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ON THE SPECTRUMS OF SOME CLASS OF SELFADJOINT SINGULAR DIFFERENTIAL OPERATORS

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Abstract. In this work, based on the Everitt-Zettl and Calkin-Gorbachuk methods in terms of boundary values all selfadjoint extensions of the minimal operator generated by some linear singular multipoint symmetric differential-operator expression for first order in the direct sum of Hilbert spaces of vector-functions on the right semi-axis are described. Later structure of the spectrum of these extensions is investigated.

1. Introduction

The general theory of selfadjoint extensions of symmetric operators in any Hilbert space and their spectral theory have deeply been investigated by many mathematicians (for example, see [1-6]). Applications of this theory to two point differential operators in Hilbert space of functions are continued today even. It is known that for the existence of selfadjoint extension of the any linear closed densely defined symmetric operator B in a Hilbert space, the necessary and sufficient condition is an equality of deficiency indices $m(B) = n(B)$, where $m(B) = dimker(B^* + i)$, $n(B) = dimker(B^* - i)$ [1]. The table is changed in the multipoint case in the following sense. Let L_1 and L_2 be minimal operators generated by the linear differential expression $l(u) = i\frac{d}{dt}$ and $m(u) = -i\frac{d}{dt}$ in the Hilbert space of functions $L^2(a, +\infty)$ and $L^2(b, +\infty)$, $a, b \in \mathbb{R}$, respectively. Consider the deficiency indices of L_1 and L_2. In this case it is known that $(m(L_1), n(L_1)) = (1, 0)$, $(m(L_2), n(L_2)) = (0, 1)$. Consequently, L_1 and L_2 are maximal symmetric operators, but are not selfadjoint [1]. However, direct sum $L = L_1 \oplus L_2$ of operators L_1 and L_2 in $L^2(a, +\infty) \oplus L^2(b, +\infty)$ of Hilbert spaces have an equal defect numbers $(1, 1)$. Then by the general theory [1] it has a selfadjoint extension. On the other hand it can be easily shown in the form that

$$u_2(b) = e^{i\varphi} u_1(a), \varphi \in [0, 2\pi), u = (u_1, u_2), \quad u_1 \in D(L_1^*), u_2 \in D(L_2^*)).$$

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Note that in the multi interval linear ordinary differential expression case the deficiency indices may be different for each interval, but equal in the direct sum Hilbert spaces from the different intervals. The selfadjoint extension theory for any order linear ordinary differential expression case is known from famous work of W.N. Everitt and A. Zettl [7] for any number of finite and infinite intervals of real-axis. This theory is based on the Glazman-Krein-Naimark Theorem. In formations on the selfadjoint extensions, the direct and complete characterizations for the Sturm-Liouville differential expression in finite or infinite interval with interior points or endpoints singularities can be found in the significant monograph of A.Zettl [8].

Lastly, note that many problems arising in the modeling of processes of multi-particle quantum mechanics, quantum field theory, in the physics of rigid bodies and etc. support to study selfadjoint extensions of symmetric differential operators in direct sum of Hilbert spaces (see [8] and references in it).

In this work in second section, by the methods of Everitt-Zettl and Calkin-Gorbachuk Theories, all selfadjoint extensions of the minimal operator generated by linear multipoint singular symmetric differential-operator expression of first order in the direct sum of Hilbert spaces $L^2(H_1,(a,+\infty))\oplus L^2(H_2,(b,+\infty))$, described, where H_1, H_2 are a separable Hilbert spaces with condition $0 < dimH_1 = dimH_2$ and $a, b \in \mathbb{R}$, in terms of boundary values. In third section the spectrum of such extensions is researched.

2. Description of Selfadjoint Extensions

Let H_1, H_2 be separable Hilbert space with $0 < dimH_1 = dimH_2 \leq \infty$ and $a, b \in \mathbb{R}$, $a < b$. In the Hilbert space $L^2(H_1,(a,+\infty))\oplus L^2(H_2,(b,+\infty))$ of vector-functions considers the following linear multipoint differential-operator expressions

$$l(u) = (l_1(u_1), l_2(u_2)), \quad where \ \ u = (u_1, u_2),$$

$$l_1(u_1) = iu_1^{'}(t) + A_1u_1(t), \quad t \in (a,+\infty),$$

$$l_2(u_2) = -iu_2^{'}(t) + A_2u_2(t), \quad t \in (b,+\infty),$$

where $A_k : D(A_k) \subset H_k \to H_k$ are linear selfadjoint operators in H_k, $k = 1, 2$. In the linear manifold $D(A_k) \subset H_k$ introduces the inner product in form

$$(f, g)_{k,+} = (A_kf, A_kg)_{H_k} + (f, g)_{H_k}, \quad f, g \in D(A_k), \quad k = 1, 2.$$

For $k = 1, 2$ $D(A_k)$ is a Hilbert space under the positive norm $\|\cdot\|_{k,+}$ respect to the Hilbert space H_k. It is denoted by $H_{k,+}$, $k = 1, 2$. Denote the Hilbert spaces with the negative norm by $H_{k,-}$, $k = 1, 2$. It is clear that an operator A_k is continuous from $H_{k,+}$ to H_k and that its adjoint operator $\tilde{A}_k : H_k \to H_{k,-}$ is an extension of the operator A_k, $k = 1, 2$. On the other hand, the operator $\tilde{A}_k : H_k \subset H_{k,-} \to H_{k,-}$ $k = 1, 2$ are linear selfadjoint operators.

In $L^2(H_1,(a,+\infty))\oplus L^2(H_2,(b,+\infty))$ define

$$\tilde{l}(u) = (\tilde{l}_1(u_1), \tilde{l}_2(u_2)), \tag{2.1}$$

where $u = (u_1, u_2)$, $\tilde{l}_1(u_1) = iu_1^{'}(t) + \tilde{A}_1 u_1(t)$, $t \in (a, +\infty)$, $\tilde{l}_2(u_2) = -iu_2^{'}(t) + \tilde{A}_2 u_2(t)$, $t \in (b, +\infty)$.

The minimal L_{10} (L_{20}) and maximal L_1 (L_2) operators generated by differential-operator expression $\tilde{l}_1(\cdot)$ ($\tilde{l}_2(\cdot)$) in $L^2(H_1, (a, +\infty))$ ($L^2(H_2, (b, +\infty))$) have been investigated in [5] and here established that the minimal operator L_{10} (L_{20}) is not selfadjoint in $L^2(H_1, (a, +\infty))$ ($L^2(H_2, (b, +\infty))$). The operators defined by $L_0 = L_{10} \oplus L_{20}$ and $L = L_1 \oplus L_2$ in $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$ are called minimal and maximal (multipoint) operators generated by the differential expression (2.1), respectively. Note that the operator L_0 is a symmetric operator in $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$. On the other hand, it is clear that

$$m(L_{10}) = 0, \quad n(L_{10}) = dimH_1,$$

$$m(L_{20}) = dimH_2, \quad n(L_{20}) = 0.$$

Consequently, $m(L_0) = dimH_2 > 0$, $n(L_0) = dimH_1 > 0$. So the minimal operator L_0 in $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$ has a selfadjoint extension [1].

In first note that the following proposition which validity of this clear can be easily proved.

Proposition 2.1. *Let us L_{n0}, M_{n0} and K_{n0} be minimal operators generated by linear differential expressions*

$$l_n(u_n) = (-1)^{n-1} iu_n^{'}(t) + A_n u_n(t), \quad t \in (a_n, +\infty), \quad a_n \in \mathbb{R},$$

$$m_n(u_n) = iu_n^{'}(t) + B_n u_n(t), \quad t \in (-\infty, b_n), \quad b_n \in \mathbb{R},$$

$$k_n(u_n) = iu_n^{'}(t) + C_n u_n(t), \quad t \in (c_n, +\infty), \quad c_n \in \mathbb{R}, \quad n = 1, 2, ..., m,$$

where $A_n : D(A_n) \subset H_n \to H_n$, $B_n : D(B_n) \subset H_n \to H_n$, $C_n : D(C_n) \subset H_n \to H_n$ are linear selfadjoint operators in the Hilbert space of vector-functions $L^2(H_n, (a_n, +\infty))$, $L^2(H_n, (-\infty, b_n))$ and $L^2(H_n, (c_n, +\infty))$, $n = 1, 2, ..., m$, respectively and $dimH_1 = dimH_2 = ... = dimH_m \leq \infty$.

In this case:

(1) *For any $n = 1, 2, ..., m$ the minimal operators L_{n0}, M_{n0} and K_{n0} have not selfadjoint extensions in $L^2(H_n, (a_n, +\infty))$, $L^2(H_n, (-\infty, b_n))$ and $L^2(H_n, (c_n, +\infty))$, $n = 1, 2, ..., m$, respectively (see[5]).*
(2) *If m is a even integer number, then the multipoint minimal operator $L_0 = \oplus_{n=1}^{m} L_{n0}$ have a selfadjoint extension in $\oplus_{n=1}^{m} L^2(H_n, (a_n, +\infty))$;*
(3) *If m is a odd integer number, then the multipoint minimal operator $L_0 = \oplus_{n=1}^{m} L_{n0}$ is not selfadjoint extension in $\oplus_{n=1}^{m} L^2(H_n, (a_n, +\infty))$;*
(4) *The multipoint minimal operator $L_0 = M_{10} \oplus K_{10} \oplus M_{20} \oplus K_{20} \oplus .... \oplus M_{m0} \oplus K_{m0}$ is a selfadjoint operator in $\oplus_{n=1}^{m} (L^2(H_n, (-\infty, b_n))) \oplus L^2(H_n, (c_n, +\infty))$;*
(5) *The multipoint minimal operator $L_0 = M_{10} \oplus K_{10} \oplus M_{20} \oplus K_{20} \oplus .... \oplus M_{(m-1)0} \oplus K_{(m-1)0} \oplus M_{m0}$ is not selfadjoint operator in $\oplus_{n=1}^{m-1} (L^2(H_n, (-\infty, b_n))) \oplus L^2(H_n, (c_n, +\infty)) \oplus L^2(H_m, (-\infty, b_m))$;*

(6) *The multipoint minimal operator* $L_0 = M_{10} \oplus K_{10} \oplus M_{20} \oplus K_{20} \oplus \ldots. \oplus M_{(m-1)0} \oplus K_{(m-1)0} \oplus K_{m0}$ *is not selfadjoint operator in* $\oplus_{n=1}^{m-1}(L^2(H_n,(-\infty,b_n))) \oplus L^2(H_n,(c_n,+\infty)) \oplus L^2(H_m,(c_m,+\infty))$.

In this section all selfadjoint extensions of the minimal operator L_0 generated by linear multipoint symmetric differential-operator expression of first order (2.1) in the direct sum of Hilbert spaces $L^2(H_1,(a,+\infty)) \oplus L^2(H_2,(b,+\infty))$ in terms of the boundary values will be described. Note that in the Calkin-Gorbachuk theory of selfadjoint extensions of the linear symmetric densely defined closed operators co-called "space of boundary values" has an important role [3,4].

Firstly, let us recall their definition.

Definition 2.1. *[3] Let* $T : D(T) \subset H \to H$ *be a closed densely defined symmetric operator in the Hilbert space* H*, having equal finite or infinite deficiency indices. A triplet* $(\mathfrak{H},\gamma_1,\gamma_2)$*, where* $\mathfrak{H}$ *is a Hilbert space,* γ_1 *and* γ_2 *are linear mappings of* $D(T^*)$ *into* $\mathfrak{H}$*, is called a space of boundary values for the operator* T *if for any* $f,g \in D(T^*)$

$$(T^*f,g)_H - (f,T^*g)_H = (\gamma_1(f),\gamma_2(g))_{\mathfrak{H}} - (\gamma_2(f),\gamma_1(g))_{\mathfrak{H}},$$

while for any $F_1,F_2 \in \mathfrak{H}$*, there exists an element* $f \in D(T^*)$*, such that* $\gamma_1(f) = F_1$ *and* $\gamma_2(f) = F_2$*.*

Note that any symmetric operator with equal deficiency indices has at least one space of boundary values [3].

Since H_1*,* H_2 *are separable Hilbert spaces and* $dimH_1 = dimH_2$*, then it is known that there exist a isometric isomorphism* $V : H_1 \to H_2$ *such that* $VH_1 = H_2$*.*

In this case the following proposition is true.

Lemma 2.1. *The triplet* (H_2,γ_1,γ_2) *is a space of boundary values of the minimal operator* L_0 *in* $L^2(H_1,(a,+\infty)) \oplus L^2(H_2,(b,+\infty))$*, where*

$$\gamma_1 : D(L_0^*) \to H_1, \gamma_1(u) = \frac{1}{i\sqrt{2}}\left(Vu_1(a) + u_2(b)\right), u \in D(L_0^*),$$

$$\gamma_2 : D(L_0^*) \to H_1, \gamma_2(u) = \frac{1}{\sqrt{2}}\left(Vu_1(a) - u_2(b)\right),\ u \in D(L_0^*).$$

Proof. For arbitrary $u = (u_1,u_2)$ and $v = (v_1,v_2)$ in $D(L)$ the validity of following equality

$$(Lu,v)_{L^2(H_1,(a,+\infty))\oplus L^2(H_2,(b,+\infty))} - (u,Lv)_{L^2(H_1,(a,+\infty))\oplus L^2(H_2,(b,+\infty))}$$
$$= (\gamma_1(u),\gamma_2(v))_{H_1} - (\gamma_2(u),\gamma_1(v))_{H_1}$$

can be easily verified. Now for any given elements $f,g \in H_1$, we will find the function $u = (u_1,u_2) \in D(L)$ such that

$$\gamma_1(u) = \frac{1}{i\sqrt{2}} \left(Vu_1\left(a\right) + u_2\left(b\right)\right) = f \quad and \quad \gamma_2\left(u\right) = \frac{1}{\sqrt{2}} \left(Vu_1\left(a\right) - u_2\left(b\right)\right) = g$$

that is,

$$u_1\left(a\right) = V^{-1}(if+g)/\sqrt{2} \quad and \quad u_2\left(b\right) = (if-g)/\sqrt{2}.$$

If we choose these functions $u_1\left(t\right), u_2\left(t\right)$ in form

$$u_1\left(t\right) = e^{(a-t)/2}V^{-1}(if+g)/\sqrt{2}, \quad t>a,$$

$$u_2\left(t\right) = e^{(b-t)/2}(if-g)/\sqrt{2}, \quad t>b,$$

then it is clear that $(u_1, u_2) \in D(L)$ and $\gamma_1\left(u\right) = f$, $\gamma_2\left(u\right) = g$. □

Furthermore, using the method in [1, 3] the following result can be deduced.

Theorem 2.2. *If $\tilde{L}$ is a selfadjoint extension of the minimal operator L_0 in $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$, then it generates by differential-operator expression (2.1) and boundary condition*

$$u_2(b) = WVu_1(a)$$

where $W : H_2 \to H_2$ is a unitary operator. Moreover, the unitary operator W is determined uniquely by the extension $\tilde{L}$; i.e., $\tilde{L} = L_W$ and vice versa.

Remark 2.1. *With similar ideas the selfadjoint extensions of minimal operator generated by multipoint differential-operator expression in $\oplus_{p=1}^{n} L^2(H_p, (a_p, \infty)) \oplus \oplus_{j=1}^{k} L^2(G_j, (b_j, \infty))$ with condition $0 < \sum_{p=1}^{n} dimH_p = \sum_{j=1}^{k} dimG_j$, can be described*

$$l(u) = (l_1(u_1), l_2(u_2), ..., l_n(u_n); m_1(v_1), m_2(v_2), ..., m_k(v_k)),$$

where $u = (u_1, u_2, ..., u_n; v_1, v_2, ..., v_k)$,

$$l_p(u_p) = iu_p^{'}(t) + A_p u_p(t), \quad t \in (a_p, \infty), \quad p = 1, 2, ..., n;$$

$$m_j(v_j) = -iu_j^{'}(t) + B_j u_j(t), \quad t \in (b_j, \infty), \quad j = 1, 2, ..., k,$$

$A_p : D(A_p) \subset H_p \to H_p$ and $B_j : D(B_j) \subset G_j \to G_j$ are linear selfadjoint operators in Hilbert spaces H_k, $p = 1, 2, ..., n$ and G_j, $j = 1, 2, ..., k$, respectively.

3. The Spectrum of the Normal Extensions

In this section the structure of the spectrum of the selfadjoint extension L_W in $L^2(H_1, (a, +\infty)) \oplus L^2(H_2, (b, +\infty))$ will be investigated.

First, we will prove the following result.

Theorem 3.1. *The point spectrum of selfadjoint extension L_W is empty, i.e.,*

$$\sigma_p\left(L_W\right) = \emptyset.$$

Proof. Let us consider the following problem

$$\tilde{l}(u) = \lambda u(t), \lambda \in \mathbb{R},$$
$$u_2(b) = WVu_1(a),$$

where $W : H_2 \to H_2$ is a unitary operator. Then

$$(\tilde{l}_1(u_1), \tilde{l}_2(u_2) = \lambda(u_1, u_2), \quad u_2(b) = WVu_1(a),$$

and we have

$$\begin{gathered} \tilde{l}_1(u_1) = iu_1^{'}(t) + \tilde{A}_1 u_1(t) = \lambda u_1(t), \quad t \in (a, +\infty), \\ \tilde{l}_2(u_2) = -iu_2^{'}(t) + \tilde{A}_2 u_2(t) = \lambda u_2(t), \quad t \in (b, +\infty), \\ u_2(b) = WVu_1(a), \quad \lambda \in \mathbb{R}. \end{gathered}$$

The general differential solution of this problem is

$$\begin{gathered} u_1(\lambda; t) = e^{i(\tilde{A}_1 - \lambda)(t-a)} f_\lambda, \quad f_\lambda \in H_1, \quad t \in (a, +\infty), \\ u_2(\lambda; t) = e^{-i(\tilde{A}_2 - \lambda)(t-b)} g_\lambda, \quad g_\lambda \in H_2, \quad t \in (b, +\infty). \end{gathered}$$

Boundary condition is in form $g_\lambda = WVf_\lambda$. In order to show $u_1(\lambda; t) \in L^2(H_1, (a, +\infty))$ and $u_2(\lambda; t) \in L^2(H_2, (b, +\infty))$, the necessary and sufficient conditions are $f_\lambda = g_\lambda = h_\lambda = 0$. So for every operator W we have $\sigma_p(L_W) = \emptyset$, where $\sigma_p(L_W)$ denotes the point spectrum of L_W. □

Since residual spectrum of any selfadjoint operator in any Hilbert space is empty, then it is sufficient to investigate the continuous spectrum of the selfadjoint extensions L_W of the minimal operator L_0.

Now, we will study continuous spectrum of the selfadjoint extension L_W, where $\sigma_c(L_W)$ denotes the continuous spectrum of L_W.

Theorem 3.2. *The continuous spectrum of any selfadjoint extension L_W is $\sigma_c(L_W) = \mathbb{R}$.*

Proof. For $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda > 0$, norm of the resolvent operator of the L_W is of the form

$$\|R_\lambda(L_W)f(t)\|^2_{L^2(H_1,(a,\infty))\oplus L^2(H_2,(b,\infty))} = \|i\int_t^\infty e^{i(\tilde{A}_1-\lambda)(t-s)} f_1(s)ds\|^2_{L^2(H_1,(a,\infty))} +$$

$$+\|e^{i(\lambda-\tilde{A}_1)(t-b)} g_\lambda + i\int_b^t e^{i(\lambda-\tilde{A}_1)(t-s)} f_2(s)ds\|^2_{L^2(H_2,(b,\infty))}$$

where $f = (f_1, f_2) \in L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))$,

$$g_\lambda = WV(i\int_a^\infty e^{i(\tilde{A}_1-\lambda)(a-s)} f_1(s)ds)$$

and $R_\lambda(L_W)$ shows the resolvent operator of L_W. Then, it is clear that for any f in $L^2(H_1, (a, \infty)) \oplus L^2(H_2, (b, \infty))$, the following inequality is true

$$\|R_\lambda(L_W)f(t)\|^2_{L^2} \geq \|i\int_t^\infty e^{i(\tilde{A}_1-\lambda)(t-s)} f_1(s)ds\|^2_{L^2(H_1,(a,\infty))}.$$

The vector functions $f^*(\lambda;t)$ which is of the form $f^*(\lambda;t) = (e^{i(\tilde{A}_1-\bar{\lambda})t}f, 0)$, $\lambda \in \mathbb{C}$, $\lambda_i = Im\lambda > 0$, $f \in H_1$ belong to $L^2(H_1,(a,\infty)) \oplus L^2(H_2,(b,\infty))$. Indeed,

$$\|f^*(\lambda;t)\|^2_{L^2(H_1,(a,\infty))\oplus L^2(H_2,(b,\infty))} = \int_a^\infty \|e^{i(\tilde{A}_1-\bar{\lambda})t}f\|^2_{H_1}dt$$

$$= \int_a^\infty e^{-2\lambda_i t}dt\|f\|^2_{H_2} = \frac{1}{2\lambda_i}e^{-2\lambda_i a} < \infty.$$

For such functions $f^*(\lambda;\cdot)$, we have

$$\|R_\lambda(L_W)f^*(\lambda;t)\|^2_{L^2(H_1,(a,\infty))\oplus L^2(H_2,(b,\infty))}$$

$$\begin{aligned} &\geq \|i\int_t^\infty e^{i(\tilde{A}_1-\lambda)(t-s)}e^{i(\tilde{A}_1-\bar{\lambda})s}fds\|^2_{L^2(H_1,(a,\infty))} \\ &= \|\int_t^\infty e^{-i\lambda t}e^{-2\lambda_i s}e^{i\tilde{A}_1 t}fds\|^2_{L^2(H_1,(a,\infty))} \\ &= \|e^{-i\lambda t}e^{i\tilde{A}_1 t}\int_t^\infty e^{-2\lambda_i s}fds\|^2_{L^2(H_1,(a,\infty))} \\ &= \|e^{-i\lambda t}\int_t^\infty e^{-2\lambda_i s}ds\|^2_{L^2(H_1,(a,\infty))}\|f\|^2_{H_1} \\ &= \frac{1}{4\lambda_i^2}\int_a^\infty e^{-2\lambda_i t}dt\|f\|^2_{H_1} = \frac{1}{8\lambda_i^3}e^{-2\lambda_i a}\|f\|^2_{H_1}. \end{aligned}$$

From this, we obtain

$$\|R_\lambda(L_W)f^*(\lambda;\cdot)\|^2_{L^2(H_1,(a,\infty))\oplus L^2(H_2,(b,\infty))} \geq \frac{e^{-\lambda_i a}}{2\sqrt{2}\lambda_i\sqrt{\lambda_i}}\|f\|_{H_1}$$

$$= \frac{1}{2\lambda_i}\|f^*(\lambda;\cdot)\|_{L^2(H_1,(a,\infty))\oplus L^2(H_2,(b,\infty))}$$

i.e., for $\lambda_i = Im\lambda > 0$ and $f \neq 0$, the following inequality is valid

$$\frac{\|R_\lambda(L_W)f^*(\lambda;\cdot)\|_{L^2(H_1,(a,\infty))\oplus L^2(H_2,(b,\infty))}}{\|f^*(\lambda;\cdot)\|_{L^2(H_1,(a,\infty))\oplus L^2(H_2,(b,\infty))}} \geq \frac{1}{2\lambda_i}$$

is valid. On the other hand, it is clear that

$$\|R_\lambda(L_W)\| \geq \frac{\|R_\lambda(L_W)f^*(\lambda;\cdot)\|_{L^2(H_1,(a,\infty))\oplus L^2(H_2,(b,\infty))}}{\|f^*(\lambda;\cdot)\|_{L^2(H_1,(a,\infty))\oplus L^2(H_2,(b,\infty))}}, \quad f \neq 0.$$

Consequently,

$$\|R_\lambda(L_W)\| \geq \frac{1}{2\lambda_i} for \lambda \in \mathbb{C}, \quad \lambda_i = Im\lambda > 0.$$

From last relation it is implies the validity of assertion. □

Example 3.1. *Consider the following boundary value problem in* $L^2((0,+\infty)\times(0,1))\oplus L^2((0,+\infty)\times(0,1))$

$$i\frac{\partial u(t,x)}{\partial t}-\frac{\partial^2 u(t,x)}{\partial x^2}=f(t,x),\quad t>0, x\in[0,1],$$
$$i\frac{\partial v(t,x)}{\partial t}-\frac{\partial^2 v(t,x)}{\partial x^2}=g(t,x),\quad t>0, x\in[0,1],$$
$$u_x^{'}(t,0)=u_x^{'}(t,1)=0,\quad v_x^{'}(t,0)=v_x^{'}(t,1)=0,\quad t>0,$$
$$u(0,x)=e^{i\varphi}v(0,x),\quad \varphi\in[0,2\pi).$$

By using the last theorem, we get that this boundary value problem is continuous and coincides with $\mathbb{R}$*.*

Remark 3.1. *If we take* $a=b$*, then a differential-operator expression generated by* $l(\cdot)$ *can be written in form*

$$l(u)=iJu^{'}(t)+Au(t),$$

where $J:=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, $A:=\begin{pmatrix}A_1 & 0\\ 0 & A_2\end{pmatrix}$ *in* $L^2(H_1\oplus H_2,(a,+\infty))$*. Particularly, the obtained results in this work generalizes some results which have been established in [5].*

Remark 3.2. *The similar problems was considered and analogous results has been obtained in works [9-12].*

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