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$\lambda-$ALMOST DIFFERENCE SEQUENCES OF FUZZY NUMBERS

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Abstract. In this study, we introduce several sets of sequences of fuzzy numbers using various sequences λ and μ in the class Λ and examine some inclusion relations among these sets.

1. Introduction

Zadeh [41] introduced the notion of fuzzy sets. The theory of fuzzy sets is used almost all sciences such as mathematical and physical sciences, social and management sciences, information sciences including computer sciences, biological sciences, medicine and engineering.

The concept of sequences of fuzzy numbers was introduced by Matloka [26]. Matloka [26] introduced bounded and convergent sequences of fuzzy numbers and showed that every convergent sequence of fuzzy numbers is bounded. Later on, Nanda [36] showed that the set of bounded and convergent sequences of fuzzy numbers forms complete metric space. Subrahmanyam [36] defined Cesàro summability for fuzzy numbers and he examined a few Tauberian theorems generalizing the classical results to fuzzy real numbers. For some further works in this direction we refer (Çanak ([11],[12] [13] [14]), Önder *et al.*[29], Sezer and Çanak [34]) . Recently, Nuray and Savaş [28] introduced statistical convergence of sequence of fuzzy numbers. Kwon [22] introduced strongly p -Cesàro summability of sequence of fuzzy numbers. He examined the relationship between strongly p -Cesàro summability and statistical convergence of sequence of fuzzy numbers. Subsequently, many authors enriched the sequences of fuzzy numbers (see[1],[4],[5],[7],[9],[15],[20],[37],[38],[39],[40]).

The concept of statistical convergence of number sequences was introduced by Steinhaus [35] and Fast [18] later reintroduced by Schoenberg [33] independently for real and complex sequences. Statistical convergence plays a central role in the theory of Fourier analysis, ergodic theory, approximation theory and number

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theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Connor [8], Fridy [19] and many others.

Denote $\hat{c}$ the set of all almost convergent sequences. Lorentz [24] proved that $x = (x_k) \in \hat{c}$ if and only if $\lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+m}$ exists, uniformly in m.

Later, Maddox [25] defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - L| = 0, \quad \text{uniformly in } m.$$

It can be shown that the sequence $x = (1, 0, 1, 0, 1, 0, ...)$ is strongly almost convergent to $\frac{1}{2}$. By $[\hat{c}]$ we denote the space of all strongly almost convergent sequences. It is easy to see that $c \subset [\hat{c}] \subset \hat{c} \subset \ell_\infty$ and the inclusions are strict, for example the sequence $x = (x_k) = \left((-1)^k\right)$ is almost convergent but not strongly almost convergent.

The generalized de la Vallée-Poussin mean is defined by Leindler [23] as

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be $(V, \lambda)-$summable to a number L if $t_n(x) \to L$ as $n \to \infty$. The set of all such sequences will be denoted by Λ.

The notion of λ-statistical convergence was introduced by Mursaleen [27] as follows:

Let $K \subset \mathbb{N}$ and define the $\lambda-$ density of K by

$$\delta_\lambda(K) = \lim_{n \to \infty} \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : k \in K\}|.$$

$\delta_\lambda(K)$ reduces to the asymptotic density $\delta(K)$ in case of $\lambda_n = n$ for all $n \in \mathbb{N}$ (see [27]).

A sequence $x = (x_k)$ is said to be $\lambda-$ statistically convergent to L if for every $\varepsilon > 0$ (see [27])

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0.$$

The concept of almost λ-statistical convergence was studied by Savaş [31]. Additionally, Çolak[10] introduced and studied on the sets of $\lambda-$statistical convergence and strongly Cesaro summability for sequences of complex numbers. Subsequently, Savaş [32] , Altinok *et al* [2] extended the idea of $\lambda-$statistical convergence and applied in generalized difference sequences of fuzzy numbers.

In this paper, we give some results on generalized fuzzy sequences and some inclusion theorems.

2. Definitions and Preliminaries

In this section we give the basic notions and some know definitions related to fuzzy numbers.

A fuzzy set u on $\mathbb{R}$ is called a fuzzy number if it has the following properties:

i) u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

ii) u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1, u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$;

iii) u is upper semicontinuous;

iv) $\operatorname{supp} u = cl\{x \in \mathbb{R} : u(x) > 0\}$, or denoted by $[u]^0$, is compact.

$\alpha-$level set $[u]^\alpha$ of a fuzzy number u is defined by

$$[u]^\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ \operatorname{supp} u, & \text{if } \alpha = 0. \end{cases}$$

It is clear that u is a fuzzy number if and only if $[u]^\alpha$ is a closed interval for each $\alpha \in [0,1]$ and $[u]^1 \neq \emptyset$.

A real number r can be regarded as a fuzzy number $\bar{r}$ defined by

$$\bar{r}(x) = \begin{cases} 1, & x = r \\ 0, & x \neq r \end{cases}.$$

If $u \in L(\mathbb{R})$, then u is called a fuzzy number, and $L(\mathbb{R})$ is said to be a fuzzy number space.

Let $u, v \in L(\mathbb{R})$ and the $\alpha-$level sets of fuzzy numbers u and v be $[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ and $[v]^\alpha = [\underline{v}^\alpha, \overline{v}^\alpha]$, $\alpha \in (0,1]$. Then, a partial ordering " $\leq$ " in $L(\mathbb{R})$ is defined by $u \leq v \Leftrightarrow \underline{u}^\alpha \leq \underline{v}^\alpha$ and $\overline{u}^\alpha \leq \overline{v}^\alpha$ for all $\alpha \in (0,1]$.

In order to calculate the distance between two fuzzy numbers u and v, we use the metric

$$d(u,v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha)$$

where d_H is the Hausdorff metric defined by

$$d_H([u]^\alpha, [v]^\alpha) = \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}.$$

It is known that d is a metric on $L(\mathbb{R})$, and $(L(\mathbb{R}), d)$ is a complete metric space [16].

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of all positive integers into $L(\mathbb{R})$. Thus, a sequence of fuzzy numbers (X_k) is a correspondence from the set of positive integers to a set of fuzzy numbers, i.e., to each positive integer k there corresponds a fuzzy number $X(k)$. It is more common to write X_k rather than $X(k)$ and to denote the sequence by (X_k) rather than X. The fuzzy number X_k is called the k-th term of the sequence.

The sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if there exist fuzzy numbers u and v such that $u \leq X_k \leq v$ for each $k \in \mathbb{N}$ and convergent to the fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $d(X_k, X_0) < \varepsilon$ for $k > k_0$. Let $\ell_\infty(\mathcal{F})$ and $c(\mathcal{F})$ denote the set of all bounded sequences and all convergent sequences of fuzzy numbers, respectively [26].

Nuray and Savaş [28] defined the notion of statistical convergence for sequences of fuzzy numbers:

Let $X = (X_k)$ be a sequence of fuzzy numbers. Then (X_k) is said to be statistically convergent to the fuzzy number X_0, if

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : d(X_k, X_0) \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$, where the vertical bars indicate the number of elements in the enclosed set. In this case, we write $S - \lim X_k = X_0$ or $X_k \xrightarrow{s} X_0$.

Kwon [22] defined the concept of strong $p-$Cesàro summability for sequences of fuzzy numbers as follows:

Let p be a positive real number. A sequence $X = (X_k)$ of fuzzy numbers is said to be strongly $p-$Cesàro summable to the fuzzy number X_0, if there is a fuzzy number X_0 such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} [d(X_k, X_0)]^p = 0.$$

The difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, consisting of all real valued sequences $x = (x_k)$ such that $\Delta^1 x = (x_k - x_{k+1})$ in the sequence spaces ℓ_∞, c and c_0, were defined by Kızmaz [21]. Başar and Altay [6] have recently introduced the difference sequence space bv_p of real sequences whose $\Delta-$transforms are in the space ℓ_p, where $\Delta x = (x_k - x_{k-1})$ and $1 \leq p \leq \infty$. The idea of difference sequences was generalized by Et and Çolak [17].

Let $w^{\mathcal{F}}$ be the set of all sequences of fuzzy numbers. The operator $\Delta^m : w^{\mathcal{F}} \to w^{\mathcal{F}}$ is defined by

$$\left(\Delta^0 X\right)_k = X_k, (\Delta X)_k = \Delta X_k = X_k - X_{k+1}, (\Delta^m X)_k = \Delta\left(\Delta^{m-1} X\right)_k, (m \geq 2), \text{ for all } k \in \mathbb{N}.$$

The $\lambda-$statistically convergence for difference sequences of fuzzy numbers was introduced by Altinok *et al.* [2]

Et *et al.* [5] defined the notion of Δ_λ^m- almost statistical convergence for sequences of fuzzy numbers:

Let $\lambda = (\lambda_n) \in \Lambda$. A sequence $X = (X_k)$ of fuzzy numbers is said to be Δ_λ^m-*almost statistically convergent* or $\hat{S}_\lambda(\Delta^m, \mathcal{F})-$convergent to X_0 if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : d(\Delta^m X_{k+i}, X_0) \geq \varepsilon\}| = 0, \text{ uniformly in } i,$$

where $I_n = [n - \lambda_n + 1, n]$. In this case we write $S_\lambda(\Delta^m, \mathcal{F})-\lim X_k = X_0$ or $X_k \to X_0(\hat{S}_\lambda(\Delta^m, \mathcal{F}))$, and $S_\lambda(\Delta^m, \mathcal{F}) = \left\{X = (X_k) : \hat{S}_\lambda(\Delta^m, \mathcal{F}) - \lim X_k = X_0 \text{ for some } X_0\right\}$.

We can give following example choosing $m = 0$.

Example 2.1. Consider the sequence $X = (X_{k+i})$ of fuzzy numbers defined by

$$X_{k+i}(x) = \begin{cases} \left.\begin{array}{cc} x-k-i, & \text{for } k+i+1 \le x \le k+i+2 \\ x-k+i+3, & \text{for } k+i+2 \le x \le k+i+3 \\ 0, & \text{otherwise} \end{array}\right\} & \text{, if } n-\left[\sqrt{\lambda_n}\right]+1 \le k+i \le n \\ \left.\begin{array}{cc} x-4, & \text{for } 4 \le x \le 5 \\ -x+6, & \text{for } 5 \le x \le 6 \\ 0, & \text{otherwise} \end{array}\right\} := X_0 & \text{, otherwise} \end{cases}$$

Then, we calculate $\alpha-$level set of this sequence as follows:

$$[X_{k+i}]^{\alpha} = \begin{cases} [k+i+\alpha+1, k+i+3-\alpha], & \text{if } n-\left[\sqrt{\lambda_n}\right]+1 \le k+i \le n \\ [\alpha-4, 6-\alpha], & \text{otherwise} \end{cases}.$$

For instance, if we take $\sqrt{\lambda_n} = \sqrt{n}$, we can write

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : d\left([X_k]^{\alpha}, [X_0]^{\alpha}\right) \ge \varepsilon\right\}\right| \le \lim_{n\to\infty} \frac{\sqrt[4]{n}}{\sqrt{n}} = 0$$

Hence, we conclude that (X_{k+i}) is $\lambda-$ statistically convergent to fuzzy number X_0, where $[X_0]^{\alpha} = [\alpha-4, 6-\alpha]$.

3. Main Results

In this section, we give some inclusion relations between the sets $\hat{S}_{\lambda}(\Delta^m, \mathcal{F})$ and $\hat{S}_{\mu}(\Delta^m, \mathcal{F})$, $\left[\hat{V}, \lambda, \mathcal{F}\right]^p_{\Delta^m}$ and $\left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m}$, $\hat{S}_{\lambda}(\Delta^m, \mathcal{F})$ and $[V, \mu, \mathcal{F}]^p_{\Delta^m}$ for various sequences λ, μ in the class Λ.

In the following by the statement " for all $n \in \mathbb{N}_{n_0}$" we mean "for all $n \in \mathbb{N}$ except finite numbers of positive integers" where $\mathbb{N}_{n_0} = \{n_0, n_0+1, n_0+2, ...\}$ for some $n_0 \in \mathbb{N} = \{1, 2, 3, ...\}$.

Definition 3.1. Let $X = (X_k)$ be a sequence of fuzzy numbers and p be a positive real number. Then the set $\left[\hat{V}, \lambda, p\right]^{\mathcal{F}}_{\Delta^m}$ is defined by

$$\left[\hat{V}, \lambda, \mathcal{F}\right]^p_{\Delta^m} = \left\{X = (X_k) \in w^{\mathcal{F}} : \frac{1}{\lambda_n} \sum_{k \in I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \to 0, \right\}.$$

If $\lambda_n = n$ then we will write $\left[\hat{C}, 1, \mathcal{F}\right]^p_{\Delta^m}$ instead of $\left[\hat{V}, \lambda, \mathcal{F}\right]^p_{\Delta^m}$.

Theorem 3.2 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \le \mu_n$ for all $n \ge n_0$.

(i) If

$$\liminf_{n\to\infty} \frac{\lambda_n}{\mu_n} > 0 \tag{1}$$

then $\hat{S}_{\mu}(\Delta^m, \mathcal{F}) \subseteq \hat{S}_{\lambda}(\Delta^m, \mathcal{F})$.

(ii) If

$$\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = 1 \tag{2}$$

then $\hat{S}_{\lambda}(\Delta^m, \mathcal{F}) = \hat{S}_{\mu}(\Delta^m, \mathcal{F})$.

Proof *(i)* Let (1) hold such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$ and $X = (X_k)$ be a sequence of fuzzy numbers. Then we get $I_n \subset J_n$ and

$$|\{k \in J_n : d(\Delta^m X_{k+i}, X_0) \geq \varepsilon\}| \geq |\{k \in I_n : d(\Delta^m X_{k+i}, X_0) \geq \varepsilon\}|$$

for $\varepsilon > 0$ and so we obtain

$$\frac{1}{\mu_n} |\{k \in J_n : d(\Delta^m X_{k+i}, X_0) \geq \varepsilon\}| \geq \frac{\lambda_n}{\mu_n} \frac{1}{\lambda_n} |\{k \in I_n : d(\Delta^m X_{k+i}, X_0) \geq \varepsilon\}|$$

for all $n \geq n_0$, where $J_n = [n - \mu_n + 1, n]$. Now using (1) and taking the limit as $n \to \infty$ in the last inequality we have $\hat{S}_{\mu}(\Delta^m, \mathcal{F}) \subseteq \hat{S}_{\lambda}(\Delta^m, \mathcal{F})$.

(ii) Consider the sequence $X = (X_k)$ of fuzzy numbers. Let $(X_k) \in S_{\lambda}(\Delta^m, \mathcal{F})$ and (2) hold. Since $I_n \subset J_n$, we write

$$\begin{aligned}
\frac{1}{\mu_n} |\{k \in J_n : d(\Delta^m X_{k+i}, X_0) \geq \varepsilon\}| &= \frac{1}{\mu_n} |\{n - \mu_n + 1 \leq k \leq n - \lambda_n : d(\Delta^m X_{k+i}, X_0) \geq \varepsilon\}| \\
&+ \frac{1}{\mu_n} |\{k \in I_n : d(\Delta^m X_{k+i}, X_0) \geq \varepsilon\}| \\
&\leq \frac{\mu_n - \lambda_n}{\mu_n} + \frac{1}{\lambda_n} |\{k \in I_n : d(\Delta^m X_{k+i}, X_0) \geq \varepsilon\}| \\
&\leq \left(1 - \frac{\lambda_n}{\mu_n}\right) + \frac{1}{\lambda_n} |\{k \in I_n : d(\Delta^m X_{k+i}, X_0) \geq \varepsilon\}|
\end{aligned}$$

for $\varepsilon > 0$ and for all $n \geq n_0$. Since $(X_k) \in \hat{S}_{\lambda}(\Delta^m, \mathcal{F})$ and $\lim_n \frac{\lambda_n}{\mu_n} = 1$, the right hand side of the last inequality tends to 0 as $n \to \infty$. It means that $(X_k) \in \hat{S}_{\mu}(\Delta^m, \mathcal{F})$ and therefore we have $\hat{S}_{\lambda}(\Delta^m, \mathcal{F}) \subseteq \hat{S}_{\mu}(\Delta^m, \mathcal{F})$. Since (2) holds, then the we have equation (1) and hence $\hat{S}_{\lambda}(\Delta^m, \mathcal{F}) = \hat{S}_{\mu}(\Delta^m, \mathcal{F})$.

If we take $\mu = (\mu_n) = (n)$ in the statement (ii) of Theorem 3.2, then· we have the following result.

Corollary 3.3 Let $\lambda = (\lambda_n)$ be a sequence in Λ. If $\lim_n \frac{\lambda_n}{n} = 1$ then we have $\hat{S}_{\lambda}(\Delta^m, \mathcal{F}) = \hat{S}(\Delta^m, \mathcal{F})$.

Theorem 3.4 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$.

(i) If $\liminf_{n\to\infty} \frac{\lambda_n}{\mu_n} > 0$ holds then $\left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m} \subseteq \left[\hat{V}, \lambda, \mathcal{F}\right]^p_{\Delta^m}$.

(ii) Let $X = (X_k)$ be a Δ^m- bounded sequence of fuzzy number. If $\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = 1$, then $\left[\hat{V}, \lambda, \mathcal{F}\right]^p_{\Delta^m} \subseteq \left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m}$.

Proof (i) Suppose that $\liminf_{n\to\infty} \frac{\lambda_n}{\mu_n} > 0$ fulfills such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$. Then we may write $I_n \subseteq J_n$ and obtain

$$\frac{1}{\mu_n} \sum_{k\in J_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \geq \frac{1}{\mu_n} \sum_{k\in I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p$$

for all $n \geq n_0$. So we get

$$\frac{1}{\mu_n} \sum_{k\in J_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \geq \frac{\lambda_n}{\mu_n} \frac{1}{\lambda_n} \sum_{k\in I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p.$$

Now using (1) and taking limit as $n \to \infty$ in the last inequality, we derive $\left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m} \subseteq \left[\hat{V}, \lambda, \mathcal{F}\right]^p_{\Delta^m}$.

(ii) Now suppose that $X = (X_k)$ is Δ^m- bounded sequence of fuzzy number. Let $(X_k) \in \left[\hat{V}, \lambda, \mathcal{F}\right]^p_{\Delta}$ and (2) hold. Then there exists some $M > 0$ such that $d\left(\Delta^m X_{k+i}, X_0\right) \leq M$ for all k. Now, since $\lambda_n \leq \mu_n$ and so that $\frac{1}{\mu_n} \leq \frac{1}{\lambda_n}$, and $I_n \subset J_n$ for all $n \geq n_0$, we may write

$$\begin{aligned}
\frac{1}{\mu_n} \sum_{k\in J_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p &= \frac{1}{\mu_n} \sum_{k=J_n-I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p + \frac{1}{\mu_n} \sum_{k\in I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \\
&\leq \frac{\mu_n - \lambda_n}{\mu_n} M^p + \frac{1}{\mu_n} \sum_{k\in I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \\
&\leq \left(1 - \frac{\lambda_n}{\mu_n}\right) M^p + \frac{1}{\lambda_n} \sum_{k\in I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p
\end{aligned}$$

for all $n \geq n_0$. Since $(X_k) \in \left[\hat{V}, \lambda, \mathcal{F}\right]^p_{\Delta^m}$ and $\lim_{n} \frac{\lambda_n}{\mu_n} = 1$, the right hand side of last inequality tends to 0 as $n \to \infty$, that is $(X_k) \in \left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m}$. This completes proof.

From Theorem 3.4 we have the following result.

Corollary 3.5 Let $X = (X_k)$ be a Δ^m- bounded sequence of fuzzy number, $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$. If $\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = 1$ holds then we have $\left[\hat{V}, \lambda, \mathcal{F}\right]^p_{\Delta^m} = \left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m}$.

Theorem 3.6 Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in Λ such that $\lambda_n \leq \mu_n$ for all $n \geq n_0$.

(i) If $\liminf\limits_{n\to\infty} \frac{\lambda_n}{\mu_n} > 0$ holds then

$$X_k \to X_0 \left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m} \Longrightarrow X_k \to X_0 \left(\hat{S}_\lambda \left(\Delta^m, \mathcal{F}\right)\right).$$

(ii) Let $X = (X_k)$ be a Δ^m- bounded sequence of fuzzy number, and $X_k \to X_0 \left(\hat{S}_\lambda \left(\Delta^m, \mathcal{F}\right)\right)$ then $X_k \to X_0 \left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m}$, whenever $\lim\limits_{n\to\infty} \frac{\lambda_n}{\mu_n} = 1$.

(iii) Let $X = (X_k)$ be a Δ^m- bounded sequence of fuzzy number, $\lim\limits_{n\to\infty} \frac{\lambda_n}{\mu_n} = 1$ holds then $\hat{S}_\lambda \left(\Delta^m, \mathcal{F}\right) = \left[\hat{V}, \mu, \mathcal{F}\right]_{\Delta^m}$.

Proof (i) Let us assume that $\varepsilon > 0$ and $X_k \to X_0 \left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m}$. Then, we get

$$\sum_{k\in J_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \geq \sum_{k\in I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \geq \sum_{\substack{k\in I_n \\ [d(\Delta X_k, X_0)]\geq\varepsilon}} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p$$
$$\geq \varepsilon^p \left|\left\{k \in I_n : \left[d\left(\Delta^m X_{k+i}, X_0\right)\right] \geq \varepsilon\right\}\right|$$

for every $\varepsilon > 0$ and so that

$$\frac{1}{\mu_n} \sum_{k\in J_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \geq \frac{\lambda_n}{\mu_n} \frac{1}{\lambda_n} \left|\left\{k \in I_n : \left[d\left(\Delta^m X_{k+i}, X_0\right)\right] \geq \varepsilon\right\}\right| \varepsilon^p$$

for all $n \geq n_0$. Then using (1) and taking limit as $n \to \infty$ in the last inequality, we deduce $X_k \to X_0 \left(\hat{S}_\lambda \left(\Delta^m\right)\right)$ whenever $X_k \to X_0 \left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m}$.

(ii) Let $X_k \to X_0 \left(\hat{S}_\lambda \left(\Delta^m\right)\right)$ and $X = (X_k)$ be a Δ^m- bounded sequence of fuzzy number. Then there exists some $M > 0$ such that $d\left(\Delta^m X_{k+i}, X_0\right) \leq M$ for all k. Since $\frac{1}{\mu_n} \leq \frac{1}{\lambda_n}$, then we obtain

$$\begin{aligned}
\frac{1}{\mu_n} \sum_{k\in J_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p &= \frac{1}{\mu_n} \sum_{k\in J_n - I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p + \frac{1}{\mu_n} \sum_{k\in I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \\
&\leq \frac{\mu_n - \lambda_n}{\mu_n} M^p + \frac{1}{\mu_n} \sum_{k\in I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \\
&\leq \left(1 - \frac{\lambda_n}{\mu_n}\right) M^p + \frac{1}{\lambda_n} \sum_{k\in I_n} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \\
&\leq \left(1 - \frac{\lambda_n}{\mu_n}\right) M^p + \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ d(\Delta^m X_{k+i}, X_0)\geq\varepsilon}} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \\
&+ \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ d(\Delta^m X_{k+i}, X_0)<\varepsilon}} \left[d\left(\Delta^m X_{k+i}, X_0\right)\right]^p \\
&\leq \left(1 - \frac{\lambda_n}{\mu_n}\right) M^p + \frac{M^p}{\lambda_n} \left|\left\{s \in I_n : d\left(\Delta^m X_{k+i}, X_0\right)\right\}\right| + \varepsilon^p
\end{aligned}$$

for every $\varepsilon > 0$ and for all $n \geq n_0$. Using (2) we get $X_k \to X_0 \left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m}$ whenever $X_k \to X_0 \left(\hat{S}_\lambda \left(\Delta^m, \mathcal{F}\right)\right)$. Therefore, we have $\hat{S}_\lambda \left(\Delta^m, \mathcal{F}\right) \subseteq \left[\hat{V}, \mu, \mathcal{F}\right]^p_{\Delta^m}$.

(iii) The proof follows from (i) and (ii).

From the first statements of both Theorem 3.2 and Theorem 3.6 we obtain the following result.

Corollary 3.7 If $\liminf_{n\to\infty} \frac{\lambda_n}{\mu_n} > 0$ then $\hat{S}_\mu \left(\Delta^m, \mathcal{F}\right) \cap [V, \mu, \mathcal{F}]^p_{\Delta^m} \subset \hat{S}_\lambda \left(\Delta^m, \mathcal{F}\right)$.

If we take $\mu_n = n$ for all $n \in \mathbb{N}$ in Theorem 3.6 then, since $\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = 1$ implies that $\liminf_{n\to\infty} \frac{\lambda_n}{\mu_n} = 1 > 0$, we have the following result.

Corollary 3.8 Let $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1$. Then

(i) Let $X = (X_k)$ be a Δ^m- bounded sequence of fuzzy number and $X_k \to X_0 \left(\hat{S}_\lambda \left(\Delta^m, \mathcal{F}\right)\right)$, then $X_k \to X_0 \left[\hat{C}, 1, \mathcal{F}\right]^p_{\Delta^m}$.

(ii) If $X_k \to X_0 \left[\hat{C}, 1, \mathcal{F}\right]^p_{\Delta^m}$, then $X_k \to X_0 \left(\hat{S}_\lambda \left(\Delta^m, \mathcal{F}\right)\right)$.

References

[1] Altin, Y. , Et, M. Basarir, M. On some generalized difference sequences of fuzzy numbers, Kuwait J. Sci. Engrg. 34 1A, 1–14 (2007).

[2] Altinok, H., Çolak, R. and Et, M. $\lambda-$Difference sequence spaces of fuzzy numbers, Fuzzy Sets and Systems 160(21), 3128-3139, (2009).

[3] Altinok, H., Çolak, R. Almost lacunary statistical and strongly almost lacunary convergence of generalized difference sequences of fuzzy numbers, J. Fuzzy Math.17(4) 951–967, (2009), .

[4] Altinok, H., Çolak, R. and Altin,Y. On the class of λ-statistically convergent difference sequences of fuzzy numbers, Soft Computing, 16 (6) 1029-1034, (2012).

[5] Et, M.; Altin, Y., Altinok, H. On almost statistical convergence of generalized difference sequences of fuzzy numbers, Math. Model. Anal. 10(4) 345–352, (2005).

[6] Başar, F. and Altay, B. On the space of sequences of p-bounded variation and related matrix mappings, Ukrainian Math. J. 55(1), 136–147, (2003).

[7] Aytar, S. and Pehlivan, S. Statistically convergence of sequences of fuzzy numbers and sequences of $\alpha-$cuts, International J. General Systems 1-7, (2007).

[8] Connor J.S. The statistical and strong p-Cesàro convergence of sequences, Analysis, 8, 47-63, (1988).

[9] Colak, R., Altin, Y., Mursaleen M. On some sets of difference sequences of fuzzy numbers, Soft Computing, 15(4), 787-793, (2011).

[10] Çolak, R. On $\lambda-$Statistical Convergence, Conference on Summability and Applications, Commerce Univ., May 12-13, 2011, Istanbul, TURKEY.

[11] Çanak, İ. Tauberian theorems for Cesàro summability of sequences of fuzzy numbers, J. Intell. Fuzzy Syst., 27 (2), 937-942, (2014).

[12] Çanak, İ. On the Riesz mean of sequences of fuzzy real numbers, J. Intell.Fuzzy Syst., 26 (6), 2685-2688, (2014).

[13] Çanak, İ. Some conditions under which slow oscillation of a sequence of fuzzy numbers follows from Cesàro summability of its generator sequence, Iran.J. Fuzzy Syst., 11 (4) 15-22, (2014).
[14] Çanak, İ. Hölder summability method of fuzzy numbers and a Tauberian theorem, Iran. J. Fuzzy Syst., 11 (4) 87-93, (2014).
[15] Das, N. R.; Choudhury, Ajanta. Boundedness of fuzzy real-valued sequences, Bull. Calcutta Math. Soc. 90 (1) 35–44, (1998).
[16] Diamond, P. and Kloeden, P. Metric spaces of fuzzy sets, Fuzzy Sets and Systems 35 (2) 241-249, (1990).
[17] Et, M. and Çolak, R. On some generalized difference sequence spaces, Soochow J. Math. 21(4), 377-386, (1995).
[18] Fast H. Sur la convergence statistique, Colloq. Math., 2, 241-244, (1951).
[19] Fridy J. On statistical convergence, Analysis, 5, 301-313, (1985).
[20] Gökhan, A. Et, M. Mursaleen, M. Almost lacunary statistical and strongly almost lacunary convergence of sequences of fuzzy numbers, Math. Comput. Modelling, 49 (3-4), 548–555, (2009).
[21] Kızmaz, H. On certain sequence spaces, Canad. Math. Bull. 24(2), 169-176, (1981).
[22] Kwon, J.S. On statistical and $p-$Cesàro convergence of fuzzy numbers, Korean J. Comput.&Appl. Math. 7 (1), 195-203, (2000).
[23] Leindler L. Über die de la Vallée-Pousinsche Summierbarkeit allgemeiner Orthogonalreihen, Acta Math. Acad. Sci. Hungar., 16, 375-387, (1965).
[24] Lorentz, G. G. A contribution to the theory of divergent sequences, Acta Math. 80 167-190,(1948).
[25] Maddox, I. J. Spaces of Strongly Summable Sequences, Quart. J. Math. Oxford , 18(2) 345-355, (1967).
[26] Matloka, M. Sequences of fuzzy numbers, BUSEFAL 28, 28-37, (1986).
[27] Mursaleen, M. $\lambda-$statistical convergence, Math. Slovaca, 50(1), 111-115, (2000).
[28] Nuray, F. and Savaş, E. Statistical convergence of sequences of fuzzy real numbers, Math. Slovaca 45 (3) 269-273, (1995).
[29] Önder, Z. Sezer, S. A. Çanak, İ. A Tauberian theorem for the weighted mean method of summability of sequences of fuzzy numbers, J. Intell. Fuzzy Syst., 28, 1403-1409, (2015).
[30] Šalát T. On statistically convergent sequences of real numbers, Math. Slovaca, 30, 139-150, (1980).
[31] Savaş E. Strong almost convergence and almost λ-statistical convergence, Hokkaido Math. Jour., 29 (3), 531-536, (2000).
[32] Savaş, E. On strongly λ-summable sequences of fuzzy numbers, Inform. Sci. 125 no. 1-4, 181–186 (2000).
[33] Schoenberg I. J., The integrability of certain functions and related summability methods, Amer. Math. Monthly, 66, 361-375, (1959).
[34] Sezer, S.A. and Çanak, İ. Power series methods of summability for series of fuzzy numbers and related Tauberian theorems, Soft Comput., DOI 10.1007/s00500- 015-1840-0
[35] Steinhaus H. Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2, 73-74, (1951).
[36] Subrahmanyam, P. V. Cesàro summability for fuzzy real numbers. p-adic analysis, summability theory, fuzzy analysis and applications (INCOPASFA) (Chennai, 1998). J. Anal. 7, 159–168, (1999).
[37] Tripathy, B.C. and A.J. Dutta. On fuzzy real-valued double sequence spaces ${}_2l^p_{\mathcal{F}}$, Math. Comput. Modelling 46 (9-10), 1294-1299, (2007).
[38] Tripathy, B.C. and Sarma, B. Sequences spaces of fuzzy real numbers defined by Orlicz functions, Math. Slovaca, 58 (5), 621-628, (2008).
[39] Tripathy, B.C. and Baruah, A. Nörlund and Riesz mean of sequences of fuzzy real numbers, Appl. Math. Lett. 23 (5), 651-655, (2010).

[40] Tuncer, A.N.and Babaarslan, F. $\lambda-$Statistical limit points of order $\beta-$of sequences of fuzzy numbers, Positivity, 19, 385-394, (2015).

[41] Zadeh, L. A. Fuzzy sets, Inform and Control, 8, 338-353, (1965).

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