INVERSE SINGULAR SPECTRAL PROBLEM VIA HOCSHTADT-LIEBERMAN METHOD

ERDAL BAS

ABSTRACT. In this paper, it is proved that if $q(x)$ is defined on $\left(\frac{\pi}{2}, \pi\right]$, then just one spectrum is enough to determine $q(x)$ on the interval $(0, \frac{\pi}{2}]$ for the Sturm-Lioville equation having singularity type $\frac{\delta}{x^p}$, on $(0, \pi]$, i.e. if the potential on the half-interval is known, then, it is necessary to reconstruct the potential on the whole interval.

1. INTRODUCTION

Inverse problems of spectral analysis consist in recovering operators from their spectral data. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other sciences. Inverse problems also play an important role in mathematical physics. An inverse eigenvalue problem (IEP) concerns the reconstruction of a matrix from prescribed spectral data. We must point out immediately that there is a well developed and critically important counterpart of eigenvalue problem associated with differential systems. The inverse problem is just as important as the direct problem in applications [1].

Inverse spectral problem theory has a long history. In 1929, Ambarzumjan firstly showed the following results [2]:

Theorem 1. *If $n^2\pi^2$ is the spectra set of the boundary value problem*

$$-y'' + q(x)y = \lambda y, \quad x \in [0,1]$$

with boundary conditions

$$y'(0) = y'(1) = 0,$$

then $q(x) \equiv 0$ in $[0,1]$.

Furthermore, Borg [3] proved that two spectra uniquely determine the potential $q(x)$. Tikhonov [4] proved that there is unique of the solution of the problem of electromagnetic sounding . Marchenko [5] showed that spectra of the one singular

Sturm-Liouville equation are determined by two various uniquely. Later, Krein [6] gave solution of the inverse Sturm-Liouville problem. Gelfand -Levitan [7] showed an algorithm for construction of $q(x); h$ and H.

The first result on the half-inverse problem is due to Hochstadt and Lieberman [8], who proved the following remarkable theorem:

Theorem 2. *Let $h_0, h_1 \in \mathbb{R}$, and assume $q_1, q_2 \in L_1(0, 1)$ to be real-valued. Consider the one-dimensional Schrödinger operators H_1, H_2 in $L_2(0, 1)$ given by*

$$H_j = -\frac{d^2}{dx^2} + q_j, \quad j = 1, 2,$$

with the boundary conditions

$$y'(0) + h_0 y(0) = 0,$$
$$y'(1) + h_1 y(1) = 0.$$

Let $\sigma(H_j) = \{\lambda_{j,n}\}$ be the (necessarily simple) spectra of H_j where $j = 1, 2$. Suppose that $q_1 = q_2$ (a.e.) on $\left[0, \frac{1}{2}\right]$ and that $\lambda_{1,n} = \lambda_{2,n}$ for all n. Then $q_1 = q_2$ (a.e.) on $[0, 1]$.

Later in 1999 F.Gesztesy and B.Simon [9] gave the generalization of Hochstadt-Lieberman Theorem and in particular they proved the next theorem:

Theorem 3. *Let $H = -\frac{d^2}{dx^2} + q$ in $L_2(0, 1)$ with above boundary conditions and $h_0, h_1 \in \mathbb{R}$. Suppose q is $C^{2k}\left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$ for some $k = 0, 1, \dots$ and for some $\varepsilon > 0$. Then q on $\left[0, \frac{1}{2}\right]$, h_0, and all the eigenvalues of H except for $(k + 1)$ uniquely determine h_1 and q on all of $[0, 1]$.*

Theorem 4. *[25] The equation (1) has fundamental $\varphi(x, \lambda)$ and $\Psi(x, \lambda)$ solutions that satisfies the following asymptotic formulas*

$$\varphi(x, \lambda) = x[1 + o(1)], \qquad \varphi'(x, \lambda) = 1 + o(1)$$

$$\psi(x, \lambda) = 1 + o(1) \qquad \psi'(x, \lambda) = o\left(\frac{1}{x}\right),$$

for each eigenvalue λ and $x \to 0$ then the entire function $\varphi(x, \lambda)$ with respect to λ and $x \geqslant 0$ provides the following inequalities

$$|\varphi(x, \lambda)| \leq x e^{|\operatorname{Im}\lambda| x} \exp\left\{\int_0^x s|q(s)|\, ds\right\}$$

$$\left|\varphi(x, \lambda) - \frac{\sin \lambda x}{\lambda}\right| \leq x \int_0^x s|q(s)|\, ds \exp\left\{|\operatorname{Im}\lambda| x + \int_0^x s|q(s)|\, ds\right\}$$

$$|\lambda\varphi(x, \lambda) - \sin \lambda x| \leq \left[\sigma_1(0) - \sigma_1\left(\frac{1}{\lambda}\right)\right] \exp\left\{|\operatorname{Im}\lambda| x + \int_0^x s|q(s)|\, ds\right\}$$

where

$$\sigma_1(x) = \int_x^\pi \sigma(s)\,ds, \qquad \sigma(x) = \int_x^\pi |q(s)|\,ds.$$

Since 1929, various kinds of spectral problems were considered by numerous authors (see [10]-[26]). In this section we collect some important facts which are needed in this research.

2. PRELIMINARIES

Lemma 5. (***Riemann-Lebesgue's Lemma***) *[27] If f is Lebesque integrable on* $[-\pi, \pi]$, *then*

$$\lim_{n\to\infty} \int_{-\pi}^\pi f(x)\cos nx\,dx = 0 = \lim_{n\to\infty} \int_{-\pi}^\pi f(x)\sin nx\,dx.$$

Theorem 6. (***Liouville's Theorem***) *[19] If* $f : C \to C$ *is entire and bounded, then f(z) is constant throughout the plane.*

Consider the singular Sturm-Liouville problem

$$Ly = -y'' + \left[\frac{\delta}{x^p} + q_0(x)\right] y = \mu y \qquad (\mu = \lambda^2,\, 0 < x \leqslant \pi) \tag{1}$$

$$y(0) = 0, \tag{2}$$

$$y(\pi, \lambda)\cos\alpha + y'(\pi, \lambda)\sin\alpha = 0, \tag{3}$$

where $\int_0^\pi x\,|q(x)|\,dx < \infty$, $\delta =$constant, $q_0(x) \in L_2(0,\pi]$, $1 < p < 2$, $q(x) = \frac{\delta}{x^p} + q_0(x)$. The spectrum of problem (1)-(3) consisting of the eigenvalues $\{\lambda_n\}$ are real and simple.

If condition (3) is replaced by

$$y(\pi, \lambda)\cos\gamma + y'(\pi, \lambda)\sin\gamma = 0, \tag{4}$$

and let $\left\{\tilde{\lambda}_n\right\}$ be a spectrum of problem (1), (2), (4). Also assume that $\sin(\alpha - \gamma) \neq 0$.

Before proving the main theorem, let us mention some necessary data which will be used later. We consider the following different from problem (1)-(3).

$$\tilde{L}y = -\tilde{y}'' + \left[\frac{\delta}{x^p} + \tilde{q}_0(x)\right] \tilde{y} = \mu\tilde{y} \qquad (\mu = \lambda^2,\, 0 < x \leqslant \pi) \tag{5}$$

$$\tilde{y}(0) = 0, \tag{6}$$

$$\tilde{y}(\pi, \lambda)\cos\beta + \tilde{y}'(\pi, \lambda)\sin\beta = 0, \tag{7}$$

and their asymptotic formulas [11-13],

$$\varphi(x,\lambda) = \frac{\sin \lambda x}{\lambda} + \frac{\sin \lambda x}{2\lambda^2} \int_0^x \sin 2\lambda t \left[\frac{\delta}{t^p} + q_0(t)\right] dt$$

$$- \frac{\cos \lambda x}{\lambda^2} \int_0^x \sin^2 \lambda t \left[\frac{\delta}{t^p} + q_0(t)\right] dt + O\left(\frac{e^{|\operatorname{Im} \lambda|x}}{|\lambda|^{5-2p}}\right) \tag{8}$$

$$\varphi'(x,\lambda) = \cos \lambda x + \frac{\cos \lambda x}{2\lambda} \int_0^x \sin 2\lambda t \left[\frac{\delta}{t^p} + q_0(t)\right] dt$$

$$- \frac{\sin \lambda x}{\lambda} \int_0^x \sin^2 \lambda t \left[\frac{\delta}{t^p} + q_0(t)\right] dt + O\left(\frac{e^{|\operatorname{Im} \lambda|x}}{|\lambda|^{4-2p}}\right). \tag{9}$$

We use short case of formulas (8), (9) for ease of operation. It can be shown that there exists kernels $K(x,s)$, and $\tilde{K}(x,s)$ which are continuous on $(0,\pi]$ such that by using the transformation operator, solutions of problems (1-3) and (5-7) are

$$y(x,\lambda) = \frac{\sin \lambda x}{\lambda} + \int_0^x K(x,s) \frac{\sin \lambda s}{\lambda} ds, \tag{10}$$

$$\tilde{y}(x,\lambda) = \frac{\sin \lambda x}{\lambda} + \int_0^x \tilde{K}(x,s) \frac{\sin \lambda s}{\lambda} ds, \tag{11}$$

respectively, where the kernel $K(x,s)$, $\tilde{K}(x,s)$ are solutions of the equations [11-13],

$$\frac{\partial^2 K}{\partial x^2} - \left[\frac{\delta}{x^p} + \tilde{q}_0(x)\right] K = \frac{\partial^2 K}{\partial s^2} - \left[\frac{\delta}{s^p} + q_0(s)\right] K, \tag{12}$$

with the boundary conditions,

$$K(x,x) = \frac{1}{2} \int_0^x (\tilde{q}_0(s) - q_0(s)) ds, \tag{13}$$

$$K(x,0) = 0. \tag{14}$$

Now, merely one of the kinds of the inverse problem is solved. We give the solution of the inverse problem for special singularity potential as follows. Hochstadt-Lieberman solved half inverse Sturm Liouville problem for regular type [8]. Similarly, numerous mathematicans study inverse problem by using Hochstadt-Lieberman methods [18,22]. In this study we have solved half inverse problem considering (1)-(3) problem. The problem which has this type singularity is really

interesting, hence our results are different according to the studies that indicated in literature.

3. MAIN RESULTS

We give the following theorem which is the main result of the paper.

Theorem 7. *Let $\{\lambda_n\}$ be a spectrum of both problems (1)-(3) and (5),(2),(3). If $q_0(x) = \tilde{q}_0(x)$ on the half interval $\left(\frac{\pi}{2}, \pi\right]$, then $q_0(x) = \tilde{q}_0(x)$ for all $x\epsilon(0, \pi]$.*

Proof. By using equations (10) and (11), we obtain that

$$y\tilde{y} = \frac{\sin^2 \lambda x}{\lambda^2} + \int\limits_0^x \left(\tilde{K}(x,s) + K(x,s)\right) \frac{\sin \lambda x}{\lambda} \frac{\sin \lambda s}{\lambda} ds$$

$$+ \int\limits_0^x K(x,s) \frac{\sin \lambda s}{\lambda} ds \times \int\limits_0^x \tilde{K}(x,t) \frac{\sin \lambda t}{\lambda} dt \qquad (15)$$

Let us extend with respect to the second argument that the range of $K(x,s)$ and $\tilde{K}(x,s)$, then, by using the trigonometric addition formulas and change of variables, we obtain that

$$y\tilde{y} = \frac{1}{2\lambda^2} \left[1 - \cos 2\lambda x + \int\limits_0^x \tilde{\tilde{K}}(x,r) \cos 2\lambda r dr \right] \qquad (16)$$

where

$$\tilde{\tilde{K}}(x,s) = 2 \left[K(x,x-2r) + \tilde{K}(x,x-2r) + \int\limits_{-x+2r}^x K(x,t)\tilde{K}(x,t-2r) dt \right.$$

$$\left. + \int\limits_{-x}^{x-2r} K(x,t)\tilde{K}(x,t+2r) dt \right]. \qquad (17)$$

Now, we define a function as follows:

$$w(\lambda) = y(\pi,\lambda)\cos\alpha + y'(\pi,\lambda)\sin\alpha. \qquad (18)$$

The eigenvalues of L or $\tilde{L}$ are zeros for $w(\lambda)$. If we consider the asymptotic formulas of y and $\tilde{y}$, $w(\lambda)$ is entire function of λ.

Also note that the equation

$$(\tilde{y}y' - \tilde{y}'y) \Big|_{0+\varepsilon}^\pi + \int\limits_{0+\varepsilon}^\pi (\tilde{q}_0(x) - q_0(x)) \tilde{y}y dx = 0, \qquad \varepsilon \to 0. \qquad (19)$$

can be easily checked. By virtue of equations (3) and (7), we have

$$\left[\tilde{y}\left(\pi,\lambda\right)y'\left(\pi,\lambda\right)-\tilde{y}'\left(\pi,\lambda\right)y\left(\pi,\lambda\right)\right]+\int_{0}^{\pi/2}\left(\tilde{q}_{0}\left(x\right)-q_{0}\left(x\right)\right)\tilde{y}ydx=0 \qquad (20)$$

Considering boundary conditions (2) and (6), the first term of the equation (20) is zero. Suppose that

$$Q=\tilde{q}_{0}-q_{0}$$

and

$$H\left(\lambda\right)=\int_{0}^{\pi/2}Q\tilde{y}\left(x,\lambda_{n}\right)y\left(x,\lambda_{n}\right)dx \qquad (21)$$

Taking into account properties of y and $\tilde{y}$, if $\lambda=\lambda_{n}$, we say that $H\left(\lambda\right)$ is an entire function for $\lambda=\lambda_{n.}$

We rewrite equation (20) as

$$\int_{0}^{\pi/2}\left(\tilde{q}_{0}-q_{0}\right)\tilde{y}ydx=0.$$

Then

$$H\left(\lambda_{n}\right)=0. \qquad (22)$$

Furthermore, using equations (10) and (21) for $0<x\leq\pi$, we have

$$\left|H\left(\lambda\right)\right|\leq Me^{2\pi\left|\operatorname{Im}\lambda\right|}, \qquad (23)$$

where M is constant. By virtue of equations (18) and (21), we write a rational function as

$$\Omega\left(\lambda\right)=\frac{H\left(\lambda\right)}{w\left(\lambda\right)}, \qquad (24)$$

where $\Omega\left(\lambda\right)$ is an entire function. Using asymptotic forms, we can rewrite last equation as follows:

$$\left|\Omega\left(\lambda\right)\right|=O\left(\frac{1}{\left|\lambda\right|^{4-2p}}\right). \qquad (25)$$

From the Liouville Theorem, we get

$$\Omega\left(\lambda\right)=0,\qquad H\left(\lambda\right)=0 \qquad (26)$$

for all λ. Let's continue to prove, now, substituting equation (16) into equation (21), we obtain

$$\frac{1}{2\lambda^{2}}\left\{\int_{0}^{\pi/2}Q\left[1-\cos 2\lambda x+\int_{0}^{x}\overset{\approx}{K}\left(x,r\right)\cos 2\lambda r dr\right]dx\right\}=0$$

$$= \int\limits_{0}^{\pi/2} Q \left[1 - \cos 2\lambda x + \int\limits_{0}^{x} \overset{\approx}{K}(x,r) \cos 2\lambda r dr \right] dx = 0$$

$$= \int\limits_{0}^{\pi/2} Q \left[1 - \cos 2\lambda x \right] dx + \int\limits_{0}^{\frac{\pi}{2}} Q(x) \int\limits_{0}^{x} \overset{\approx}{K}(x,r) \cos 2\lambda r dr dx = 0 \qquad (27)$$

Letting $\lambda \to \infty$ for all λ, and $(0 < x < \frac{\pi}{2})$, by means of Riemann-Lebesgue lemma and applying some straight forward computations, we can find that

$$\int\limits_{0}^{\pi/2} Q(x)\, dx = 0 \qquad (28)$$

and

$$-\int\limits_{0}^{\pi/2} \cos 2\lambda x \left[Q(r) - \int\limits_{r}^{\pi/2} Q(x) \overset{\approx}{K}(x,r)\, dx \right] dr = 0. \qquad (29)$$

Taking into account the completeness of the function $\cos 2\lambda x$, we can write that

$$Q(r) - \int\limits_{r}^{\pi/2} Q(x) \overset{\approx}{K}(x,r)\, dx = 0, \qquad 0 < x < \pi/2. \qquad (30)$$

Since equation (30) is Volterra integral equation, it has only trivial solution, $Q(x) = 0$. Therefore

$$Q(x) = \tilde{q}_0(x) - q_0(x) = 0.$$

almost everywhere on $(0, \pi]$. This completes the proof. $\qquad\qquad\square$

4. CONCLUSION

As a result, we prove that if the potentials of Sturm-Liouville problem having special singularity coincides on a half interval then the potentials are also coincides on the whole interval according to the only one spectrum.

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