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SENSITIVITY ANALYSIS FOR A PARAMETRIC MULTI-VALUED IMPLICIT QUASI VARIATIONAL-LIKE INCLUSION

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Abstract. In this paper, using proximal-point mapping of strongly maximal P-η-monotone mapping and the property of the fixed-point set of multi-valued contractive mapping, we study the behaviour and sensitivity analysis of the solution set of a parametric generalized implicit quasi-variational-like inclusion involving strongly maximal P-η-monotone mapping in real Hilbert space. Further, under suitable conditions, we discuss the Lipschitz continuity of the solution set with respect to the parameter. The technique and results presented in this paper can be viewed as extension of the techniques and corresponding results given in [2,7-10,20,21].

1. Introduction

Variational inequality theory has become very effective and powerful tool for studying a wide range of problems arising in mechanics, contact problems in elasticity, optimization and control problems, management science, operation research, general equilibrium problems in economics and transportation, unilateral, obstacle, moving boundary valued problems etc. Variational inequalities have been generalized and extended in different directions using novel and innovative techniques.

In recent years, much attention has been given to develop general techniques for the sensitivity analysis of solution set of various classes of variational inequalities (inclusions). From the mathematical and engineering point of view, sensitivity properties of various classes of variational inequalities can provide new insight concerning the problem being studied and stimulate ideas for solving problems. The

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sensitivity analysis of solution set for variational inequalities have been studied extensively by many authors using quite different techniques. By using the projection technique, Dafermos [5], Mukherjee and Verma [17], Noor [19] and Yen [23] studied the sensitivity analysis of solution of some classes of variational inequalities with single-valued mappings. By using the implicit function approach that makes use of so-called normal mappings, Robinson [22] studied the sensitivity analysis of solutions for variational inequalities in finite-dimensional spaces. By using resolvent operator technique, Adly [1], Noor [20], and Agarwal *et al.* [2] studied the sensitivity analysis of solution of some classes of quasi-variational inclusions involving single-valued mappings.

Recently, by using projection and resolvent techniques, Ding and Luo [8], Liu *et al.* [16], Park and Jeong [21], Ding [6,7], Khan [11] Kazmi and Khan [13,14] and Huang *et al.* [10] studied the behaviour and sensitivity analysis of solution set for some classes of generalized variational inequalities (inclusions) involving multi-valued mappings.

The method based on proximal-point mapping is a generalization of projection method and has been widely used to study the existence of solution and to develop iterative schemes of variational (-like) inclusions. Recently Chidume, Kazmi and Zegeye [4], Fang and Huang [9], and Kazmi and Khan [12] has introduced the notion of η-proximal point mapping, P-proximal point mapping and P-η-proximal point mapping and used these to study the various classes of variational (-like) inclusions.

Motivated by recent work going in this direction, we introduce the notion of strongly maximal P-η-monotone mapping and discuss some of its properties. Further, we define strongly P-η-proximal mapping for strongly maximal P-η-monotone, an extension of η-proximal mappings [4], P-proximal mappings [9], strongly P-proximal mappings [24], P-η-proximal mapping [12] and classical proximal mapping in Hilbert space, and prove that it is single-valued and Lipschitz continuous. Further, we consider a parametric multi-valued implicit quasi-variational-like inclusion problem (in short PMIQVLIP) in real Hilbert space. Further, using strongly P-η-proximal mapping technique and the property of the fixed point set of multi-valued mapping, we study the behaviour and sensitivity analysis of the solution set for PMIQVLIP. Further, the Lipschitz continuity of solution set of PMIQVLIP is proved under some suitable conditions. The results presented in this chapter generalize and improve the results given by many authors, see for example [2,7,10,20,21].

2. Strongly P-η-proximal mappings

We assume that H is a real Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$; $CB(H)$ is the family of all nonempty closed and bounded subsets of H; $C(H)$ is the family of all nonempty compact subsets of H; 2^H is the power set of

H; $\mathcal{H}(\cdot,\cdot)$ is the Hausdroff metric on $C(H)$, defined by

$$\mathcal{H}(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} d(x,y), \sup_{y\in B}\inf_{x\in A} d(x,y)\},\ A,B\in C(H).$$

First, we review and define the following concepts:

Definition 2.1[12]. Let $\eta : H\times H \to H$ be a mapping. Then a mapping $P : H \to H$ is said to be

(i) *η-monotone* if

$$\langle P(x) - P(y),\, \eta(x,y)\rangle \ \geq\ 0,\ \forall\, x,y\in H;$$

(ii) *strictly η-monotone* if

$$\langle P(x) - P(y),\, \eta(x,y)\rangle \ >\ 0,\ \forall\, x,y\in H$$

and equality holds if and only if $x = y$;

(iii) *δ-strongly η-monotone* if there exists a constant $\delta > 0$ such that

$$\langle P(x) - P(y),\, \eta(x,y)\rangle \ \geq\ \delta\|x-y\|^2,\ \forall\, x,y\in H.$$

Definition 2.2[4]. A mapping $\eta : H\times H \to H$ is said to be *τ-Lipschitz continuous*, if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \ \leq\ \tau\|x-y\|,\ \forall\, x,y\in H.$$

Definition 2.3[4]. Let $\eta : H\times H \to H$ be a single-valued mapping. Then a multi-valued mapping $M : H \to 2^H$ is said to be

(i) *η-monotone* if

$$\langle u - v,\, \eta(x,y)\rangle \ \geq\ 0,\ \forall\, x,y\in H,\ \forall\, u\in M(x),\ \forall\, v\in M(y);$$

(ii) *strictly η-monotone* if

$$\langle u - v,\, \eta(x,y)\rangle \ \geq\ 0,\ \forall\, x,y\in H,\ \forall\, u\in M(x),\ \forall\, v\in M(y),$$

and equality holds if and only if $x = y$.

(iii) *γ-strongly η-monotone* if there exists a constant $\gamma > 0$ such that

$$\langle u - v,\, \eta(x,y)\rangle \ \geq\ \gamma\|x-y\|^2,\ \forall\, x,y\in H,\ \forall\, u\in M(x),\ \forall\, v\in M(y);$$

(iv) *maximal η-monotone* if M is η-monotone and $(I+\rho M)(H) = H$ for any $\rho > 0$, where I stands for identity mapping.

Remark 2.1. If $\eta(x,y) \equiv x - y$, $\forall\, x,y\in H$, then from Definitions 2.1 and 2.3, we recover the usual definitions of monotonicity of mappings P and M.

Definition 2.4[12]. Let $\eta : H\times H \to H$ and $P : H \to H$ be mappings. Then a multi-valued mapping $M : H \to 2^H$ is said to be *maximal P-η-monotone* if M is η-monotone and $(P+\rho M)(H) = H$ for any $\rho > 0$.

Definition 2.5. Let $\eta : H \times H \to H$ and $P : H \to H$ be mappings. A multi-valued mapping $M : H \to 2^H$ is said to be *γ-strongly maximal P-η-monotone* if M is γ-strongly η-monotone and $(P + \rho M)H = H$ for any $\rho > 0$.

Remark 2.2.

(i) If M is η-monotone, Definition 2.5 reduces to Definition 2.4 given by Kazmi and Khan [12] in Banach space.

(ii) If M is η-monotone and $P \equiv I$, Definition 2.5 reduces to definition of maximal η-monotone mapping given by Chidume *et al.* [4] in Banach space.

(iii) If $\eta(x, y) \equiv x - y$ for all $x, y \in H$ then Definition 2.5 reduces to the definition of γ-strongly maximal monotone mapping given by Zeng *et al.* [24].

(iv) If $\eta(x, y) \equiv x - y$ for all $x, y \in H$ and M is monotone, Definition 2.5 reduces to the definition of maximal P-monotone mapping given by Fang and Huang [9].

The following theorem gives some properties of γ-strongly maximal P-η-monotone mappings.

Theorem 2.1. Let $\eta : H \times H \to H$ be a mapping; let $P : H \to H$ be a strictly η-monotone mapping and let $M : H \to 2^H$ be a γ-strongly maximal P-η-monotone multi-valued mapping. Then

(a) $\langle u - v, \eta(x, y)\rangle \geq 0, \ \forall (v, y) \in \text{Graph}(M)$ implies $(u, x) \in \text{Graph}(M)$, where $\text{Graph}(M) := \{(u, x) \in H \times H : u \in M(x)\}$;

(b) the mapping $(P + \rho M)^{-1}$ is single-valued for all $\rho > 0$.

Proof (a). Proof is on similar lines of proof of Theorem 2.1(a)[12].

Proof (b). For any given $z \in H$ and constant $\rho > 0$, let $x, y \in (P + \rho M)^{-1}(z)$. Then

$$\rho^{-1}(z - P(x)) \in M(x)$$

and

$$\rho^{-1}(z - P(y)) \in M(y).$$

Now

$$\begin{aligned} 0 &= \rho\langle \rho^{-1}(z-P(x)) - \rho^{-1}(z-P(y)), \eta(x,y)\rangle \\ &\quad + \langle P(x)-P(y), \eta(x,y)\rangle \\ &\geq \gamma\|x-y\|^2 + \langle P(x)-P(y), \eta(x,y)\rangle \\ &\geq \gamma\|x-y\|^2, \end{aligned}$$

using γ-strongly η-monotonicity of M and strictly η-monotonicity of P. Hence, preceeding inequality implies that $x = y$, which implies $(P+\rho M)^{-1}$ is single-valued. This completes the proof of (b).

By Theorem 2.1, we define strongly P-η-proximal mapping for a γ-strongly maximal η-monotone mapping M as follows:

$$R^M_{P,\eta}(z) \;=\; (P+\rho M)^{-1},\ \forall\, z \in H, \tag{2.3}$$

where $\rho > 0$ is a constant, $\eta : H \times H \to H$ is a mapping and $P : H \to H$ is a strictly η-monotone mapping.

Remark 2.3.

(i) If $\eta(x,y) \equiv x-y$ for all $x, y \in H$ and M is η-monotone, then strongly P-η-proximal mapping reduces to P-proximal mapping given by Fang and Huang [9].

(ii) If $P \equiv I$ and M is η-monotone, then strongly P-η-proximal mapping reduces to η-proximal mapping given by Chidume *et al.* [4] in Banach space.

Next, we prove that strongly P-η-proximal mapping is Lipschitz continuous.

Theorem 2.2. Let $P : H \to H$ be a δ-strongly η-accretive mapping; let $\eta : H \times H \to H$ be a τ-Lipschitz continuous mapping and let $M : H \to 2^H$ be a γ-strongly maximal η-monotone multi-valued mapping. Then strongly P-η-proximal mapping $R^M_{P,\eta}$ is $\dfrac{\tau}{\delta+\rho\gamma}$-Lipschitz continuous, that is,

$$\|R^M_{P,\eta}(x) - R^M_{P,\eta}(y)\| \;\leq\; \frac{\tau}{\delta+\rho\gamma}\,\|x-y\|,\ \forall\, x, y \in H.$$

Proof. Let $x, y \in H$. From definition of $R^M_{P,\eta}$, we have $R^M_{P,\eta}(x) = (P+\rho M)^{-1}(x)$. This implies that

$$\rho^{-1}(x - P(R^M_{P,\eta}(x))) \in M(R^M_{P,\eta}(x)).$$

Similarly, we have

$$\rho^{-1}(y - P(R^M_{P,\eta}(y))) \in M(R^M_{P,\eta}(y)).$$

Since M is γ-strongly η-monotone, we obtain

$$\gamma\|R^M_{P,\eta}(x) - R^M_{P,\eta}(y)\|$$
$$\leq\ \rho^{-1}\langle (x - P(R^M_{P,\eta}(x))) - (y - P(R^M_{P,\eta}(y))),\ \eta(R^M_{P,\eta}(x),\ R^M_{P,\eta}(y))\rangle$$

$= \rho^{-1}\langle x-y,\, \eta(R^M_{P,\eta}(x),\, R^M_{P,\eta}(y))\rangle - \rho^{-1}\langle P(R^M_{P,\eta}(x)) - P(R^M_{P,\eta}(y)),\, \eta(R^M_{P,\eta}(x),\, R^M_{P,\eta}(y))\rangle.$

Since P is δ-strongly η-monotone and η is τ-Lipschitz continuous, then the preceding inequality implies

$$\rho\gamma\|R^M_{P,\eta}(x) - R^M_{P,\eta}(y)\| \;\le\; \tau\|x-y\|\,\|R^M_{P,\eta}(x) - R^M_{P,\eta}(y)\| - \delta\|R^M_{P,\eta}(x) - R^M_{P,\eta}(y)\|^2,$$

which gives

$$\|R^M_{P,\eta}(x) - R^M_{P,\eta}(y)\| \;\le\; \frac{\tau}{\delta+\rho\gamma}\|x-y\| \;\forall\, x,y \in H.$$

This completes the proof.

Remark 2.4.

(i) Theorems 2.1-2.2 generalize Proposition 2.1 and Theorems 2.1-2.2 [9] and corresponding result in [24].

(ii) Lemmas 2.6 and 2.8 [4] and Theorems 2.1-2.2 [12] can be extended using the technique of Theorems 2.1-2.2.

3. Formulation of problem

Let Λ and Ω be open subsets of a real Hilbert space H such that (Λ, d_1) and (Λ, d_2) are metric spaces, in which the parameters λ and μ takes values, respectively.

Let $P : H \to H$; $\eta : H \times H \to H$; $N, M : H \times H \times \Omega \to H$; $g, m : H \times \Lambda \to H$ be single-valued mappings such that $g \not\equiv 0$ and let $A, B, C, D : H \times \Omega \to C(H)$ and $F : H \times \Lambda \to C(H)$ be multi-valued mappings. Suppose that $W : H \times H \times \Lambda \to 2^H$ is a multi-valued mapping such that for each $(t, \lambda) \in H \times \Lambda$, $W(\cdot, t, \lambda) : H \to 2^H$ is strongly maximal P-η-monotone and $\text{range}(g-m)(H \times \{\lambda\}) \cap \text{domain}\, W(\cdot, y, \lambda) \neq \emptyset$, where $(g-m)(x, \lambda) = g(x, \lambda) - m(x, \lambda)$ for any $(x, \lambda) \in H \times \lambda$. For each $(f, \lambda, \mu) \in H \times \Lambda \times \Omega$, we consider the parametric multi-valued implicit quasi-variational-like inclusion problem (PMIQVLIP):

Find $x = x(\lambda, \mu) \in H$, $u = u(x, \mu) \in A(x, \mu)$, $v = v(x, \mu) \in B(x, \mu)$, $w = w(x, \mu) = c(x, \mu)$, $y = y(x, \mu) \in D(x, \mu)$ and $z = z(x, \lambda) \in F(x, \lambda)$ such that $(g-m)(x, \lambda) \in \text{domain}\, W(\cdot, z, \lambda)$ and

$$f \in N(u, v, \mu) - M(w, y, \mu) + W((g-m)(x, \lambda), z, \lambda). \tag{3.1}$$

Some special cases:

(1) If $E \equiv H$; $(\Lambda, d_1) \equiv (\Lambda, d_2)$; $P \equiv I$, an identity mapping; $\eta(x, t) \equiv x - t$ for all $x, t \in H$, and $W(\cdot, z, \lambda)$ is maximal monotone for each fixed $(z, \lambda) \in H \times \Lambda$, then PMIQVLIP (3.1) reduces to the problem:

Find $x = x(\lambda) \in H$, $u = u(x, \lambda) \in A(x, \lambda)$, $v = v(x, \lambda) \in B(x, \lambda)$, $w = w(x, \lambda) \in C(x, \lambda)$, $y = y(x, \lambda) \in D(x, \lambda)$ and $z = z(x, \lambda) \in F(x, \lambda)$ such that $(g-m)(x, \lambda) \in \text{domain}\, W(\cdot, z, \lambda)$ and

$$f \in N(u, v, \lambda) - M(w, y, \lambda) + W((g-m)(x, \lambda), z, \lambda),$$

which has been studied by Liu *et al.* [16].

(2) If $E \equiv H$; $(\Lambda, d_1) \equiv (\Omega, d_2)$; $P \equiv I$; $\eta(x,t) \equiv x - t$ for all $x, t \in H$; $M \equiv 0$; $f \equiv 0$, and $W(\cdot, z, \lambda)$ is maximal monotone for each fixed $(z,\lambda) \in H \times \Lambda$, then PMIQVLIP (3.1) reduces to the problem:

Find $x = x(\lambda) \in H$, $u = u(x,\lambda) \in A(x,\lambda)$, $v = v(x,\lambda) \in B(x,\lambda)$ and $z = z(x,\lambda) \in F(x,\lambda)$ such that $(g-m)(x,\lambda) \in \text{domain}\, W(\cdot, z, \lambda)$ and

$$0 \in N(u, v, \lambda) + W((g-m)(x,\lambda), z, \lambda),$$

which has been studied by Ding [6].

(3) If $E \equiv H$; $(\Lambda, d_1) \equiv (\Omega, d_2)$; $P \equiv I$; $\eta(x,t) \equiv x - t$ for all $x, t \in H$; $f \equiv 0$; $M \equiv C \equiv D \equiv m \equiv 0$; $g \equiv I$ and $A(x,\lambda) \equiv B(x,\lambda) \equiv F(x,\lambda) \equiv x$ for all $(x,\lambda) \in H \times \Lambda$ and $W(\cdot, z, \lambda)$ is maximal monotone for each fixed $(z,\lambda) \in H \times \Lambda$. Then PMIQVLIP (3.1) reduces to the problem:

Find $x \in H$ such that

$$0 \in N(x, x, \lambda) + W(x, x, \lambda),$$

which has been studied by Agarwal *et al.* [2].

(4) If $E \equiv H$; $(\Lambda, d_1) \equiv (\Omega, d_2)$; $P \equiv I$; $\eta(x,t) \equiv x - t$ for all $x, t \in H$; $f \equiv 0$; $M \equiv B \equiv C \equiv D \equiv E \equiv m \equiv 0$; $A(x,\lambda) \equiv x$ for all $(x,\lambda) \in H \times \Lambda$; $N(x,t,\lambda) \equiv N_1(x,\lambda)$ and $W(x,y,\lambda) \equiv W_1(x,\lambda)$, for all $(x,y,\lambda) \in H \times H \times \Lambda$, where $N_1, W_1 : H \times \Lambda \to 2^H$, be such that $W_1(\cdot,\lambda)$ is maximal monotone for each fixed $\lambda \in \Lambda$, then PMIQVLIP (3.1) reduces to the problem:

$$0 \in N_1(x,\lambda) + W_1(g(x,\lambda),\lambda),$$

which has been studied by Adly [1].

For a suitable choices of the mappings $A, B, C, D, F, N, M, W, g, P, m, \eta$, it is easy to see that PMIQVLIP (3.1) includes a number of known classes of quasi-variational-like inequalities (inclusions) studied by many authors as special cases see for example [1-6,8,16,19-21] and the references therein.

Now, for each fixed $(\lambda,\mu) \in \Lambda \times \Omega$, the solution set $S(\lambda,\mu)$ of PMIQVLIP (3.1) is denoted as

$$\begin{aligned} S(\lambda,\mu) := \Big\{ & x = x(\lambda,\mu) \in H : \ u = u(x,\mu) \in A(x,\mu),\ v = v(x,\mu) \in B(x,\mu) \\ & w = w(x,\mu) \in C(x,\mu),\ y = y(x,\mu) \in D(x,\mu),\ z = z(x,\lambda) \in F(x,\lambda), \\ & \text{such that } f \in N(u,v,\mu) - M(w,y,\mu) + W((g-m)(x,\lambda), z, \lambda) \Big\}. \end{aligned} \tag{3.2}$$

The aim of the paper is to study the behaviour and sensitivity analysis of the solution set $S(\lambda,\mu)$, and the conditions on mappings $A, B, C, D, F, N, M, W, g, P, m, \eta$ under which the solution set $S(\lambda,\mu)$ of PMIQVLIP (3.1) is nonempty and Lipschitz continuous with respect to the parameters $\lambda \in \Lambda$, $\mu \in \Omega$.

We need the following results:

Lemma 3.1[18]. Let (X,d) be a complete metric space. Suppose that $Q : X \to C(X)$ satisfies

$$\mathcal{H}(Q(x),\, Q(t) \;\le\; \nu d(x,y),\; \forall\, x,t \in X,$$

where $\nu \in (0,1)$ is a constant. Then the mapping Q has fixed point in E.

Lemma 3.2[15]. Let (X,d) be a complete metric space and let $T_1, T_2 : X \to C(X)$ be θ-$\mathcal{H}$-contraction mappings, then

$$\mathcal{H}(F(T_1),\, F(T_2)) \;\le\; (1-\theta)^{-1} \sup_{x\in X} \mathcal{H}(T_1(x),\, T_2(x)),$$

where $F(T_1)$ and $F(T_2)$ are the sets of fixed points of T_1 and T_2, respectively.

4. Sensitivity analysis of the solution set $S(\lambda,\mu)$

First, we define the following concepts.

Definition 4.1[13]. A mapping $g : H \times \Lambda \to H$ is said to be

(i) (L_g, l_g)-*mixed Lipschitz continuous*, if there exist constants $L_g, l_g > 0$ such that

$$\|g(x_1,\lambda_1) - g(x_2,\lambda_2)\| \;\le\; L_g\|x_1 - x_2\| + l_g\|\lambda_1 - \lambda_2\|,\; \forall\,(x_1,\lambda_1),\,(x_2,\lambda_2) \in H \times \Lambda;$$

(ii) s-*strongly monotone*, if there exists a constant $s > 0$ such that

$$\langle g(x_1,\lambda) - g(x_2,\lambda),\, x_1 - x_2\rangle \;\ge\; s\|x_1 - x_2\|^2,\; \forall\,(x_1,\lambda),(x_2,\lambda) \in H \times \Lambda.$$

Definition 4.2[13]. A multi-valued mapping $A : H \times \Omega \to C(H)$ is said to be (L_A, l_A)-$\mathcal{H}$-*mixed Lipschitz continuous*, if there exist constants $L_A, l_A > 0$ such that

$$\mathcal{H}(A(x_1,\mu_1), A(x_2,\mu_2)) \;\le\; L_A\|x_1 - x_2\| + l_A\|\mu_1 - \mu_2\|,\; \forall\,(x_1,\mu_1),(x_2,\mu_2) \in H \times \Omega.$$

Definition 4.3[13]. Let $A, B : H \times \Omega \to C(H)$ be multi-valued mappings. A mapping $N : H \times H \times \Omega \to H$ is said to be

(i) $(L_{(N,1)}, L_{(N,2)}, l_N)$-*mixed Lipschitz continuous*, if there exist constants $L_{(N,1)}$, $L_{(N,2)}, l_N > 0$ such that

$$\|N(x_1,y_1,\mu_1) - N(x_2,y_2,\mu_2)\| \;\le\; L_{(N,1)}\|x_1 - x_2\| + L_{(N,2)}\|y_1 - y_2\| + l_N\|\mu_1 - \mu_2\|,$$

$$\forall\,(x_1,y_1,\mu_1),(x_2,y_2,\mu_2) \in H \times H \times \Omega;$$

(ii) ξ-*strongly mixed monotone with respect to* A *and* B, if there exists a constant $\xi > 0$ such that

$$\langle N(u_1,v_1,\mu) - N(u_2,v_2,\mu),\, x - y\rangle \;\ge\; \xi\|x - y\|^2,$$

$$\forall\, x,y \in H,\; \mu \in \Omega,\; u_1 \in A(x,\mu),\; u_2 \in A(y,\mu),\; v_1 \in B(x,\mu),\; v_2 \in B(y,\mu);$$

(iii) *σ-generalized mixed pseudocontractive with respect to A and B*, if there exists a constant $\sigma > 0$ such that

$$\langle N(u_1, v_1, \mu) - N(u_2, v_2, \mu),\, x - y\rangle \ \leq\ \sigma\|x - y\|^2,$$

$$\forall\ x, y \in H,\ \mu \in \Omega,\ u_1 \in A(x, \mu),\ u_2 \in A(y, \mu),\ v_1 \in B(x, \mu),\ v_2 \in B(y, \mu);$$

Now, we have the following fixed-point formulation of PMIQVLIP (3.1).

Lemma 4.1. For each $(f, \lambda, \mu) \in H \times \Lambda \times \Omega$, (x, u, v, w, y, z) with $x \in x(\lambda, \mu) \in H$, $u = u(x, \mu) \in A(x, \mu)$, $v = v(x, \mu) \in B(x, \mu)$, $w = w(x, \mu) \in C(x, \mu)$, $y = y(x, \mu) \in D(x, \mu)$ and $z = z(x, \lambda) \in F(x, \lambda)$ such that $(g - m)(x, \lambda) \in \text{domain}\, W(\cdot, z, \lambda)$ is a solution of PMIQVLIP (3.1) if and only if the multi-valued mapping $G : H\times\Lambda\times\Omega \to 2^H$ defined by

$$G(t, \lambda, \mu) = \bigcup_{u\in A(t,\mu),v\in B(t,\mu),w\in C(t,\mu),y\in D(t,\mu),z\in F(x,\lambda)} \Big[t-(g-m)(t, \lambda)$$

$$+R_{P,\eta}^{W(\cdot,z,\lambda)}[P \circ (g - m)(t, \lambda) - \rho N(u, v, \mu) + \rho M(w, y, \mu) + \rho f]\Big],\ t \in H, \qquad (4.1)$$

has a fixed point, where $P : H \to H$; $P \circ (g - m)$ denotes P composition of $(g - m)$; $R_{P,\eta}^{W(\cdot,z,\lambda)} = (P + \rho W(\cdot, z, \lambda))^{-1}$ and $\rho > 0$ is a constant.

Proof. For each $(f, \lambda, \mu) \in H\times\Lambda\times\Omega$, PMIQVLIP (3.1) has a solution (x, u, v, w, y, z) if and only if

$$f \in N(u, v, \mu)-M(w, y, \mu)+W((g-m)(x, \lambda), z, \lambda)$$

$$\Leftrightarrow\ P\circ(g-m)(x, \lambda)-\rho N(u, v, \mu)+\rho M(w, y, \mu)+\rho f \in (P+\rho W(\cdot, z, \lambda))((g-m)(x, \lambda)).$$

Since for each $(z, \lambda) \in H \times \Lambda$, $W(\cdot, z, \lambda)$ is maximal strongly P-η-monotone, by definition of strongly P-η-proximal mapping $R_{P,\eta}^{W(\cdot,z,\lambda)}$ of $W(\cdot, z, \lambda)$, preceding inclusion holds if and only if

$$(g - m)(x, \lambda)\ =\ R_{P,\eta}^{W(\cdot,z,\lambda)}[P \circ (g - m)(x, \lambda) - \rho N(u, v, \mu) + \rho M(w, y, \mu) + \rho f],$$

that is $x \in G(x, \lambda, \mu)$. This completes the proof.

Theorem 4.1. Let the multi-valued mappings $A, B, C, D : H \times \Omega \to C(H)$ and $F : H \times \Lambda \to C(H)$ be $\mathcal{H}$-Lipschitz continuous in the first arguments with constant L_A, L_B, L_C, L_D and L_F, respectively; let the mappings $\eta : H \times H \to H$ be τ-Lipschitz continuous and $P : H \to H$ be δ-strongly η-monotone. Let the mappings $g, m : H \times \Lambda \to H$ be such that $(g - m)$ is s-strongly monotone and $L_{(g-m)}$-Lipschitz continuous in the first argument and let the mapping $P \circ (g - m)$ be r-strongly monotone and $L_{P\circ(g-m)}$-Lipschitz continuous in the first argument; let the mapping $N : H \times H \times \Omega \to H$ be ξ-strongly mixed monotone with respect to A and B and $(L_{(N,1)}, L_{(N,2)})$-mixed Lipschitz continuous in first two arguments and let the mapping $M : H \times H \times \Omega \to H$ be σ-generalized mixed pseudocontractive with respect to C and D, and $(L_{(M,1)}, L_{(M,2)})$-mixed Lipschitz continuous in first two arguments. Suppose that the multi-valued mapping $W : H \times H \times \Lambda \to 2^H$ is such that for each $(t, \lambda) \in H \times \Lambda$, $W(\cdot, t, \lambda) : H \to 2^H$ is γ-strongly maximal P-η-monotone with range $(g - m)(H \times \{\lambda\}) \cap \text{domain}\, W(\cdot, t, \lambda) \neq \emptyset$. Suppose that there exist constants $k_1, k_2 > 0$ such that

$$\|R_{P,\eta}^{W(\cdot,x_1,\lambda_1)}(t) - R_{P,\eta}^{W(\cdot,x_2,\lambda_2)}(t)\| \leq k_1\|x_1 - x_2\| + k_2\|\lambda_1 - \lambda_2\|, \quad (4.2)$$
$$\forall\, x_1, x_2, t \in H;\ \lambda_1, \lambda_2 \in \Lambda,$$

and suppose for $\rho > 0$, the following condition holds:

$$\theta = q + \epsilon(\rho); \quad (4.3)$$

where

$$q := k_1 L_F + \sqrt{1 - 2s + L^2_{(g-m)}};\ \epsilon(\rho) := \frac{\tau}{\delta + \rho\gamma}[p + \sqrt{1 - 2\rho(\xi - \sigma) + 2\rho^2(L_N^2 + L_M^2)}];$$

$$p := \sqrt{1 - 2r + L^2_{p\circ(g-m)}};\ L_N := L_A L_{(N,1)} + L_B L_{(N,2)};\ L_M := L_C L_{(M,1)} + L_D L_{(M,2)};$$

$$\left|\rho - \frac{\xi - \sigma + (e\delta - p)e\gamma}{2(L_N^2 + L_M^2) - e^2\gamma^2}\right| < \frac{\sqrt{[\xi - \sigma - (e\delta - p)e\gamma^2] - [2(L_N^2 + L_M^2 - e^2\gamma^2][1 - (e\delta - p)^2]}}{2(L_N^2 + L_M^2) - e^2\gamma^2},$$

$$\xi - \sigma > (e\delta - p)e\gamma + \sqrt{[2(L_N^2 + L_M^2) - e^2\gamma^2][1 - (e\delta - p)^2]};\ \xi > \sigma; \quad (4.4)$$

$2(L_N^2 + L_M^2) > e^2\nu^2$; $(e\delta - p)$, $e := (1-q)/\tau, q \in (0, 1)$.

Then, for each fixed $f \in H$, the multi-valued mapping G defined by (4.1) is a compact-valued uniform θ-$\mathcal{H}$-contraction mapping with respect to $(\lambda, \mu) \in \Lambda \times \Omega$, where θ is given by (4.3)-(4.4). Moreover, for each $(\lambda, \mu) \in \Lambda \times \Omega$, the solution set $S(\lambda, \mu)$ of PMIQVLIP (3.1) is nonempty and closed.

Proof. Let (x, λ, μ) be an arbitrary element of $H \times \Lambda \times \Omega$.. Since A, B, C, D, F are compact-valued, then for any sequences $\{u_n\} \subset A(x, \mu)$, $\{v_n\} \subset B(x, \mu)$, $\{w_n\} \subset C(x, \mu)$, $\{y_n\} \subset D(x, \mu)$, $\{z_n\} \subset F(x, \lambda)$, there exist subsequences $\{u_{n_i}\} \subset \{u_n\}$, $\{v_{n_i}\} \subset \{v_n\}$, $\{w_{n_i}\} \subset \{w_n\}$, $\{y_{n_i}\} \subset \{y_n\}$, $\{z_{n_i}\} \subset \{z_n\}$ and elements $u \in A(x, \mu)$, $v \in B(x, \mu)$, $w \in C(x, \mu)$, $y \in D(x, \mu)$, $z \in F(x, \lambda)$ such that $u_{n_i} \to u$, $v_{n_i} \to v$, $w_{n_i} \to w$, $y_{n_i} \to y$, $z_{n_i} \to z$ as $i \to \infty$. By using Theorem 2.2, (4.2) and the mixed Lipschitz continuity of N and M, we estimate

$$\|R_{P,\eta}^{W(\cdot,z_{n_i},\lambda)}[P\circ(g-m)(x,\lambda) - \rho N(u_{n_i}, v_{n_i}, \mu) + \rho M(w_{n_i}, y_{n_i}, \mu) + \rho f]$$
$$- R_{P,\eta}^{W(\cdot,z,\lambda)}[P\circ(g-m)(x,\lambda) - \rho N(u, v, \mu) + \rho M(w, y, \mu) + \rho f]\|$$
$$\leq \|R_{P,\eta}^{W(.,z_{n_i},\lambda)}[P\circ(g-m)(x,\lambda) - \rho N(u_{n_i}, v_{n_i}, \mu) + \rho M(w_{n_i}, y_{n_i}, \mu) + \rho f]$$
$$- R_{P,\eta}^{W(\cdot,z,\lambda)}[P\circ(g-m)(x,\lambda) - \rho N(u_{n_i}, v_{n_i}, \mu) + \rho M(w_{n_i}, y_{n_i}, \mu) + \rho f]\|$$
$$+ \|R_{P,\eta}^{W(\cdot,z,\lambda)}[P\circ(g-m)(x,\lambda) - \rho N(u_{n_i}, v_{n_i}, \mu) + \rho M(w_{n_i}, y_{n_i}, \mu) + \rho f]$$
$$- R_{P,\eta}^{W(\cdot,z,\lambda)}[P\circ(g-m)(x,\lambda) - \rho N(u, v, \mu) + \rho M(w, y, \mu) + \rho f]\|$$

$$\leq k_1\|z_{n_i}-z\|+\rho\,\frac{\tau}{\delta+\rho\gamma}\Big[\|N(u_{n_i},v_{n_i},\mu)-N(u,v,\mu)\|+\|M(w_{n_i},y_{n_i},\mu)-M(w,y,\mu)\|\Big]$$
$$\leq k_1\|z_{n_i}-z\|+\rho\,\frac{\tau}{\delta+\rho\gamma}\Big[L_{(N,1)}\|u_{n_i}-u\|+L_{(N,2)}\|v_{n_i}-v\|+L_{(M,1)}\|w_{n_i}-w\|+L_{(M,2)}\|y_{n_i}-y\|\Big]$$
$$\to 0, \text{ as } i\to\infty. \tag{4.5}$$

Thus (4.1) and (4.5) yield that $G(x,\lambda,\mu)\in C(H)$.

Now, for each fixed $(\lambda,\mu)\in\Lambda\times\Omega$, we prove that $G(x,\lambda,\mu)$ is a uniform θ-$\mathcal{H}$-contraction mapping. Let (x_1,λ,μ), (x_2,λ,μ) be arbitrary elements of $H\times\Lambda\times\Omega$ and any $t_1\in G(x_1,\lambda,\mu)$, there exist $u_1=u_1(x_1,\mu)\in A(x_1,\mu)$, $v_1=v_1(x_1,\mu)\in B(x_1,\mu)$, $w_1=w_1(x_1,\mu)\in C(x_1,\mu)$, $y_1=y_1(x_1,\mu)\in D(x_1,\mu)$ and $z_1=z_1(x_1,\lambda)\in F(x_1,\lambda)$ such that

$$t_1 = x_1-(g-m)(x_1,\lambda)+R_{P,\eta}^{W(\cdot,z_1,\lambda)}[P\circ(g-m)(x_1,\lambda)-\rho N(u_1,v_1,\mu)+\rho M(w_1,y_1,\mu)+\rho f]. \tag{4.6}$$

It follows from the compactness of $A(x_2,\mu)$, $B(x_2,\mu)$, $C((x_2,\mu)$, $D(x_2,\mu)$ and $F(x_2,\lambda)$ and $\mathcal{H}$-Lipschitz continuity of A,B,C,D,F that there exist $u_2=u_2(x_2,\mu)\in A(x_2,\mu)$, $v_2=v_2(x_2,\mu)\in B(x_2,\mu)$, $w_2=w_2(x_2,\mu)\in C(x_2,\mu)$, $y_2=y_2(x_2,\mu)\in D(x_2,\mu)$ and $z_2=z_2(x_2,\lambda)\in F(x_2,\lambda)$ such that

$$\begin{aligned}
\|u_1-u_2\| &\leq \mathcal{H}(A(x_1,\mu),\,A(x_2,\mu)) \leq L_A\|x_1-x_2\|,\\
\|v_1-v_2\| &\leq \mathcal{H}(B(x_1,\mu),\,B(x_2,\mu)) \leq L_B\|x_1-x_2\|,\\
\|w_1-w_2\| &\leq \mathcal{H}(C(x_1,\mu),\,C(x_2,\mu)) \leq L_C\|x_1-x_2\|,\\
\|y_1-y_2\| &\leq \mathcal{H}(D(x_1,\mu),\,D(x_2,\mu)) \leq L_D\|x_1-x_2\|,\\
\|z_1-z_2\| &\leq \mathcal{H}(F(x_1,\lambda),\,F(x_2,\lambda)) \leq L_F\|x_1-x_2\|.
\end{aligned} \tag{4.7}$$

Let

$$t_2 = x_2-(g-m)(x_2,\lambda)+R_{P,\eta}^{W(\cdot,z_2,\lambda)}[P\circ(g-m)(x_2,\lambda)-\rho N(u_2,v_2,\mu)+\rho M(w_2,y_2,\mu)+\rho f], \tag{4.8}$$

then we have $t_2\in G(x_2,\lambda,\mu)$.

Next, using Theorem 2.2 and (4.1), we estimate

$$\begin{aligned}
\|t_1-t_2\| \leq\ & \|x_1-x_2-((g-m)(x_1,\lambda)-(g-m)(x_2,\lambda))\|\\
&+\|R_{P,\eta}^{W(\cdot,z_1,\lambda)}[P\circ(g-m)(x_1,\lambda)-\rho N(u_1,v_1,\mu)+\rho M(w_1,y_1,\mu)+\rho f]\\
&-R_{P,\eta}^{W(\cdot,z_2,\lambda)}[P\circ(g-m)(x_1,\lambda)-\rho N(u_1,v_1,\mu)+\rho M(w_1,y_1,\mu)+\rho f]\|\\
&+\|R_{P,\eta}^{W(\cdot,z_2,\lambda)}[P\circ(g-m)(x_1,\lambda)-\rho N(u_1,v_1,\mu)+\rho M(w_1,y_1,\mu)+\rho f]\\
&-R_{P,\eta}^{W(\cdot,z_2,\lambda)}[P\circ(g-m)(x_2,\lambda)-\rho N(u_2,v_2,\mu)+\rho M(w_2,y_2,\mu)+\rho f]\|
\end{aligned}$$

$$\leq \|x_1 - x_2 - ((g-m)(x_1,\lambda) - (g-m)(x_2,\lambda))\| + k_1\|z_1 - z_2\| + \frac{\tau}{\delta+\rho\gamma}\|x_1 - x_2 - (P\circ(g-m)(x_1,\lambda) - P\circ(g-m)(x_2,\lambda) + \|x_1 - x_2 - \rho(N(u_1,v_1,\mu) - N(u_2,v_2,\mu) - M(w_1,y_1,\mu) + M(w_2,y_2,\mu))\|\Big]. \tag{4.9}$$

Since $(g-m)$ is s-strongly monotone and $L_{(g-m)}$-Lipschitz continuous, we have

$$\|x_1-x_2-((g-m)(x_1,\lambda)-(g-m)(x_2,\lambda))\|^2 \leq \|x_1-x_2\|^2-2\langle(g-m)(x_1,\lambda)-(g-m)(x_2,\lambda), x_1-x_2\rangle+\|(g-m)(x_1,\lambda)-(g-m)(x_2,\lambda)\|^2 \leq (1-2s+L^2_{(g-m)})\|x_1-x_2\|^2. \tag{4.10}$$

Similarly, since $P\circ(g-m)$ is r-strongly monotone and $L_{P\circ(g-m)}$-Lipschitz continuous, we have

$$\|x_1-x_2-(P\circ(g-m)(x_1,\lambda)-P\circ(g-m)(x_2,\lambda))\|^2 \leq (1-2r+L^2_{P\circ(g-m)})\|x_1-x_2\|^2. \tag{4.11}$$

Since N is $(L_{(N,1)}, L_{(N,2)})$-mixed Lipschitz continuous; M is $(L_{(M,1)}, L_{(M,2)})$-mixed Lipschitz continuous and the multi-valued mappings A, B, C, D are $\mathcal{H}$-Lipschitz continuous, we have

$$\|N(u_1,v_1,\mu) - N(u_2,v_2,\mu)\| \leq L_{(N,1)}\|u_1 - u_2\| + L_{(N,2)}\|v_1 - v_2\| \leq L_{(N,1)}\mathcal{H}(A(x_1,\mu), A(x_2,\mu)) + L_{(N,2)}\mathcal{H}(B(x_1,\mu), B(x_2,\mu)) \leq (L_A L_{(N,1)} + L_B L_{(N,2)})\|x_1 - x_2\|, \tag{4.12}$$

and

$$\|M(w_1,y_1,\mu) - N(w_2,y_2,\mu)\| \leq (L_C L_{(M,1)} + L_D L_{(M,2)})\|x_1 - x_2\|. \tag{4.13}$$

Further, since N is ξ-strongly mixed monotone with respect to A and B, M is σ-generalized mixed pseudocontractive with respect to C and D then, using $\|a+b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$, we have

$$\|x_1-x_2-\rho(N(u_1,v_1,\mu)-N(u_2,v_2,\mu)-M(w_1,y_1,\mu)+M(w_2,y_2,\mu))\|^2 \leq \|x_1-x_2\|^2-2\rho\Big[\langle N(u_1,v_1,\mu)-N(u_2,v_2,\mu),\ x_1-x_2\rangle - \langle M(w_1,y_1,\mu) - M(w_2,y_2,\mu),\ x_1 - x_2\rangle\Big] + 2\rho^2\Big[\|N(u_1,v_1,\mu) - N(u_2,v_2,\mu)\|^2 + \|M(w_1,y_1,\mu)-M(w_2,y_2,\mu)\|^2\Big] \leq \|x_1-x_2\|^2-2\rho(\xi-\sigma)\|x_1-x_2\|^2 + 2\rho^2[(L_A L_{(N,1)} + L_B L_{(N,2)})^2 + (L_C L_{(M,1)} + L_D L_{(M,2)})^2\Big]\|x_1 - x_2\|^2 \leq \Big(1-2\rho(\xi-\sigma)+2\rho^2\Big[(L_A L_{(N,1)}+L_{(N,2)})^2+(L_C L_{(M,1)}+L_D L_{(M,2)})^2\Big]\Big)\|x_1-x_2\|^2. \tag{4.14}$$

Now, from (4.9)-(4.14), we have

$$\|t_1 - t_2\| \ \leq \ \theta \, \|x_1 - x_2\|, \tag{4.15}$$

where

$$\theta = q + \epsilon(\rho); \ \ q := k_1 L_F + \sqrt{1 - 2s + L^2_{(g-m)}};$$

$$\epsilon(\rho) := \frac{\tau}{\delta + \rho\gamma}\Big[\sqrt{1 - 2s + L^2_{P\circ(g-m)}} + \sqrt{1 - 2\rho(\xi - \sigma) + 2\rho^2(L_N^2 + L_M^2)};$$

$$L_N := (L_A L_{(N,1)} + L_B L_{(N,2)}); \ \ L_M := (L_C L_{(M,1)} + L_D L_{(M,2)}).$$

Hence, we have

$$d(t_1, G(x_2, \lambda, \mu)) \ = \ \inf_{t_2 \in G(x_2,\lambda,\mu)} \|t_1 - t_2\| \ \leq \ \theta\|x_1 - x_2\|.$$

Since $t_1 \in G(x_1, \lambda, \mu)$ is arbitrary, we obtain

$$\sup_{t_1 \in G(x_1,\lambda,\mu)} d(t_1, G(x_2, \lambda, \mu)) \ \leq \ \theta\|x_1 - x_2\|.$$

By using same argument, we can prove

$$\sup_{t_2 \in G(x_2,\lambda,\mu)} d(G(x_1, G(x_1, \lambda, \mu), t_2) \ \leq \ \theta\|x_1 - x_2\|.$$

By the definition of the Hausdorff metric $\mathcal{H}$ on $C(H)$, we have

$$\mathcal{H}\Big(G(x_1, \lambda, \mu), G(x_2, \lambda, \mu)\Big) \ \leq \ \theta\|x_1 - x_2\|, \tag{4.16}$$

that is, $G(x, \lambda, \mu)$ is a uniform θ-$\mathcal{H}$-contraction mapping with respect to $(\lambda, \mu) \in \Lambda \times \Omega$. Also, it follows from condition (4.3)-(4.4) that $\theta < 1$ and hence $G(x, \lambda, \mu)$ is a multi-valued contraction mapping which is uniform with respect to $(\lambda, \mu) \in \Lambda \times \Omega$. By Lemma 3.1 for each $(\lambda, \mu) \in \Lambda \times \Omega$, $G(x, \lambda, \mu)$ has a fixed point $x = x(\lambda, \mu) \in H$, that is, $x = x(\lambda, \mu) \in G(x, \lambda, \mu)$ and hence Lemma 4.1 ensure that $S(\lambda, \mu) \neq \emptyset$. Further, for any sequence $\{x_n\} \subset S(\lambda, \mu)$ with $\lim\limits_{n\to\infty} x_n = x_0$, we have $x_n \in G(x_n, \lambda, \mu)$ for all $n \geq 1$. By virtue of (4.16), we have

$$\begin{aligned} d(x_0, G(x_0, \lambda, \mu)) \ &\leq \ \|x_0 - x_n\| + \mathcal{H}\Big(G(x_n, \lambda, \mu), G(x_0, \lambda, \mu)\Big) \\ &\leq \ (1 + \theta)\|x_n - x_0\| \to 0 \text{ as } n \to \infty, \end{aligned}$$

that is, $x_0 \in G(x_0, \lambda, \mu)$ and hence $x_0 \in S(\lambda, \mu)$. Thus $S(\lambda, \mu)$ is closed in H. This completes the proof.

5. Lipschitz continuity

Now, we prove that the solution set $S(\lambda,\mu)$ of PMIQVLIP (3.1) is $\mathcal{H}$-Lipschitz continuity for each $(\lambda,\mu)\in\Lambda\times\Omega$.

Theorem 5.1. Let the multi-valued mappings A,B,C,D and F be $\mathcal{H}$-mixed Lipschitz continuous with pairs of constants (L_A,l_A), (L_B,l_B), (L_C,l_C), (L_D,l_D) and (L_F,l_F), respectively. Let the mappings η,P be the same as in Theorem 4.1; let the mapping $(g-m)$ be s-strongly monotone and $(L_{(g-m)},l_{(g-m)})$-Lipschitz continuous; let the mapping $P\circ(g-m)$ be r-strongly monotone and $(L_{P\circ(g-m)},l_{P\circ(g-m)})$-Lipschitz continuous. Let the mapping N be ξ-strongly mixed monotone with respect to A and B and $(L_{(N,1)},L_{(N,2)},l_N)$-mixed Lipschitz continuous, and let the mapping M be σ-generalized mixed pseudomonotone with respect to C and D, and $(L_{(M,1)},L_{(M,2)},l_M)$-mixed Lipschitz continuous. Suppose that the multi-valued mapping W is same as in Theorem 4.1 and conditions (4.2),(4.3),(4.4) hold, then for each $(\lambda,\mu)\in\Lambda\times\Omega$, the solution set $S(\lambda,\mu)$ of PMIQVLIP (3.1) is a $\mathcal{H}$-Lipschitz continuous mapping from $\Lambda\times\Omega$ into H.

Proof. For each (λ,μ), $(\bar\lambda,\bar\mu)\in\Lambda\times\Omega$, it follows from Theorem 4.1 that $S(\lambda,\mu)$ and $S(\bar\lambda,\bar\mu)$ are both nonempty and closed subsets of H. It also follows from Theorem 4.1 that $G(x,\lambda,\mu)$ and $G(x,\bar\lambda,\bar\mu)$ both are multi-valued θ-$\mathcal{H}$-contraction mappings with same contractive constant $\theta\in(0,1)$. By Lemma 3.2, we obtain

$$\mathcal{H}(S(\lambda,\mu),\,S(\bar\lambda,\bar\mu))\ \le\ \left(\frac{1}{1-\theta}\right)\sup_{x\in H}\ \mathcal{H}(G(x,\lambda,\mu),G(x,\bar\lambda,\bar\mu)),\tag{5.1}$$

where θ is given by (4.3)-(4.4).

Now, for any $a\in G(x,\lambda,\mu)$, there exist $u=u(x,\mu)\in A(x,\mu)$, $v=v(x,\mu)\in B(x,\mu)$, $w=w(x,\mu)\in C(x,\mu)$, $y=y(x,\mu)\in D(x,\mu)$ and $z=z(x,\lambda)\in F(x,\lambda)$ satisfying

$$a\ =\ x-(g-m)(x,\lambda)+R_{P,\eta}^{W(\cdot,z,\lambda)}[P\circ(g-m)(x,\lambda)-\rho N(u,v,\mu)+\rho M(w,y,\mu)+\rho f].\tag{5.2}$$

It is easy to see that there exist $\bar u=u(x,\bar\mu)\in A(x,\bar\mu)$, $\bar v=v(x,\bar\mu)\in B(x,\bar\mu)$, $\bar w=w(x,\bar\mu)\in C(x,\bar\mu)$, $\bar y=y(x,\bar\mu)\in D(x,\bar\mu)$ and $\bar z=z(x,\bar\lambda)\in F(x,\bar\lambda)$ such that

$$\begin{aligned}
\|u-\bar u\| &\le \mathcal{H}(A(x,\mu),\,A(x,\bar\mu))\ \le\ l_A\|\mu-\bar\mu\|,\\
\|v-\bar v\| &\le \mathcal{H}(B(x,\mu),\,B(x,\bar\mu))\ \le\ l_B\|\mu-\bar\mu\|,\\
\|w-\bar w\| &\le \mathcal{H}(C(x,\lambda),\,C(x,\bar\mu))\ \le\ l_C\|\mu-\bar\mu\|,\\
\|y-\bar y\| &\le \mathcal{H}(D(x,\mu),\,D(x,\bar\mu))\ \le\ l_D\|\mu-\bar\mu\|,\\
\|z-\bar z\| &\le \mathcal{H}(F(x,\lambda),F(x,\bar\lambda))\ \le\ l_F\|\lambda-\bar\lambda\|.
\end{aligned}\tag{5.3}$$

Let

$$b = x-(g-m)(x,\bar{\lambda})+R_{P,\eta}^{W(\cdot,\bar{z},\bar{\lambda})}\Big[P\circ(g-m)(x,\bar{\lambda})-\rho N(\bar{u},\bar{v},\bar{\mu})+\rho M(\bar{w},\bar{y},\bar{\mu})+\rho f\Big]. \tag{5.4}$$

Clearly, $b \in G(x,\bar{\lambda},\bar{\mu})$.

Since N and M are mixed Lipschitz continuous and in view of (4.3) and (5.1)-(5.4) and with $t = P\circ(g-m)(x,\bar{\lambda}) - \rho N(\bar{u},\bar{v},\bar{\mu}) + \rho M(\bar{w},\bar{y},\bar{\mu})$, we have

$$\begin{aligned}
\|a-b\| \;\leq\; & \|(g-m)(x,\lambda)-(g-m)(x,\bar{\lambda})\| \\
& +\|R_{P,\eta}^{W(\cdot,z,\lambda)}[P\circ(g-m)(x,\lambda)-\rho N(u,v,\mu)+\rho M(w,y,\mu)+\rho f] - R_{P,\eta}^{W(\cdot,z,\lambda)}(t)\| \\
& +\|R_{P,\eta}^{W(\cdot,z,\lambda)}(t)-R_{P,\eta}^{W(\cdot,\bar{z},\lambda)}(t)\|+\|R_{P,\eta}^{W(\cdot,\bar{z},\lambda)}(t)-R_{P,\eta}^{W(\cdot,\bar{z},\bar{\lambda})}(t)\| \\
\leq\; & \|(g-m)(x,\lambda)-(g-m)(x,\bar{\lambda})\|+\frac{\tau}{\delta+\rho\gamma}\Big[\|P\circ(g-m)(x,\lambda)-P\circ(g-m)(x,\bar{\lambda})\| \\
& +\rho\|N(u,v,\mu)-N(\bar{u},\bar{v},\bar{\mu})\|+\rho\|M(w,y,\mu)-M(\bar{w},\bar{y},\bar{\mu})\|\Big] \\
& +k_1\|z-\bar{z}\|+k_2\|\lambda-\bar{\lambda}\| \\
\leq\; & l_{(g-m)}\|\lambda-\bar{\lambda}\|+\frac{\tau}{\delta+\rho\gamma}l_{P\circ(g-m)}\|\lambda-\bar{\lambda}\|+\rho\Big(l_A L_{(N,1)}+l_B L_{(N,2)}+l_N \\
& +l_C L_{(M,1)}+l_D L_{(M,2)}+l_M\Big)\|\mu-\bar{\mu}\|\Big]+k_1 l_F\|\lambda-\bar{\lambda}\|+k_2\|\lambda-\bar{\lambda}\| \\
\leq\; & \theta_1(\|\lambda-\bar{\lambda}\|+\|\mu-\bar{\mu}\|),
\end{aligned}$$

where

$$\theta_1 := \max\{(l_{(g-m)}+k_1 l_F+k_2+\frac{\tau}{\delta+\rho\gamma}\,L_{P\circ(g-m)}),$$

$$\frac{\tau}{\delta+\rho\gamma}(l_A L_{(N,1)}+l_B L_{(N,2)}+l_N+l_C L_{(M,1)}+l_D L_{(M,2)}+l_M)\}$$

Hence, we obtain

$$\sup_{a\in G(x,\lambda,\bar{\mu})} d(a,G(x,\bar{\lambda},\mu)) \;\leq\; \theta_1\|\lambda,\mu)-(\bar{\lambda},\bar{\mu})\|_*,$$

where $\|(\lambda,\mu)\|_* = \|\lambda\|+\|\mu\|$.

By using similar argument, we have

$$\sup_{b\in G(x,\bar{\lambda},\bar{\mu})} d(G(x,\lambda,\mu),b) \;\leq\; \theta_1\|(\lambda,\mu)-(\bar{\lambda},\bar{\mu})\|_*.$$

Hence, it follows that

$$\mathcal{H}(G(x,\lambda,\mu),G(x,\bar{\lambda},\bar{\mu})) \leq \theta_1\|(\lambda,\mu)-(\bar{\lambda},\bar{\mu})\|_*,\ \forall\,(x,\lambda,\mu),(x,\bar{\lambda},\bar{\mu})\in H\times\Lambda\times\Omega.$$

By Lemma 5.1, we obtain

$$\mathcal{H}(S(\lambda,\mu),S(\bar{\lambda},\bar{\mu})) \;\leq\; \Big(\frac{\theta_1}{1-\theta}\Big)\|(\lambda,\mu)-(\bar{\lambda},\bar{\mu})\|_*,$$

which implies that $S(\lambda, \mu)$ is $\mathcal{H}$-Lipschitz continuous in $(\lambda, \mu) \in \Lambda \times \Omega$, and this completes the proof.

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