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APPROXIMATION BY CHLODOWSKY TYPE q-JAKIMOVSKI-LEVIATAN OPERATORS

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Abstract. This paper deals with the Chlodowsky type q-Jakimovski-Leviatan operators. We first establish approximation properties and rate of convergence results for these operators. Our main purpose is to give a theorem on the rate of convergence of the r^{th} $q-$derivative of the operators.

1. Introduction

In 1969, Jakimovski and Leviatan [8] introduced a new Favard-Szasz type operators by means of Appell polynomials $p_k(x) = \sum_{i=0}^{k} a_i \frac{x^{k-i}}{(k-i)!}$ $(k \in \mathbb{N})$ which satisfy the identity

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1}$$

Here $g(u) = \sum_{n=0}^{\infty} a_n u^n$ is an analytic function in the disc $|u| < r$, $(r > 1)$ and $g(1) \neq 0$. In [8], the authors considered the operator

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{2}$$

for $f \in E[0,\infty)$ where $E[0,\infty)$ denotes the set of functions that satisfy the property $|f(x)| \leq \beta e^{\alpha x}$ for some finite constants $\alpha, \beta \geq 0$. They studied approximation properties of these operators as well as some results due to Szasz. Later in [6], Ciupa defined a sequence of linear operators as

$$P_{n,t}(f;x) = \frac{e^{-nt}}{g(1)} \sum_{k=0}^{\infty} p_k(nt) f\left(x + \frac{k}{n}\right) \tag{3}$$

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and established approximation properties and rate of convergence for these operators by using modulus of continuity. In 2010, Atakut and Büyükyazıcı [2] studied some approximation properties of Stancu type generalization of the Favard-Szàsz operators which is given by

$$P_{n,t}^{\alpha,\beta}(f;x)=\frac{e^{-nt}}{g(1)}\sum_{k=0}^{\infty}p_k(nt)f\left(x+\frac{k+\alpha}{n+\beta}\right).$$

Another Stancu type generalization is given by Sucu and Varma [15] by means of the Sheffer polynomials. They obtained convergence properties of the operators and estimated the rate of convergence by using classical and second modulus of continuity. In [16], Sucu et. al. constructed a new sequence of linear positive operators that generalize Szasz operators including Boas-Buck-type polynomials. They establish a convergence theorem for these operators.

Chlodowsky type generalization of Jakimovski-Leviatan operators is investigated in [7]. These operators are defined as

$$P_n^*(f;x)=\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)f\left(\frac{k}{n}b_n\right) \tag{4}$$

with b_n a positive increasing sequence with the properties

$$\lim_{n\to\infty}b_n=\infty, \qquad \lim_{n\to\infty}\frac{b_n}{n}=0. \tag{5}$$

The authors obtained some local approximation results and studied some convergence properties in weighted spaces using weighted Korovkin-type theorems. Very recently Kantorovich type generalization of Jakimovski-Leviatan operators are constructed in [5]. Authors studied the convergence of the operators in a weighted space of function on positive semi axis.

In the last two decades quantum-calculus has attracted very much attention in the approximation theory. Beginning in 1997 with Philips [14], a great number of studies are performed related to this subject and still there are many authors working on this subject. Lupaş [11] was the first to define a $q-$generalization of Bernstein-operators, then Philips introduced another generalization of Bernstein operators based on $q-$integers and it is known as $q-$Bernstein operators in literature. These operators motivated many author to study further in this direction and a great number of studies have been done about the q-generalizations of other linear positive operators.

Here, related to our work, we shall mention a few studies on q-generalizations of some operators.

In 2008, Aral [3] defined a new operator called q-Szasz-Mirakyan operators, as

$$S_n^*(f;q,x)=E_q^{\left(-\frac{[n]}{b_n}x\right)}\sum_{k=0}^{\infty}f\left(\frac{[k]b_n}{[n]}\right)\frac{\left([n]\,x\right)^k}{[k]!\left(b_n\right)^k} \tag{6}$$

for $0 < q < 1$, where $0 \leq x \leq \alpha_q(n)$, $\alpha_q(n) = \dfrac{b_n}{(1-q)[n]}$, $f \in C(R_0)$ and (b_n) is a sequence of positive numbers such that $\lim\limits_{n\to\infty} b_n = \infty$. Approximation properties of these operators are obtained by means of the weighted Korovkin-type theorem and rate of convergence is computed. Also a representation for the r^{th} q-derivative of q-Szasz-Mirakyan operators is given in terms of q-differences and divided differences.

In [1], Atakut and Büyükyazıcı introduced a q-analogue of Favard-Szasz type operators related to the q-Appell polynomials as

$$L_n(f;q,x) = \frac{E_q^{(-[n]t)}}{A(1)} \sum_{k=0}^{\infty} \frac{P_k(q;[n]t)}{[k]!} f\left(x + \frac{[k]}{[n]}\right) \tag{7}$$

The authors proved approximation theorems and the rate of convergence theorems for these operators. Later in [9] a Stancu type generalization of the above q-Favard-Szasz operators are defined as

$$T_{n,t}^{\alpha,\beta}(f;q;x) = \frac{E_q^{(-[n]t)}}{g(1)} \sum_{k=0}^{\infty} \frac{p_k(q;[n]t)}{[k]!} f\left(x + \frac{[k]+\alpha}{[n]+\beta}\right).$$

The approximation properties and rates of convergence results for these operators are obtained in the statistical sense.

Very recently, A. Karaisa [10] defined Chlodowsky type generalization of the q-Favard-Szasz operators as follows:

$$P_n^*(f;q,x) = \frac{E_q^{(-\frac{[n]}{b_n}x)}}{A(1)} \sum_{k=0}^{\infty} \frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!} f\left(\frac{[k]}{[n]} b_n\right) \tag{8}$$

where $q \in (0,1)$, (b_n) is a positive increasing sequence with the properties

$$\lim_{n\to\infty} b_n = \infty, \qquad \lim_{n\to\infty} \frac{b_n}{[n]} = 0. \tag{9}$$

Here $\{p_n(q;.)\}_{n\geq 0}$ is a q-Appell polynomial set which is generated by

$$A(t)e_q^{(xt)} = \sum_{n\geq 0} p_n(q;x) \frac{t^n}{[n]!} \tag{10}$$

and $A(t)$ is defined by

$$A(t) = \sum_{n\geq 0} a_k t^k, \qquad a_0 = 1.$$

The author studied the weighted statistical approximation properties of the operators via Korovkin type approximation theorem and computed the rate of statistical convergence by using modulus of continuity. In [12], authors also studied weighted approximation and error estimation of these operators.

Before giving our main results let us recall some basic notations from q-calculus.

For any real number $q > 0$, the q-integer and the q-factorial of a nonnegative integer k are defined as

$$[k]_q := [k] = \begin{cases} \dfrac{1-q^k}{1-q} \ , & q \neq 1 \\ k \ , & q = 1. \end{cases}$$

$$[k]_q! := [k]! = \begin{cases} [k]_q \, [k-1]_q \ldots [1]_q \ , & k = 1, 2, .. \\ 1 \ , & k = 0 \end{cases}$$

respectively. For the integers n and k, the q-binomial coefficients are also defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]_q!}{[k]_q! \, [n-k]_q!} \qquad (n \geq k \geq 0).$$

The q-derivative of a function $f(x)$ with respect to x is

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

and higher q-derivatives are defined as

$$D_q^0(f(x)) = f(x), \quad D_q^n(f(x)) = D_q(D_q^{n-1}(f(x)), \quad n = 1, 2, 3, \ldots$$

The q-derivative of the product of the functions $f(x)$ and $g(x)$ are defined as

$$D_q(f(x)g(x)) = f(qx) D_q g(x) + g(x) D_q f(x). \tag{11}$$

By symmetry we can interchange $f(x)$ and $g(x)$ and write the equivalent form of the above equality as

$$D_q(f(x)g(x)) = f(x) D_q g(x) + g(qx) D_q f(x). \tag{12}$$

The two q-analogues of the classical exponential function e^x are defined by

$$e_q^x = \sum_{j=0}^{\infty} \frac{x^j}{[j]!}$$

and

$$E_q^x = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{[j]!}.$$

It is clear that these two analogues satisfy the following properties:

$$D_q e_q^{ax} = a e_q^{ax} \text{ and } D_q E_q^{ax} = a E_q^{aqx}, \tag{13}$$

$$e_q^x E_q^{-x} = E_q^x e_q^{-x} = 1. \tag{14}$$

For any real function f, the q-difference operators are defined as

$$\begin{aligned} \Delta_q^0 f_j &= f_j \\ \Delta_q^{k+1} f_j &= \Delta_q^k f_{j+1} - q^k \Delta_q^k f_j \end{aligned} \tag{15}$$

where $f_j = f\left(\frac{[j]}{[n]}b_n\right)$ and $j, n, k \in \mathbb{N}$ (see [3]).

Now we recall some statements that give the relation between divided differences and the k^{th} $q-$ difference of a function and also the relation between the $q-$ difference of a function and its $q-$derivatives.

Lemma 1. *(See [4]) For all $j, k \geq 0$,*

$$f[x_j, ...x_{j+k}] = \frac{\Delta_q^k f(x_j)}{q^{k(2j+k-1)/2}[k]!h^k}$$

where $x_j = x_0 + [j]h$ and $h > 0$ is an arbitrary constant.

Corollary 2. *(See [4]) Let the function f and its first $(n-1)$ $q-$derivatives be continuous, and $D_q^n(f)$ exist in the open interval (a, b). Then there exists $\widehat{q} \in (0, 1)$ such that, for all $q \in (\widehat{q}, 1) \cup (1, \widehat{q}^{-1})$,*

$$\frac{\Delta_q^n f(x_0)}{q^{n(n-1)/2}h^n} = D_q^n(f)(\xi_x)$$

where ξ_x is in the interval containing $x_{0,}...x_n$ and $x_j = x_0 + [j]h$.

In this study our main aim is to examine the r^{th} $q-$derivative of the operator $P_n^*(f; q, x)$ defined in (8). We first investigate approximation properties of these new operators with the help of Korovkin's Theorem and obtain rate of convergence results by means of modulus of continuity. Finally we give a statement about the rate of convergence of the r^{th} $q-$derivative of the operator.

2. Main Results

In order to give the approximation theorem for the sequence $\{P_n^*(f; q, x)\}$, we shall need the following Lemma.

Lemma 3. *For any $n \in \mathbb{N}$,*

$$P_n^*(e_0; q, x) = 1 \tag{16}$$

$$P_n^*(e_1; q, x) = x + E_q^{\left(-\frac{[n]}{b_n}x\right)} e_q^{\left(q\frac{[n]}{b_n}x\right)} \frac{b_n}{[n]} \frac{D_q(A(1))}{A(1)} \tag{17}$$

$$\begin{aligned} P_n^*(e_2; q, x) &= qx^2 + \frac{b_n}{[n]}x + E_q^{\left(-\frac{[n]}{b_n}x\right)} e_q^{\left(q\frac{[n]}{b_n}x\right)} \left\{ \left(\frac{b_n}{[n]}\right)^2 \frac{D_q(A(1))}{A(1)} \right. \\ &\quad \left. + \frac{b_n}{[n]}qx\frac{D_q(A(1))}{A(1)} + \left(\frac{b_n}{[n]}\right)^2 q\frac{D_q^2(A(1))}{A(1)} + \frac{b_n}{[n]}q^2x\frac{D_q(A(q))}{A(1)} \right\} \end{aligned} \tag{18}$$

for all $x \in [0, \infty)$.

Proof. We have,

$$A(1)e_q^{\left(\frac{[n]}{b_n}x\right)} = \sum_{k=0}^{\infty} \frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!} \tag{19}$$

$$A(1)\frac{[n]}{b_n}xe_q^{\left(\frac{[n]}{b_n}x\right)} + D_q(A(1))e_q^{\left(q\frac{[n]}{b_n}x\right)} = \sum_{k=0}^{\infty} \frac{p_{k+1}(q;\frac{[n]}{b_n}x)}{[k]!} \tag{20}$$

and

$$A(1)\left(\frac{[n]}{b_n}\right)^2 x^2 e_q^{\left(\frac{[n]}{b_n}x\right)} + \frac{[n]}{b_n}xe_q^{\left(q\frac{[n]}{b_n}x\right)}D_q(A(1)) + D_q^2(A(1))e_q^{\left(q\frac{[n]}{b_n}x\right)}$$

$$+ D_q(A(q))\frac{q[n]}{b_n}xe_q^{\left(q\frac{[n]}{b_n}x\right)} = \sum_{k=0}^{\infty} \frac{p_{k+2}(q;\frac{[n]}{b_n}x)}{[k]!}. \tag{21}$$

Identities (16) and (17) are obvious from (19) and (20), respectively. (See [10]) One gets the equality (18) from the identity $[k] = 1 + q[k-1]$ and from (21). □

Remark 4. *For the special case $q = 1$, we have*

$$\begin{aligned} P_n^*(e_0;1,x) &= P_n^*(e_0;x) \\ P_n^*(e_1;1,x) &= P_n^*(e_1;x) \\ P_n^*(e_2;1,x) &= P_n^*(e_2;x). \end{aligned}$$

where $P_n^*(e_0;x), P_n^*(e_1;x)$ and $P_n^*(e_2;x)$ are given explicitly in [7].

Theorem 5. *Let*

$$C^*[0,\infty) = \{f \in C[0,\infty) : |f(x)| \le e^{\gamma x} \text{ for any } x \ge 0 \text{ and certain } \gamma \text{ finite}\}.$$

If $f \in C^*[0,\infty)$, then

$$\lim_{n\to\infty} P_n^*(f;q,x) = f(x)$$

uniformly on each compact $[0,a] \subset R$.

Proof. The proof is obvious from the Korovkin's Theorem. □

Now we compute the rate of convergence of $P_n^*(f;q,x)$ by means of modulus of continuity $w(f:\delta)$ which is defined as

$$w(f;\delta) = \sup_{\substack{t,x\in[0,\infty) \\ |t-x|\le\delta}} |f(t) - f(x)|.$$

A necessary and sufficient condition for a function $f \in C[0,a]$ is

$$\lim_{\delta\to 0} w(f;\delta) = 0, \tag{22}$$

and it is well known that for any $\delta > 0$ and each $t \in [0, a]$

$$|f(t) - f(x)| \leq w(f; \delta)\left(1 + \frac{|t - x|}{\delta}\right). \tag{23}$$

Before giving the theorem on the rate of convergence of the operator $L_n^*(f; q, x)$, let us first investigate its second central moment:

$$\begin{aligned} P_n^*((e_1 - x)^2; q, x) &= P_n^*(e_2; q, x) - 2xP_n^*(e_1; q, x) + x^2 P_n^*(e_0; q, x) \\ &= (q-1)x^2 + \frac{b_n}{[n]}x + E_q^{\left(-\frac{[n]}{b_n}x\right)} e_q^{\left(q\frac{[n]}{b_n}x\right)} \left\{ \left(\frac{b_n}{[n]}\right)^2 \left(\frac{D_q(A(1))}{A(1)}\right.\right. \\ &+ \left. q\frac{D_q^2(A(1))}{A(1)}\right) + \frac{b_n}{[n]}x\left((q-2)\frac{D_q(A(1))}{A(1)} + q^2\frac{D_q(A(q))}{A(1)}\right)\Bigg\} \end{aligned} \tag{24}$$

Theorem 6. *Let (q_n) denote a sequence satisfying $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. For any function $f \in C^*[0, \infty)$, if $\lim_{n\to\infty} \frac{b_n}{[n]} = 0$, then*

$$|P_n^*(f; q, x) - f(x)| \leq 2w(f; \delta_n(x))$$

where

$$\begin{aligned} \delta_n(x) &= \Bigg\{(q-1)x^2 + \frac{b_n}{[n]}x + E_q^{\left(-\frac{[n]}{b_n}x\right)} e_q^{\left(q\frac{[n]}{b_n}x\right)} \left\{ \left(\frac{b_n}{[n]}\right)^2 \left(\frac{D_q(A(1))}{A(1)}\right.\right. \\ &+ \left. q\frac{D_q^2(A(1))}{A(1)}\right) + \frac{b_n}{[n]}x\left((q-2)\frac{D_q(A(1))}{A(1)} + q^2\frac{D_q(A(q))}{A(1)}\right)\Bigg\}\Bigg\}^{1/2}. \end{aligned}$$

Proof. For the proof see [10] (Theorem 4.1) with $P_n^*((e_1-x)^2; q, x)$ given in (24). □

Lastly we give our main theorem on the rate of convergence of the r^{th} $q-$derivative of the operator $P_n^*(f; q, x)$ $(D_q^r P_n^*(f; q, x))$ to the r^{th} $q-$derivative of the function f $(D_q^r f)$.

Corollary 7. *For each integer $r > 0$*

$$D_q^r P_n^*(f; q, x) = \frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{A(1)} \sum_{k=0}^{\infty} \frac{p_k(q; \frac{[n]}{b_n}x)}{[k]!} \left(\frac{[n]}{b_n}\right)^r \Delta_q^r f\left(\frac{[k]}{[n]}b_n\right) \tag{25}$$

Proof. Applying the q-differential operator to (8) and using (11) and (12) we have

$$\begin{aligned} D_q\left(P_n^*(f; q, x)\right) &= -\frac{[n]}{b_n}\frac{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}{A(1)} \sum_{k=0}^{\infty} \frac{p_k(q; \frac{[n]}{b_n}x)}{[k]!} f\left(\frac{[k]}{[n]}b_n\right) \\ &+ \frac{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}{A(1)} \sum_{k=0}^{\infty} \frac{D_q\left(p_k(q; \frac{[n]}{b_n}x)\right)}{[k]!} f\left(\frac{[k]}{[n]}b_n\right) \end{aligned}$$

Computing the second sum in the right hand side of the above inequality, we find

$$\sum_{k=0}^{\infty}\frac{D_q\left(p_k(q;\frac{[n]}{b_n}x)\right)}{[k]!}f\left(\frac{[k]}{[n]}b_n\right)=\frac{[n]}{b_n}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}f\left(\frac{[k+1]}{[n]}b_n\right).$$

Hence we get

$$\begin{aligned}D_q\left(P_n^*(f;q,x)\right) &= \frac{[n]}{b_n}\frac{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}{A(1)}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}\left(f\left(\frac{[k+1]}{[n]}b_n\right)-f\left(\frac{[k]}{[n]}b_n\right)\right)\\ &= \frac{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}{A(1)}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}\frac{[n]}{b_n}\Delta_q^1 f\left(\frac{[k]}{[n]}b_n\right).\end{aligned}$$

Similarly

$$\begin{aligned}&D_q^2\left(P_n^*(f;q,x)\right)=D_q\left(D_q\left(P_n^*(f;q,x)\right)\right)\\ &=-q\left(\frac{[n]}{b_n}\right)^2\frac{E_q^{\left(-\frac{[n]}{b_n}q^2x\right)}}{A(1)}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}\left(f\left(\frac{[k+1]}{[n]}b_n\right)-f\left(\frac{[k]}{[n]}b_n\right)\right)\\ &+\left(\frac{[n]}{b_n}\right)^2\frac{E_q^{\left(-\frac{[n]}{b_n}q^2x\right)}}{A(1)}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}\left(f\left(\frac{[k+2]}{[n]}b_n\right)-f\left(\frac{[k+1]}{[n]}b_n\right)\right)\\ &=\left(\frac{[n]}{b_n}\right)^2\frac{E_q^{\left(-\frac{[n]}{b_n}q^2x\right)}}{A(1)}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}\Delta_q^2 f\left(\frac{[k]}{[n]}b_n\right).\end{aligned}$$

Applying the $q-$differential operator to (25), we find

$$\begin{aligned}D_q\left(D_q^r\left(P_n^*(f;q,x)\right)\right) &= D_q^{r+1}\left(P_n^*(f;q,x)\right)\\ &= -q^r\left(\frac{[n]}{b_n}\right)^{r+1}\frac{E_q^{\left(-\frac{[n]}{b_n}q^{r+1}x\right)}}{A(1)}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}\Delta_q^r f\left(\frac{[k]}{[n]}b_n\right)\\ &\quad+\left(\frac{[n]}{b_n}\right)^{r+1}\frac{E_q^{\left(-\frac{[n]}{b_n}q^{r+1}x\right)}}{A(1)}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}\Delta_q^r f\left(\frac{[k+1]}{[n]}b_n\right)\\ &= \frac{E_q^{\left(-\frac{[n]}{b_n}q^{r+1}x\right)}}{A(1)}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}\left(\frac{[n]}{b_n}\right)^{r+1}\Delta_q^{r+1} f\left(\frac{[k]}{[n]}b_n\right).\end{aligned}$$

When k is replaced by $k+1$ (25) holds and the proof is completed. □

Using the relation between divided differences and q-difference given in Lemma 1, we can write

$$\Delta_q^r f\left(\frac{[k]}{[n]}b_n\right) = \left(\frac{b_n}{[n]}\right)^r [r]! q^{rk} q^{r(r-1)/2}\left[\frac{[k]}{[n]}b_n, \frac{[k+1]}{[n]}b_n, ... \frac{[k+r]}{[n]}b_n; f\right].$$

Then for each integer $r > 0$, we can rewrite (25) as

$$\begin{aligned} D_q^r P_n^*(f;q,x) &= q^{r(r-1)/2}[r]!\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{A(1)} \\ &\quad \times \sum_{k=0}^{\infty}\frac{p_k(q;\dfrac{[n]}{b_n}x)}{[k]!}q^{rk}\left[\frac{[k]}{[n]}b_n, \frac{[k+1]}{[n]}b_n, ..., \frac{[k+r]}{[n]}b_n; f\right]. \end{aligned}$$

In order to prove our last theorem we need the following theorem.

Theorem 8. *(See* [13]*) Let* $C^{r+1}[a,b]$ *be the space of* $(r+1)$*-times continuously differentiable functions and* $f \in C^{r+1}[a,b]$*. If* $x_i \geq y_i$ *for all* $i = 0, 1, ...r$ *and* $\sum_{i=0}^{r}(x_i - y_i) \neq 0$*, then there exists* $\hat{q} \in (0,1)$ *and* $\xi \in (a,b)$ *so that for all* $q \in (\hat{q}, 1) \cup (1, \hat{q}^{-1})$

$$f[x_0, ...x_r] - f[y_0, ...y_r] = \frac{D_q^{(r+1)}(\xi)}{(r+1)!}\sum_{i=0}^{r}(x_i - y_i). \tag{26}$$

Proof. The proof is the $q-$analogue of the the proof of Theorem 2.1 in [13] and can be done similarly. $\square$

Theorem 9. *Let* $f \in C^{r+1}[0, b_n)$ *with* $\lim_{n\to\infty} b_n = \infty$*. If* $D_q^{(n+1)}(\xi) \geq 0$ $(n = 0, ...r)$ *then we have,*

$$\left| D_q^r\left(P_n^*(f;q,x)\right) - q^{r(r-1)/2}\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}x\right)}}D_q^r f(x)\right|$$

$$\begin{aligned} &\leq 2q^{r(r-1)/2}\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}w\left(D_q^r f, \delta_n + \frac{[r]}{[n]}b_n\right) \\ &\quad + q^{r(r-1)/2}\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}x\right)}}\left(w\left(D_q^r f, \frac{[r]}{[n]}b_n\right)\right). \end{aligned}$$

Proof. From Theorem 8, by considering $D_q^{(n+1)}(\xi) \geq 0$, we can write

$$\begin{aligned}
& D_q^r\left(P_n^*(f;q,x)\right) \\
= \; & q^{r(r-1)/2}[r]!\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{A(1)}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}q^{rk}\left[\frac{[k]}{[n]}b_n, \frac{[k+1]}{[n]}b_n, \ldots \frac{[k+r]}{[n]}b_n; f\right] \\
\leq \; & q^{r(r-1)/2}[r]!\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{A(1)}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}\left[\frac{[k]}{[n]}b_n, \frac{([k]+[1])}{[n]}b_n, \ldots \frac{([k]+[r])}{[n]}b_n; f\right] \\
= \; & q^{r(r-1)/2}[r]!\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{A(1)}\sum_{k=0}^{\infty}\frac{p_k(q;\frac{[n]}{b_n}x)}{[k]!}\phi\left(\frac{[k]}{[n]}b_n\right) \\
= \; & q^{r(r-1)/2}[r]!\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}x\right)}}P_n^*(\phi;q,x)
\end{aligned}$$

where $\phi(x) = \left[x, x+\frac{[1]}{[n]}b_n, x+\frac{[2]}{[n]}b_n, ..., x+\frac{[r]}{[n]}b_n; f\right]$. Hence, we have

$$\begin{aligned}
& \left|D_q^r\left(P_n^*(f;q,x)\right) - q^{r(r-1)/2}\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}D_q^r f(x)\right| \\
\leq \; & \left|q^{r(r-1)/2}[r]!\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}P_n^*(\phi;q,x) - q^{r(r-1)/2}\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}D_q^r f(x)\right| \\
\leq \; & q^{r(r-1)/2}[r]!\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}\left|P_n^*(\phi;q,x) - \phi(x)\right| \\
& + \left|q^{r(r-1)/2}[r]!\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}\phi(x) - q^{r(r-1)/2}\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}D_q^r f(x)\right| \\
= \; & I_1 + I_2. \qquad (27)
\end{aligned}$$

From Theorem 6, we can write

$$I_1 \leq 2q^{r(r-1)/2}[r]!\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}w(\phi;\delta_n(x)).$$

We also have

$$\begin{aligned}|\phi(x+h)-\phi(x)| &= \Bigg|\left[x+h, x+h+\frac{[1]}{[n]}b_n, x+h+\frac{[2]}{[n]}b_n, ..., x+h+\frac{[r]}{[n]}b_n; f\right] \\ &\quad -\left[x, x+\frac{[1]}{[n]}b_n, x+\frac{[2]}{[n]}b_n, ..., x+\frac{[r]}{[n]}b_n; f\right]\Bigg| \qquad (28)\end{aligned}$$

The connection between $q-$differences $\Delta_q^k f(x_0)$ and the k^{th} $q-$derivative of the function f , $D_q^k(f)$, was given in Corollary 2. If we take h as $\dfrac{b_n}{[n]}$, we get

$$\frac{\Delta_q^k f(x_0)}{q^{k(k-1)/2}[k]!\left(\dfrac{b_n}{[n]}\right)^k} = f[x_0, ...x_k] = \frac{D_q^k f(\xi)}{[k]!}$$

where $\xi \in (x_0, x_k)$ and $x_j = x + [j]\frac{b_n}{[n]}$. Using this equality in (28) we can write, for $\theta_1, \theta_2 \in (0,1)$,

$$\begin{aligned}|\phi(x+h)-\phi(x)| &= \frac{1}{[r]!}\left|D_q^r f(x+h+\frac{[r]}{[n]}b_n\theta_1) - D_q^r f(x+\frac{[r]}{[n]}b_n\theta_2)\right| \\ &\leq \frac{1}{[r]!}w\left(D_q^r f, h+\frac{[r]}{[n]}b_n\left|\theta_1-\theta_2\right|\right) \\ &\leq \frac{1}{[r]!}w\left(D_q^r f, h+\frac{[r]}{[n]}b_n\right).\end{aligned}$$

If we take $h = \delta_n$, we get

$$|\phi(x+\delta_n)-\phi(x)| \leq \frac{1}{[r]!}w\left(D_q^r f, \delta_n+\frac{[r]}{[n]}b_n\right)$$

from which we can write

$$w(\phi, \delta_n) \leq \frac{1}{[r]!}w\left(D_q^r f, \delta_n+\frac{[r]}{[n]}b_n\right).$$

Hence we have

$$\begin{aligned}I_1 &\leq 2q^{r(r-1)/2}[r]!\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}w(\phi;\delta_n(x)) \\ &\leq 2q^{r(r-1)/2}\frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}}w\left(D_q^r f, \delta_n+\frac{[r]}{[n]}b_n\right). \qquad (29)\end{aligned}$$

Now let us consider I_2. We have

$$\begin{aligned}
I_2 &= \left| q^{r(r-1)/2}[r]! \frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}} \phi(x) - q^{r(r-1)/2} \frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}} D_q^r f(x) \right| \\
&= \left| q^{r(r-1)/2}[r]! \frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}} \left[x, x+\frac{[1]}{[n]}b_n, x+\frac{[2]}{[n]}b_n, ..., x+\frac{[r]}{[n]}b_n; f \right] \right. \\
&\quad \left. - q^{r(r-1)/2} \frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}} D_q^r f(x) \right| \\
&= \left| q^{r(r-1)/2} \frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}} \left(D_q^r f(x+\frac{[r]}{[n]}b_n\theta_3) - D_q^r f(x) \right) \right| \\
&\leq q^{r(r-1)/2} \frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}} w\left(D_q^r f, \frac{[r]}{[n]}b_n\theta_3 \right), \qquad \theta_3 \in (0,1) \qquad (30) \\
&\leq q^{r(r-1)/2} \frac{E_q^{\left(-\frac{[n]}{b_n}q^r x\right)}}{E_q^{\left(-\frac{[n]}{b_n}qx\right)}} w\left(D_q^r f, \frac{[r]}{[n]}b_n \right).
\end{aligned}$$

Lastly substituting (29) and (30) into (27) we get the desired result and the proof of the theorem is completed. □

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