will show that



is a quadratic R-module.

QM1) We know that $\beta_1 x : X \to R$ is a nil(2)-module. We have also

$$\beta_1 xy(l_1,l_3) = \beta_1 x(\partial_2(l_1),\delta_2(l_3)) = \beta_1(f_2(\partial_2(l_1))) = \beta_1(\beta_2(f_1(l_1))) = \beta_1\beta_2(f_1 l_1) = 1.$$

QM2) For $\{(c,d)\}\otimes\{(c',d')\} \in M''\otimes M''$, we define $\omega'' : M''\times M'' \to Y$ as follows:

$$\omega''(\{(c,d)\}\otimes\{(c',d')\}) = (\omega_1(\{c\}\otimes\{c'\}),\omega_2(\{d\}\otimes\{d'\})).$$

Then we have

$$\begin{aligned} y\omega''(\{(c,d)\}\otimes\{(c',d')\}) &= y(\omega_1(\{c\}\otimes\{c'\}),\omega_2(\{d\}\otimes\{d'\})) \\ &= (\partial_2\omega_1(\{c\}\otimes\{c'\}),\delta_2\omega_2(\{d\}\otimes\{d'\})) \\ &= (w_1(\{c\}\otimes\{c'\}),w_2(\{d\}\otimes\{d'\})) \\ &= (\langle c,c'\rangle,\langle d,d'\rangle) \\ &= \langle (c,d),(c',d')\rangle \\ &= w''(\{(c,d)\}\otimes\{(c',d')\}). \end{aligned}$$

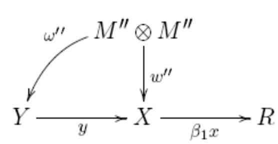
QM3) For$(l_1,l_3)\in Y,(c,d)\in X$, we get

$$\begin{aligned} &\omega''(\{y(l_1,l_3)\}\otimes\{(c,d)\}\{(c,d)\}\otimes\{y(l_1,l_3)\}) \\ &\quad=\omega''(\{(\partial_2 l_1,\delta_2 l_3)\}\otimes\{(c,d)\}\{(c,d)\}\otimes\{(\partial_2 l_1,\delta_2 l_3)\} \\ &\quad=(\omega_1(\{\partial_2 l_1\}\otimes\{c\}),\omega_2(\{d\}\otimes\{\delta_2 l_3\}))(\omega_1(\{c\}\otimes\{\partial_2 l_1\}),\omega_2(\{\delta_2 l_3\}\otimes\{d\})) \\ &\quad=(\omega_1(\{\partial_2 l_1\}\otimes\{c\})(\{c\}\otimes\{\partial_2 l_1\}),\omega_2(\{d\}\otimes\{\delta_2 l_3\}))((\{\delta_2 l_3\}\otimes\{d\})) \\ &\quad=(l_1^{-1}l_1^{\partial_1(c)},l_3^{-1}l_3^{\delta_1(d)})=(l_1^{-1}l_1^{\beta_1 f_1(c)},l_3^{-1}l_3^{\beta_1 g_1(d)}) \\ &\quad=(l_1^{-1}l_1^{\beta_1 f_1(c)},l_3^{-1}l_3^{\beta_1 f_1(c)})=(l_1,l_3)^{-1}(l_1,l_3)^{\beta_1 f_1(c)} \\ &\quad=(l_1,l_3)^{-1}(l_1,l_3)^{\beta_1 x(c,d)}. \end{aligned}$$

QM4) For $(l_1,l_3),(l_1',l_2')\in Y$, we obtain

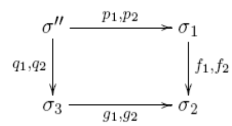
$$\begin{aligned} \omega''(\{y(l_1,l_3)\}\otimes\{y(l_1',l_3')\}) &= \omega''(\{(\partial_2 l_1,\delta_2 l_3)\}\otimes\{(\partial_2 l_1',\delta_2 l_3')\}) \\ &= (\omega_1(\{\partial_2 l_1\}\otimes\{\partial_2 l_1'\},\omega_2(\{\delta_2 l_3\}\otimes\{\delta_2 l_3'\})) \\ &= ([l_1,l_1'],[l_3,l_3']) \\ &= [(l_1,l_3),(l_1',l_3')]. \end{aligned}$$

Thus the diagram

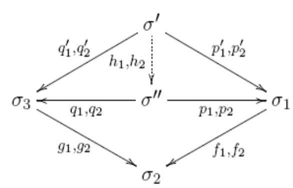


is a quadratic R-module.

There are two induced morphisms $p_1, p_2 : \sigma'' \to \sigma_1$ and $q_1, q_2 : \sigma'' \to \sigma_3$ given by projections; note that $fp = gq$, and this shows that the diagram



is commutative and the morphisms p and q satisfy the universal property: let $(p'_1, p'_2) : \sigma' \to \sigma_1$ and $(q'_1, q'_2) : \sigma' \to \sigma_3$ be any morphisms of quadratic modules with $fp' = gq'$ and $\sigma' : L' \to A \to R$, then there exists a unique morphism $(h_1, h_2) : \sigma' \to \sigma''$ given by $h_1(x') = (p'_1(x'), q'_1(x'))$ and $h_2(y') = (p'_2(y'), q'_2(y'))$ for $x' \in A, y' \in L'$ such that the diagram



is commutative; i.e. $p_1h_1 = p'_1$, $p_2h_2 = p'_2$, $q_1h_1 = q'_1$ and $q_2h_2 = q'_2$. Then we say that **Quad/R** has pullbacks. □

We note that the category of quadratic modules has terminal object, σ_t, thus the proof of the following is easy.

Proposition 2.3. **Quad/R** *has finite products.*

Proof: It is of course sufficient to prove the proposition for a family having just two members say σ_1 and σ_3. The product $\sigma_1 \sqcap \sigma_3$ will be the pullback over the terminal object σ_t where

$$\sigma_1 : C_1 \otimes C_1 \xrightarrow[\omega_1]{} L_1 \xrightarrow[\partial_2]{} C \xrightarrow[\partial_1]{} R$$

$$\sigma_3 : C_3 \otimes C_3 \xrightarrow[\omega_3]{} L_3 \xrightarrow[\delta_2]{} D \xrightarrow[\delta_1]{} R$$

$$\sigma_t : C \otimes C \xrightarrow[id]{} L_2 \xrightarrow[w]{} R \xrightarrow[id]{} R$$

and the diagram

$$\begin{array}{ccc} \sigma_1 \sqcap \sigma_3 & \xrightarrow{p_1, p_1'} & \sigma_1 \\ {\scriptstyle p_2, p_2'}\downarrow & & \downarrow{\scriptstyle f_2, \partial_1} \\ \sigma_3 & \xrightarrow[g_2, \delta_1]{} & \sigma_t \end{array}$$

is commutative i.e. $f_2 p_1 = g_2 p_2$, $\partial_1 p_1' = \delta_1 p_2'$. Then it is easy to check that

$$\sigma_1 \sqcap \sigma_3 : L_1 \sqcap L_3 \xrightarrow[\beta_2]{} C \sqcap D \xrightarrow[\beta_1]{} R$$

is a quadratic module where $\beta_1 : C \sqcap D \to R$ is given by $\beta_1(c, d) = \partial_1 p_1'(c, d) = \delta_1 p_2'(c, d)$ and $\beta_2 : L_1 \sqcap L_3 \to C \sqcap D$ is given by $\beta_2(l_1, l_3) = (\partial_2 l_1, \delta_2 l_3)$ and $\omega_{13} : ((C \sqcap D)^{cr})^{ab} \otimes ((C \sqcap D)^{cr})^{ab} \longrightarrow L_1 \sqcap L_3$ is given by $\omega_{13}(\{(c, d)\} \otimes \{(c', d')\}) = (\omega_1(\{c\} \otimes \{c'\}), \omega_3(\{d\} \otimes \{d'\}))$.

Then by induction, **Quad/R** has finite products. $\square$

Proposition 2.4. **Quad/R** *has a limit for any functor* $\mathbf{F} : \boldsymbol{\sigma} \to$ **Quad/R** *with* σ *finite.*

Proof: As **Quad/R** has finite products and equilasers, the result follows. $\square$

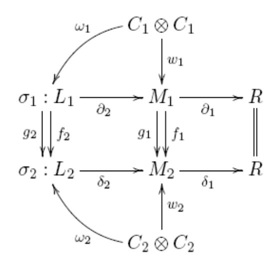
Therefore the category **Quad/R** is finitely complete, i.e. it has all finite limits.

3. Finite Colimits in Quad/R

This section will describe the construction of finite colimits of quadratic modules over a group R. First we will give the coequalizer of two morphisms then the construction of the coproducts of two quadratic R-modules.

Proposition 3.1. *In* **Quad/R** *every pair of morphisms of crossed modules with common domain and codomain has a coequilaser.*

Proof: Let $f = (f_1, f_2)$, $g = (g_1, g_2) : \sigma_1 \to \sigma_2$ be two quadratic R-module morphisms as given in the following diagram.



$$\overline{\sigma}' : \overline{L}' \xrightarrow{\overline{\delta_2}'} \overline{M}' \xrightarrow{\overline{\delta_1}'} R$$

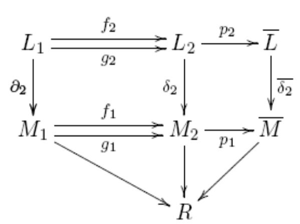
Let I be a normal subgroup of M_2 generated by elements of the form $f_1(m)-g_1(m)$, for all $m \in M_1$ and J be the normal subgroup of L_2 generated by elements of the form $f_2(l) - g_2(l)$, for all $l \in L_1$. Note that $I \subseteq \ker \delta_1$ and $\mathrm{Im}\delta_2$ is the normal subgroup of I. Set the factor groups $\overline{M} = M_2/I$ and $\overline{L} = L_2/J$. Define $\overline{\delta_1} : \overline{M} \to R$ as $\overline{\delta_1}(m_2 I) = \delta_1(m_2)$ and $\overline{\delta_2} : \overline{L} \to \overline{M}$ as $\overline{\delta_2}(l_2 J) = \delta_2(l_2)I$. In this case

$$\overline{\sigma} : \overline{L} \xrightarrow{\overline{\delta_2}} \overline{M} \xrightarrow{\overline{\delta_1}} R$$

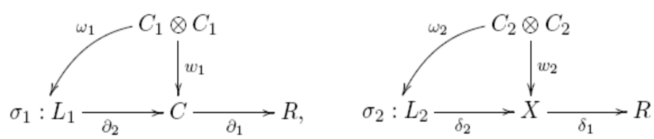
is a quadratic module with the quadratic map
$\overline{\omega} : (\overline{M})^{cr^{ab}} \otimes (\overline{M})^{cr^{ab}} \longrightarrow \overline{L}$ which is given by $\{m_2 I\} \otimes \{m_2' I\} \longmapsto \omega(\{m_2\} \otimes \{m_2'\})$.
Since $\overline{\delta_1}((m_2 I)^r) = \overline{\delta_1}({m_2}^r I) = \delta_1(\ {m_2}^r) = r^{-1}\delta_1(m_2)r = r^{-1}\overline{\delta_1}(m_2 I)r$
and for $m_2 I, m_2' I, m_2'' I \in \overline{M}$,

$$\begin{aligned}
\langle \langle m_2 I, m_2' I \rangle, m_2'' I \rangle &= \langle (m_2^{-1} I)(m_2'^{-1} I)(m_2 I)(m_2'^{\overline{\delta_1}(m_2' I)}, (m_2'' I) \rangle \\
&= \langle m_2^{-1} m_2'^{-1} m_2 m_2'^{\delta_1(m_2')} I, (m_2'' I) \rangle \\
&= \langle \langle m_2, m_2 \rangle I, m_2'' I \rangle = \langle \langle m_2, m_2' \rangle, m_2'' \rangle I \\
&= I,
\end{aligned}$$

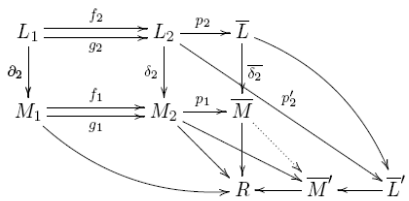
the map $\overline{\delta_1}$ is a nil(2)-module. We leave as an exercise to the reader the verification of remaining quadratic module axioms. Moreover, the induced map $p = (p_1, p_2) : \sigma_2 \to \overline{\sigma}$ is a quadratic module morphism. Namely the diagram



is commutative. Finally we will check the universal property of p. If there exist a quadratic module and a quadratic module morphism $p' = (p_1', p_2') : \sigma_2 \to \overline{\sigma}'$ then there exists a unique quadratic module morphism $\varphi = (\varphi_1', \varphi_2') : \overline{\sigma} \to \overline{\sigma}'$ which is $\varphi_2'(lJ) = p_2'(l)$ and $\varphi_1'(mJ) = p_1'(m)$ and satisfying $\varphi_1' p_1 = p_1'$ and $\varphi_2' p_2 = p_2'$.



Then p is universal morphism so we get the following commutative diagram



Therefore, p is an coequilaser of f and g. □

Construction of Coproduct

We give the construction of coproduct of two quadratic modules in the category of quadratic R-modules. Let

be two quadratic modules. Suppose that X acts on C via δ_1, so we can form the semidirect product

$$X \ltimes C = \{(x, c) : x \in X, c \in C\}$$

with the multiplication $(x, c)(x', c') = (xx'^{\delta_1 x'} c')$ for $(x, c), (x', c') \in X \ltimes C$, where an action of R on $X \ltimes C$ is given by $(x, c)^r = (x^r, c^r)$ for $r \in R$. We get the injections $i_1 : X \to X \ltimes C$, $i_1(x) = (x, 1)$ and $j_1 : C \to X \ltimes C$, $j_1(c) = (1, c)$. We define $\beta_1' : X \ltimes C \to R$ by $\beta_1'(x, c) = \delta_1 x \partial_1 c$. It is clear that β_1' is a well-defined homomorphism. Let P be a normal subgroup of $X \ltimes C$ generated by elements of the forms:

(1) $\langle\langle(x, c), (x', c')\rangle, (x'', c'')\rangle$
(2) $\langle(x, c), \langle(x', c'), (x'', c'')\rangle\rangle$

for $(x, c), (x', c'), (x'', c'') \in X \ltimes C$.

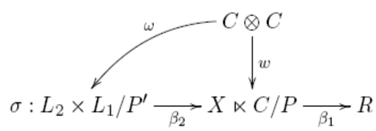
Thus we can form the factor group $X \ltimes C/P$ and we get an induced morphism $\beta_1 : X \ltimes C/P \to R$ as $\beta_1((x, c)P) = \delta_1 x \partial_1 c$. Clearly β_1 is a nil(2)-module. Furthermore, L_2 acts on L_1 via $\delta_1\delta_2$. Since $\delta_1\delta_2 = 1$, the semidirect product of L_2 and L_1 is direct product of L_2 and L_1, that is $L_2 \ltimes L_1 = L_2 \times L_1$. We get the injections $i_2 : L_2 \to L_2 \times L_1$, $i_2(l_2) = (l_2, 1)$ and $j_2 : L_1 \to L_2 \times L_1$, $j_2(l_1) = (1, l_1)$. We define the map $\beta_2' : L_2 \times L_1 \to X \ltimes C$ by $\beta_2'(l_2, l_1) = (\delta_2 l_2, \partial_2 l_1)$. β_2' is also a well-defined group homomorphism. Let P' be a normal subgroup of $L_2 \times L_1$ generated by elements of the forms

(1) $(\omega_2(\{x_1\} \otimes \{\langle x_2, x_3\rangle\}), \omega_1(\{c_1\} \otimes \{\langle c_2, c_3\rangle\}))$,
(2) $(\omega_2(\{\langle x_1, x_2\rangle\} \otimes \{x_3\}), \omega_1(\{\langle c_1, c_2\rangle\} \otimes \{c_3\}))$

for all $(x_1, c_1), (x_2, c_2), (x_3, c_3) \in X \ltimes C$.

We can form the factor group $L_2 \times L_1/P'$ and we define a map $\beta_2 : L_2 \times L_1/P' \to X \ltimes C/P$ as $\beta_2((l_2, l_1)P') = (\delta_2 l_2, \partial_2 l_1)P$. An action of $X \ltimes C/P$ on $L_2 \times L_1/P'$ is given via β_1 such that $((l_2, l_1)P'^{(x,c)P} = ((l_2, l_1)P'^{\beta_1(x,c)P} = ({l_2}^{\delta_1 x}, {l_1}^{\partial_1 c})P'$ for $(x, c)P \in X \ltimes C/P$ and $(l_2, l_1)P' \in L_2 \times L_1/P'$. Then we get the following result.

Proposition 3.2.



is a quadratic R-module.

Proof: Firstly, we define the quadratic map ω by

$$\omega(\{(x, c)P\} \otimes \{(x', c')P\}) = (\omega_2(\{x\} \otimes \{x'\}), \omega_1(\{c\} \otimes \{c'\}))P'$$

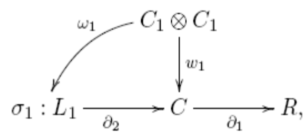
for $(x, c)P, (x', c')P \in X \ltimes C/P$.

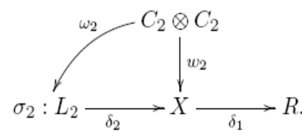
QM1) We know that β_1 is a nil(2)-module and

$$\beta_1\beta_2((l_2, l_1)P) = \beta_1((\delta_2(l_2), \partial_2(l_1))P) = \delta_1\delta_2(l_2)\partial_1\partial_2 l_1 = 1.$$

QM2) For $(x, c)P, (x', c')P \in X \ltimes C/P$ we get

$$\begin{aligned} \beta_2\omega(\{(x, c)P\} \otimes \{(x', c')P\}) &= \beta_2((\omega_2(\{x\} \otimes \{x'\}), \omega_1(\{c\} \otimes \{c'\}))P') \\ &= (\delta_2\omega_2(\{x\} \otimes \{x'\}), \partial_2\omega_1(\{c\} \otimes \{c'\}))P \\ &= (w_2(\{x\} \otimes \{x'\}), w_1(\{c\} \otimes \{c'\})P) \\ &= (\langle x, x'\rangle, \langle c, c'\rangle)P = \langle (x, c), (x', c')\rangle P \\ &= w(\{(x, c)P\} \otimes \{(x', c')P\}). \end{aligned}$$





QM3) For $(x,c)P \in X \ltimes C/P$ and $(l_2,l_1)P' \in L_2 \times L_1/P'$, we obtain

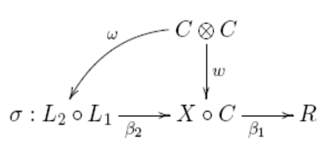
$$\begin{aligned} &\omega(\{\beta_2((l_2,l_1)P')\} \otimes \{(x,c)P\}\{(x,c)P\} \otimes \{\beta_2((l_2,l_1)P')\}) \\ =&\omega(\{(\delta_2 l_2, \partial_2 l_1)P\} \otimes \{(x,c)P\}\{(x,c)P\} \otimes \{(\delta_2 l_2, \partial_2 l_1)P\}) \\ =&(\omega_2(\{\delta_2 l_2\} \otimes \{x\}\{x\} \otimes \{\delta_2 l_2\}), \omega_1(\{\delta_2 l_2\} \otimes \{c\}\{c\} \otimes \{\delta_2 l_2\}))P' \\ =&({l_2}^{-1}{l_2}^{\delta_1(x)}, {l_1}^{-1}{l_1}^{\partial_1(c)})P' = (l_2,l_1)^{-1}({l_2}^{\delta_1 x}, {l_1}^{\partial_1 c})P' \\ =&(l_2,l_1)^{-1}(l_2,l_1)^{\beta_1(x,c)P}P'. \end{aligned}$$

QM4) For $(l_2,l_1)P', (l'_2,l'_1)P' \in L_2 \times L_1/P'$, we have

$$\begin{aligned} \omega(\{\beta_2((l_2,l_1)P')\} \otimes \{\beta_2((l'_2,l'_1)P')\}) &= \omega(\{(\delta_2 l_2, \partial_2 l_1)P\} \otimes \{(\delta_2 l'_2, \partial_2 l'_1)P\}) \\ &= (\omega_2(\{\delta_2 l_2\} \otimes \{\delta_2 l'_2\}), \omega_1(\{\partial_2 l_1\} \otimes \{\partial_2 l'_1\}))P' \\ &= ([l_2,l'_2],[l_1,l'_1])P' \\ &= [(l_2,l_1)P', (l'_2,l'_1)P']. \end{aligned}$$

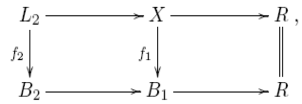
□

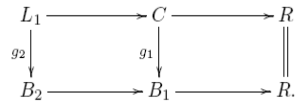
Theorem 3.3. *The constructed quadratic module*



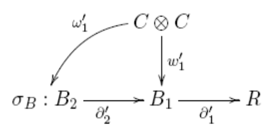
where $L_2 \circ L_1 = L_2 \times L_1/P'$ *and* $X \circ C = X \ltimes C/P$ *with the morphisms* $i = (i_1, i_2)$, $j = (j_1, j_2)$ *is the coproduct of the quadratic modules*

Proof: We will check the universal property of morphisms (i_1, j_1) into $(X \ltimes C/P, \beta_1)$ and (i_2, j_2) into $(L_2 \times L_1/P', \beta_2)$.



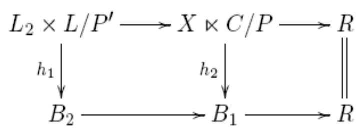


Consider an arbitrary quadratic R-module

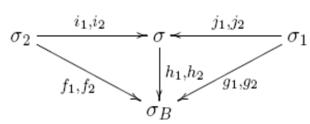


and morphisms of quadratic R-modules $f = (f_2, f_1) : \sigma_2 \to \sigma_B$, $g = (g_2, g_1) : \sigma_1 \to \sigma_B$, i.e.

Then there is a map $h = (h_1, h_2) : \sigma \to \sigma_B$ given by $h_1((l_2, l_1)P') = f_1(l_2)g_1(l_1)$, $h_2((x, c)P) = f_2(x)g_2(c)$.



It is a unique morphism of quadratic modules for the diagram



to commute. Actually we obtain

$$h_2 i_2(l_2) = h_2(l_2, 1) = f_2(l_2)g_2(1) = f_2(l_2),$$

$$h_1 i_1(x) = h_1(x, 1) = f_1(x)g_1(1) = f_1(x),$$

$$h_2 j_2(l_1) = h_2(1, l_1) = f_2(1)g_2(l_1) = g_2(l_1),$$

$$h_1 j_1(c) = h_1(1, c) = f_1(1)g_1(c) = g_1(c).$$

□

The construction of coproducts in **Quad/R** will give us a functor

$$\circ : \mathbf{Quad/R} \times \mathbf{Quad/R} \to \mathbf{Quad/R}$$

which is left adjoint to the diagonal functor

$$\triangle : \mathbf{Quad/R} \to \mathbf{Quad/R} \times \mathbf{Quad/R}.$$

Proposition 3.4. $\mathbf{Quad/R}$ *has all colimits for any functor* $\mathbf{J} : \boldsymbol{\sigma} \longrightarrow \mathbf{Quad/R}$, *i.e.* $\mathbf{Quad/R}$ *is cocomplete.*

Proof: Since $\mathbf{Quad/R}$ has coproduct and coequilaser, it is clear. □

4. Example for Coproduct of Quadratic Modules

We will give a description of the coproduct of quadratic modules in the particular case of two quadratic modules $\beta_1 : L_1 \to C \to R$ and $\beta_2 : L_2 \to X \to R$ in the useful case when $v_1(X) \subseteq \mu_1(C)$ and there is a P-equivariant section $\sigma : \mu_1 c \to C$ of μ_1.

Definition 4.1. *If C acts on the group X we define $[X, C]$ to be the subgroup of X generated by the elements $x^{-1}x^c$ for all $x \in X, c \in C$. This subgroup is called the displacement subgroup.*

Proposition 4.2. *The displacement subgroup $[X, C]$ is a normal subgroup of X.*

Proof: Let $c \in C, x, x_1 \in X$. We easily check that

$$x_1^{-1}(x^{-1}x^c)x_1 = (xx_1)^{-1}(xx_1)^c(x_1^{-1}x_1^c)^{-1} \in [X, C]$$

□

Definition 4.3. *We define $X/[X, C]$ as a quotient of X by displacement subgroup. The elements of $X/[X, C]$ is written by $[x]$. It is clear that $X/[X, C]$ is a trivial C-module since $[x^c] = [x]$.*

Proposition 4.4. *Let $\mu_1 : C \to R, \quad v_1 : X \to R$ be nil(2)-modules, so that C acts on X via μ_1. Then R acts on $X/[X, C]$ by $[x]^r = [x]^r$ for $r \in R$. Moreover this action is trivial when restricted to $\mu_1 C$.*

Proof: It is because $(x^{-1}x^c)^r = (x^{-1})^r(x^c)^r = (x^r)^{-1}(x^r)^{c^r}$ for all $x \in X, c \in C, r \in R$. The action is trivial since $[x]^{\mu_1 c} = [x^{\mu_1 c}] = [x^c] = [x]$. □

Proposition 4.5. *Let $\mu_1 : C \to R, \quad v_1 : X \to R$ be nil(2)-modules, such that $v_1(X) \subseteq \mu_1(C)$. Then $X/[X, C]$ is Abelian and therefore $\varepsilon : C \times X/[X, C] \to R$ given by $\varepsilon(c, [x]) = \mu_1 c$ is a nil(2)-module.*

Proof: Let $x_1, x_2, x_3 \in X$. Choose $c \in C$ such that $v_1 x_1 = \mu_1 c$. Then

$$
\begin{aligned}
&\langle\langle [x_1], [x_2] \rangle, [x_3] \rangle \\
&= \langle [x_1^{-1}][x_2^{-1}][x_1][x_2^{v_1 x_1}], [x_3] \rangle \\
&= [(x_2^{v_1 x_1})^{-1}][x_1^{-1}][x_2][x_1][x_3^{-1}][x_1^{-1}][x_2^{-1}][x_1][x_2^{v_1 x_1}][x_3^{v_1(x_1^{-1}x_2^{-1}x_1 x_2^{v_1 x_1})}] \\
&= ([x_2{}^{\mu_1 c}])^{-1}[x_1^{-1}][x_2][x_1][x_3^{-1}][x_1^{-1}][x_2^{-1}][x_1][x_2^{\mu_1 c}][x_3] \\
&= ([x_2^{-1}][x_1^{-1}][x_2][x_1])[x_3]^{-1}([x_2^{-1}][x_1^{-1}][x_2][x_1])^{-1}[x_3] = 1.
\end{aligned}
$$

therefore $X/[X, C]$ is Abelian. Now we will show that ε is a nil(2)-module

$$
\begin{aligned}
&\langle\langle (c_1, [x_1]), (c_2, [x_2]) \rangle, (c_3, [x_3]) \rangle \\
&= \langle (c_1, [x_1])^{-1}(c_2, [x_2])^{-1}(c_1, [x_1])(c_2, [x_2])^{\varepsilon(c_1, [x_1])}, (c_3, [x_3]) \rangle \\
&= \langle (c_1^{-1} c_2^{-1} c_1 c_2^{\mu_1 c_1}, [x_1]^{-1}[x_2]^{-1}[x_1][x_2]), (c_3, [x_3]) \rangle \\
&= \langle (\langle c_1, c_2 \rangle, 1), (c_3, [x_3]) \rangle \\
&= (\langle c_1, c_2 \rangle, 1)^{-1}(c_3, [x_3])^{-1}(\langle c_1, c_2 \rangle, 1)(c_3, [x_3])^{\varepsilon(\langle c_1, c_2 \rangle, 1)} \\
&= (\langle c_1, c_2 \rangle^{-1} c_3^{-1} \langle c_1, c_2 \rangle c_3^{\mu_1(\langle c_1, c_2 \rangle)}, 1) \\
&= (\langle\langle c_1, c_2 \rangle, c_3 \rangle, 1) = (1, 1)
\end{aligned}
$$

□

Proposition 4.6. *Let* $\beta_1 : L_1 \xrightarrow{\mu_2} C \xrightarrow{\mu_1} R$ *and* $\beta_2 : L_2 \xrightarrow{v_2} X \xrightarrow{v_1} R$ *be two quadratic modules with* $v_1(X) \subseteq \mu_1(C)$ *and* L_2 *is an Abelian group with trivial action of* $\mu_1 C$. *Let* $\sigma_1 : \mu_1(C) \longrightarrow C$ *be an R-equivariant section of* μ_1. *Then the morphisms of quadratic modules*

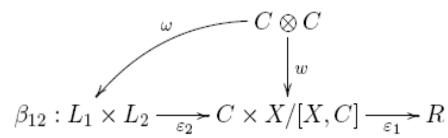
$$
\begin{array}{ll}
i_1 : C \longrightarrow C \times X/[X, C]; & c \mapsto (c, 1) \\
i_2 : L_1 \longrightarrow L_1 \times L_2; & l_1 \mapsto (l_1, 1) \\
j_1 : X \longrightarrow C \times X/[X, C]; & x \mapsto (\sigma_1 v_1 x, [x]) \\
j_2 : L_2 \longrightarrow L_1 \times L_2; & l_2 \mapsto (1, l_2)
\end{array}
$$

give a coproduct of quadratic modules. Hence the canonical morphism of quadratic modules

$$
\begin{array}{ll}
C \circ X \longrightarrow C \times X/[X, C]; & c \circ x \mapsto (c \sigma_1 v_1 x, [x]) \\
L_1 \circ L_2 \longrightarrow L_1 \times L_2; & l_1 \circ l_2 \mapsto (l_1, l_2)
\end{array}
$$

is an isomorphism.

Proof: It can be shown easily that



is a quadratic module where

$\varepsilon_1 : C \times X/[X,C] \longrightarrow R; \quad (c,[x]) \mapsto \mu_1 c$

$\varepsilon_2 : L_1 \times L_2 \longrightarrow C \times X/[X,C]; \quad \varepsilon_2(l_1,l_2) \mapsto (\mu_2 l_1, 1)$

$\omega : C \otimes C \longrightarrow L_1 \times L_2; \quad \{(c_1,[x_1])\} \otimes \{(c_2,[x_2])\} \mapsto (\omega_1(\{c_1\} \otimes \{c_2\}), 1)$

Then one can show that pairs (i_1, j_1) and (i_2, j_2) satisfies the universal property of the coproduct of quadratic modules. □

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