Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.
Volume 66, Number 1, Pages 51–65 (2017)
DOI: 10.1501/Commua1_0000000774
ISSN 1303–5991

STRONG AND WEAK CONVERGENCE OF AN ITERATIVE PROCESS FOR A FINITE FAMILY OF MULTIVALUED MAPPINGS SATISFYING THE CONDITION (C)

ISA YILDIRIM

Abstract. The aim of this paper is to introduce an iterative process with errors for a finite family of multivalued mappings satisfying the condition (C) which is weaker than nonexpansiveness. We also prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

1. Introduction

Fixed point theory is one of the most important tool of modern mathematics. This deals with the conditions which guarantee that a singlevalued mapping T of a set X into itself admits one or more fixed points, that is, points x of X which solve an operator equation $x = Tx$, called a fixed point equation. Fixed point theory serves as an essential tool for solving problems arising in various branches of mathematical analysis. These problems can be modeled by the equation $Tx = x$; where T is a nonlinear operator defined on a set equipped with some topological or order structure.

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [9] and Nadler [10]. Theory of multivalued mappings is harder than the corresponding theory of singlevalued mappings. Theory of multivalued mappings has applications in control theory, convex optimization, differential equations and economics.

Throughout this paper, the letter $\mathbb{N}$ will denote the set of natural numbers. We recall some definitions as follows:

Let X be a real Banach space. A subset E is called *proximinal* if for each $x \in X$, there exists an element $y \in E$ such that

$$d(x, y) = \inf\{\|x - z\| : z \in E\} = d(x, E).$$

Received by the editors: March 03, 2016, Accepted: July 28, 2016.
2010 *Mathematics Subject Classification.* 47H10, 47H09.
Key words and phrases. Strong and Weak Convergence, Multivalued Mapping, Condition (C).

It is known that a weakly compact convex subsets of a Banach space and closed convex subsets of a uniformly convex Banach space are proximinal. We shall denote by $CB(E), K(E)$ and $P(E)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of E, respectively. Let H be a Hausdorff metric induced by the metric d of X, that is

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}$$

for every $A, B \in CB(E)$. It is obvious that $P(E) \subseteq CB(E)$.

Let $T : E \to CB(E)$ be a multivalued mapping. An element $x \in E$ is said to be a fixed point of T, if $x \in Tx$. The set of fixed points of T will be denote by $F(T)$. Moreover, we will write $\mathcal{F} = \cap_{i=1}^{r} F(T_i)$ for the set of all common fixed points of the mappings $T_1, T_2, ..., T_r$. The mapping $T : E \longrightarrow CB(E)$ is said to be

(i) *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$, for all $x, y \in E$;

(ii) *quasi-nonexpansive* if $H(Tx, Tp) \leq \|x - p\|$, for all $x \in E$ and $p \in F(T)$.

In 2008, Suzuki [17] introduced a condition on mappings, called (C) which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. A multivalued mapping $T : E \longrightarrow CB(E)$ is said to satisfy condition (C) provided that

$$\frac{1}{2} d(x, Tx) \leq \|x - y\| \Rightarrow H(Tx, Ty) \leq \|x - y\|$$

for all $x, y \in E$.

From the above definitions, it follows that a nonexpansive mapping must be quasi-nonexpansive mapping. However, the converse of this statement is not true, in general. If $T : E \longrightarrow CB(E)$ is a multivalued nonexpansive mapping, then T satisfies the condition (C) ([1]). Moreover, if $T : E \longrightarrow CB(E)$ is a multivalued mapping which satisfies the condition (C) and has a fixed point, then T is a quasi-nonexpansive mapping ([5]).

Different iterative processes have been used to approximate fixed points of multivalued nonexpansive mappings. Among these iterative processes, Sastry and Babu [13] considered the following.

Let E be a nonempty convex subset of a Banach space X, $T : E \longrightarrow P(E)$ a multi-valued mapping with $p \in Tp$.

(i) The sequences of Mann iterates is defined by $x_1 \in K$,

$$x_{n+1} = (1 - a_n)x_n + a_n y_n, \tag{1.1}$$

where for all $n \in \mathbb{N}$ and $y_n \in Tx_n$;

(ii) The sequence of Ishikawa iterates is defined by $x_1 \in E$,

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \\ y_n = (1 - \beta_n)x_n + \beta_n z_n, \end{cases} \tag{1.2}$$

where for all $n \in \mathbb{N}$, $u_n \in Ty_n$ and $z_n \in Tx_n$.

They proved that the Mann and Ishikawa iteration processes for multivalued mapping T with a fixed point p converge to a fixed point q of T under certain conditions. They also claimed that the fixed point q may be different from p. Panyanak [12] extended result of Sastry and Babu [13] to uniformly convex Banach spaces. After, Song and Wang [15] noted that there was a gap in the proof of the main result in [12]. They further revised the gap and also gave the affirmative answer to Panyanak's open question. Shazad and Zegeye [16] extended and improved results already appeared in the papers [12, 13, 15].

Khan and Yildirim [8] further generalized the results of Song and Cho [14] and Shahzad and Zegeye [16] partly by incorporating and unifying their techniques. For results on a three step iteration process, see for example, Khan et al. [7].

Recently, Yildirim and Ozdemir ([19], [20]) proved some strong and weak convergence result for nonexpansive and quasi-nonexpansive mappings by using the following multistep iteration process: For an arbitrary fixed order $r \geq 2$,

$$\begin{cases} x_{n+1} = (1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T_1 y_{n+r-2}, \\ y_{n+r-2} = (1-\alpha_{2n})\, y_{n+r-3} + \alpha_{2n} T_2 y_{n+r-3}, \\ \vdots \\ y_{n+1} = \left(1-\alpha_{(r-1)n}\right) y_n + \alpha_{(r-1)n} T_{r-1} y_n, \\ y_n = (1-\alpha_{rn})\, x_n + \alpha_{rn} T_r x_n, \end{cases} \tag{1.3}$$

or, in short,

$$\begin{cases} x_{n+1} = (1-\alpha_{1n})\, y_{n+r-2} + \alpha_{1n} T_1 y_{n+r-2}, \\ y_{n+r-i} = (1-\alpha_{in})\, y_{n+r-(i+1)} + \alpha_{in} T_i y_{n+r-(i+1)}, \\ y_n = (1-\alpha_{rn})\, x_n + \alpha_{rn} T_r x_n, \end{cases} \tag{1.4}$$

where for all $n \in \mathbb{N}$, $\{\alpha_{1n}\}$ and $\{\alpha_{in}\}$, $i = 2, ...r$, are real sequences in $[0,1)$.

If $T_1 = T_2 = ... = T_r = T$ and $\alpha_{in} = 0$ for $i = 2, ...r$ and all $n \in \mathbb{N}$, then (1.3) reduces to (1.1).

In 2011, Eslamian and Homaeipour [6] introduced a new three-step iterative process for multivalued mappings in Banach spaces. They also proved some convergence theorems for multivalued mappings satisfying condition (C) in uniformly convex Banach spaces. Their iteration process with errors as follows:

Let E be a nonempty convex subset of a Banach space X and $T_1, T_2, T_3 : E \longrightarrow CB(E)$ be three multivalued mappings. Then for $x_1 \in E$,

$$\begin{cases} x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n u_n + \beta_n s_n^{''}, \\ y_n = (1-c_n-d_n)x_n + c_n v_n + d_n s_n^{'}, \\ w_n = (1-a_n-b_n)x_n + a_n z_n + b_n s_n, \end{cases} \tag{1.5}$$

where for all $n \in \mathbb{N}$, $u_n \in T_1 y_n, v_n \in T_2 w_n$ and $z_n \in T_3 x_n$, $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{s_n\}, \left\{s_n^{'}\right\}$ and $\left\{s_n^{''}\right\}$ are bounded sequences in X.

Inspired by the above works, we introduce the following iterative process for a finite familiy of multivalued mappings.

Let E be a nonempty convex subset of a Banach space X and $T_i : E \longrightarrow CB(E)$ $(i = 1, 2, ..., r)$ be a finite family of multivalued mappings. For an arbitrary fixed order $r \geq 2$,

$$\begin{cases} x_{n+1} = (1 - \alpha_{1n} - \beta_{1n})\, y_{n+r-2} + \alpha_{1n} z_{n,1} + \beta_{1n} u_{1n}, \\ y_{n+r-2} = (1 - \alpha_{2n} - \beta_{2n})\, y_{n+r-3} + \alpha_{2n} z_{n,2} + \beta_{2n} u_{2n}, \\ \vdots \\ y_{n+1} = \left(1 - \alpha_{(r-1)n} - \beta_{(r-1)n}\right) y_n + \alpha_{(r-1)n} z_{n,r-1} + \beta_{(r-1)n} u_{(r-1)n}, \\ y_n = (1 - \alpha_{rn} - \beta_{rn})\, x_n + \alpha_{rn} z_{n,r} + \beta_{rn} u_{rn}, \end{cases} \tag{1.6}$$

where for all $n \in \mathbb{N}$, $z_{n,r} \in T_r(x_n)$ and $z_{n,i} \in T_i\left(y_{n+r-(i+1)}\right)$ for $i = 1, 2, 3, ..., r$ and $\{\alpha_{in}\}, \{\beta_{in}\} \subset [0, 1]$ and $\{u_{in}\}$ are bounded sequences in X.

Finding common fixed points of a finite family of mappings is an important problem. Altough many algorithms have been introduced for various classes of mappings, the existence of common fixed points of a family of mappings are not known in many situations. So, it is natural to consider approximation results for such mappings.

The purpose of this paper is to study convergence of the sequence in (1.6) to a common fixed point of a finite family of multivalued mappings in uniformly convex Banach spaces. Our work is a significant generalization of the corresponding results in the literature.

2. Preliminaries

Let X be a real normed linear space. The modulus of convexity of X is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| = \|y\| = 1, \|x - y\| = \varepsilon \right\};$$

X is called *uniformly convex* if and only if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

A mapping $T : E \longrightarrow CB(E)$ is said to be *semicompact* if, for any sequence $\{x_n\}$ in E such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in E$. We note that if E is compact, then every multivalued mappings $T : E \longrightarrow CB(E)$ is semicompact.

A mapping $T : E \longrightarrow CB(E)$ is said to satisfy *condition* (I) if there is a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(t) > 0$ for all $t \in (0, \infty)$ such that

$$d(x, Tx) \geq g(d(x, F(T))).$$

Let $T_i : E \longrightarrow CB(E)$ $(i = 1, 2, ..., r)$ be a finite family of mappings. The mappings T_i for all i $(i = 1, 2, ..., r)$ are said to satisfy *condition* (II) if there exist a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(t) > 0$ for all $t \in (0, \infty)$ such that

$$\sum_{i=1}^{r} d\left(x, T_i x\right) \geq g\left(d\left(x, \mathcal{F}\right)\right),$$

where $\mathcal{F} = \cap_{i=1}^{r} F\left(T_i\right)$.

Throughout this paper, we will denote the weak convergence and the strong convergence by $\rightharpoonup$ and $\rightarrow$, respectively.

A Banach space E is said to satisfy *Opial's condition* [11] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

Examples of Banach spaces satisfying this condition are Hilbert spaces and all l^p spaces $(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition.

The mapping $T : E \longrightarrow CB(E)$ is called *demi-closed* if for every sequence $\{x_n\} \subset E$ and any $y_n \in Tx_n$ such that $x_n \rightharpoonup x$ and $y_n \rightarrow y$, we have $x \in E$ and $y \in Tx$. If the space E satisfies Opial's condition, then $I - T$ is demi-closed at 0, where $T : E \longrightarrow K(E)$ is a nonexpansive multivalued mapping ([4]).

We use the following lemmas to prove our main results.

Lemma 1. [2] *Let E be a nonempty subset of a uniformly convex Banach space X and $T : E \longrightarrow CB(E)$ be a multivalued mapping with convex-valued and satisfying the condition (C) then*

$$H(Tx, Ty) \leq 2d\left(x, Tx\right) + \|x - y\|, \quad \forall x, y \in E.$$

Lemma 2. [2] *(Demi-closed principle) Let X be a uniformly convex Banach space satisfying the Opial condition, E be a nonempty closed and convex subset of X. Let $T : E \longrightarrow CB(E)$ be a multi-valued mapping with convex-values and satisfying the condition (C). Let $\{x_n\}$ be a sequence in E such that $x_n \rightharpoonup p \in E$, and let $\lim_{n\to\infty} d\left(x_n, Tx_n\right) = 0$, then $p \in Tp$, i.e., $I - T$ is demi-closed at zero.*

Lemma 3. [1] *Let $T : E \longrightarrow CB(E)$ be a multivalued nonexpansive mapping, then T satisfies the condition (C).*

Lemma 4. [18] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq \left(1 + \delta_n\right) a_n + b_n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n\to\infty} a_n = 0$.

Lemma 5. [3] *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing and convex function $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta\varphi\left(\|x - y\|\right),$$

for all $x, y, z \in B_r$ and all $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma=1$.

3. Main Results

We start with the following lemma.

Lemma 6. *Let E be a nonempty, closed and convex subset of a uniformly convex Banach space X. Let $T_i : E \longrightarrow CB(E)$, $(i=1,2,...,r)$ be a finite family of multivalued mappings satisfying the condition (C). Assume that $\mathcal{F}=\cap_{i=1}^{r} F(T_i) \neq \emptyset$, $T_i(p)=\{p\}, (i=1,2,...,r)$ for each $p \in \mathcal{F}$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each i. Let $\{x_n\}$ be the sequence as defined in (1.6). Then $\lim_{n\to\infty} \|x_n - p\|$ exist for any $p \in \mathcal{F}$.*

Proof. Suppose that $p \in \mathcal{F}$. Since the sequences $\{u_{in}\}$ are bounded for $i=1,2,...,r$, there exists $\lambda > 0$ such that

$$\max\{\sup\|u_{1n}-p\|, \sup\|u_{2n}-p\|, ..., \sup\|u_{rn}-p\|\} \leq \lambda.$$

Using (1.6) and the condition (C), we have

$$\begin{aligned}
\|x_{n+1}-p\| &\leq (1-\alpha_{1n}-\beta_{1n})\|y_{n+r-2}-p\| + \alpha_{1n}\|z_{n,1}-p\| + \beta_{1n}\|u_{1n}-p\| \\
&\leq (1-\alpha_{1n}-\beta_{1n})\|y_{n+r-2}-p\| + \alpha_{1n} d(z_{n,1}, T_1(p)) + \beta_{1n}\|u_{1n}-p\| \\
&\leq (1-\alpha_{1n}-\beta_{1n})\|y_{n+r-2}-p\| + \alpha_{1n} H(T_1(y_{n+r-2}), T_1(p)) \\
&\quad +\beta_{1n}\|u_{1n}-p\| \\
&\leq (1-\alpha_{1n}-\beta_{1n})\|y_{n+r-2}-p\| + \alpha_{1n}\|y_{n+r-2}-p\| + \beta_{1n}\|u_{1n}-p\| \\
&= (1-\beta_{1n})\|y_{n+r-2}-p\| + \beta_{1n}\|u_{1n}-p\| \\
&\leq \|y_{n+r-2}-p\| + \beta_{1n}\lambda
\end{aligned}$$

and

$$\begin{aligned}
&\|y_{n+r-2}-p\| \\
&\quad\leq (1-\alpha_{2n}-\beta_{2n})\|y_{n+r-3}-p\| + \alpha_{2n}\|z_{n,2}-p\| + \beta_{2n}\|u_{2n}-p\| \\
&\quad\leq (1-\alpha_{2n}-\beta_{2n})\|y_{n+r-3}-p\| + \alpha_{2n} d(z_{n,2}, T_2(p)) + \beta_{2n}\|u_{2n}-p\| \\
&\quad\leq (1-\alpha_{2n}-\beta_{2n})\|y_{n+r-3}-p\| + \alpha_{2n} H(T_2(y_{n+r-3}), T_2(p)) + \beta_{2n}\|u_{2n}-p\| \\
&\quad\leq (1-\alpha_{2n}-\beta_{2n})\|y_{n+r-3}-p\| + \alpha_{2n}\|y_{n+r-3}-p\| + \beta_{2n}\|u_{2n}-p\| \\
&\quad= (1-\beta_{2n})\|y_{n+r-3}-p\| + \beta_{2n}\|u_{2n}-p\| \\
&\quad\leq \|y_{n+r-3}-p\| + \beta_{2n}\lambda.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|y_{n+1}-p\| &\leq \left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\|y_n-p\|+\alpha_{(r-1)n}\|z_{n,r-1}-p\| \\
&\quad +\beta_{(r-1)n}\left\|u_{(r-1)n}-p\right\| \\
&\leq \left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\|y_n-p\|+\alpha_{(r-1)n}d\left(z_{n,r-1},T_{r-1}(p)\right) \\
&\quad +\beta_{(r-1)n}\left\|u_{(r-1)n}-p\right\| \\
&\leq \left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\|y_n-p\|+\alpha_{(r-1)n}H\left(T_{r-1}(y_n),T_{r-1}(p)\right) \\
&\quad +\beta_{(r-1)n}\left\|u_{(r-1)n}-p\right\| \\
&\leq \left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\|y_n-p\|+\alpha_{(r-1)n}\|y_n-p\| \\
&\quad +\beta_{(r-1)n}\left\|u_{(r-1)n}-p\right\| \\
&= \left(1-\beta_{(r-1)n}\right)\|y_n-p\|+\beta_{(r-1)n}\left\|u_{(r-1)n}-p\right\| \\
&\leq \|y_n-p\|+\beta_{(r-1)n}\lambda,
\end{aligned}$$

and

$$\begin{aligned}
\|y_n-p\| &\leq (1-\alpha_{rn}-\beta_{rn})\|x_n-p\|+\alpha_{rn}\|z_{n,r}-p\|+\beta_{rn}\|u_{rn}-p\| \\
&\leq (1-\alpha_{rn}-\beta_{rn})\|x_n-p\|+\alpha_{rn}d\left(z_{n,r},T_r(p)\right)+\beta_{rn}\|u_{rn}-p\| \\
&\leq (1-\alpha_{rn}-\beta_{rn})\|x_n-p\|+\alpha_{rn}H\left(T_r(x_n),T_r(p)\right)+\beta_{rn}\|u_{rn}-p\| \\
&\leq (1-\alpha_{rn}-\beta_{rn})\|x_n-p\|+\alpha_{rn}\|x_n-p\|+\beta_{rn}\|u_{rn}-p\| \\
&\leq \|x_n-p\|+\beta_{rn}\lambda.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|x_{n+1}-p\| &\leq \|x_n-p\|+(1-\beta_{rn})\left(1-\beta_{(r-1)n}\right)...(1-\beta_{2n})\beta_{1n}\lambda \\
&\quad +(1-\beta_{rn})\left(1-\beta_{(r-1)n}\right)...(1-\beta_{3n})\beta_{2n}\lambda+... \\
&\quad +(1-\beta_{rn})\beta_{(r-1)n}\lambda+\beta_{rn}\lambda \\
&\leq \|x_n-p\|+\beta_{1n}\lambda+\beta_{2n}\lambda+...+\beta_{(r-1)n}\lambda+\beta_{rn}\lambda \\
&= \|x_n-p\|+(\beta_{1n}+\beta_{2n}+...+\beta_{(r-1)n}+\beta_{rn})\lambda \\
&\leq \|x_n-p\|+\left(\beta_{1n}+\beta_{2n}+...+\beta_{(r-1)n}+\beta_{rn}\right)\lambda \\
&= \|x_n-p\|+\mu_n \qquad (3.1)
\end{aligned}$$

where $\mu_n=\lambda\left(\beta_{1n}+\beta_{2n}+...+\beta_{(r-1)n}+\beta_{rn}\right)$. Using the fact that $\sum_{n=1}^{\infty}\mu_n<\infty$ and Lemma 4, we conclude that $\lim_{n\to\infty}\|x_n-p\|$ exist for any $p\in\mathcal{F}$. □

We now give some strong convergence theorems.

Theorem 1. *Let E be a nonempty, closed and convex subset of a uniformly convex Banach space X. Let $T_i : E \longrightarrow CB(E)$, $(i = 1, 2, ..., r)$ be a finite family of multivalued mappings with nonempty convex-values and satisfying the condition (C). Assume that $\mathcal{F} = \cap_{i=1}^{r} F(T_i) \neq \emptyset$, $T_i(p) = \{p\}$, $(i = 1, 2, ..., r)$ for each $p \in \mathcal{F}$ and T_i $(i = 1, 2, ..., r)$ satisfying the condition (II). Let $\alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., r$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each i. Then the sequence $\{x_n\}$ defined in (1.6) converges strongly to a common fixed point of T_i for $i = 1, 2, ..., r$.*

Proof. We will do our proof in two steps.

Step 1. Assume that $p \in \mathcal{F}$. By Lemma 6, $\lim_{n\to\infty} \|x_n - p\|$ exists. Since $\{x_n\}$ is bounded, there exists $r > 0$ such that $x_n - p, y_{n+r-m} - p \in B_r(0)$ all for some positive integer $m, 2 \leq m \leq r$ and $n \in \mathbb{N}$. As Step 1, there exists $\eta > 0$ such that

$$\max\left\{\sup\|u_{1n} - p\|^2, \sup\|u_{2n} - p\|^2, ..., \sup\|u_{rn} - p\|^2\right\} \leq \eta.$$

It follows from Lemma 5 that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \left(1 - \alpha_{(r-1)n} - \beta_{(r-1)n}\right)\|y_{n+r-2} - p\|^2 + \alpha_{1n}\|z_{n,1} - p\|^2 \qquad (3.2)\\
&\quad + \beta_{1n}\|u_{1n} - p\|^2 - \alpha_{1n}(1 - \alpha_{1n} - \beta_{1n})\varphi(\|y_{n+r-2} - z_{n,1}\|)\\
&\leq (1 - \alpha_{1n} - \beta_{1n})\|y_{n+r-2} - p\|^2 + \alpha_{1n} d(z_{n,1}, T_1(p))^2\\
&\quad + \beta_{1n}\|u_{1n} - p\|^2 - \alpha_{1n}(1 - \alpha_{1n} - \beta_{1n})\varphi(\|y_{n+r-2} - z_{n,1}\|)\\
&\leq (1 - \alpha_{1n} - \beta_{1n})\|y_{n+r-2} - p\|^2 + \alpha_{1n} H(T_1(y_{n+r-2}), T_1(p))^2\\
&\quad + \beta_{1n}\|u_{1n} - p\|^2 - \alpha_{1n}(1 - \alpha_{1n} - \beta_{1n})\varphi(\|y_{n+r-2} - z_{n,1}\|)\\
&\leq (1 - \beta_{1n})\|y_{n+r-2} - p\|^2 + \beta_{1n}\|u_{1n} - p\|^2\\
&\quad - \alpha_{1n}(1 - \alpha_{1n} - \beta_{1n})\varphi(\|y_{n+r-2} - z_{n,1}\|)\\
&\leq \|y_{n+r-2} - p\|^2 + \beta_{1n}\eta - \alpha_{1n}(1 - \alpha_{1n} - \beta_{1n})\varphi(\|y_{n+r-2} - z_{n,1}\|)
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+r-2} - p\|^2 &\leq (1 - \alpha_{2n} - \beta_{2n})\|y_{n+r-3} - p\|^2 + \alpha_{2n}\|z_{n,2} - p\|^2 \qquad (3.3)\\
&\quad + \beta_{2n}\|u_{2n} - p\|^2 - \alpha_{2n}(1 - \alpha_{2n} - \beta_{2n})\varphi(\|y_{n+r-3} - z_{n,2}\|)\\
&\leq (1 - \alpha_{2n} - \beta_{2n})\|y_{n+r-3} - p\|^2 + \alpha_{2n} d(z_{n,2}, T_2(p))^2\\
&\quad + \beta_{2n}\|u_{2n} - p\|^2 - \alpha_{2n}(1 - \alpha_{2n} - \beta_{2n})\varphi(\|y_{n+r-3} - z_{n,2}\|)\\
&\leq (1 - \alpha_{2n} - \beta_{2n})\|y_{n+r-3} - p\|^2 + \alpha_{2n} H(T_2(y_{n+r-3}), T_2(p))^2\\
&\quad + \beta_{2n}\|u_{2n} - p\|^2 - \alpha_{2n}(1 - \alpha_{2n} - \beta_{2n})\varphi(\|y_{n+r-3} - z_{n,2}\|)\\
&\leq (1 - \alpha_{2n} - \beta_{2n})\|y_{n+r-3} - p\|^2 + \alpha_{2n}\|y_{n+r-3} - p\|^2\\
&\quad + \beta_{2n}\|u_{2n} - p\|^2 - \alpha_{2n}(1 - \alpha_{2n} - \beta_{2n})\varphi(\|y_{n+r-3} - z_{n,2}\|)\\
&\leq \|y_{n+r-3} - p\|^2 + \beta_{2n}\eta\\
&\quad - \alpha_{2n}(1 - \alpha_{2n} - \beta_{2n})\varphi(\|y_{n+r-3} - z_{n,2}\|).
\end{aligned}$$

Again, we apply Lemma 5 to conclude that

$$
\begin{aligned}
&\|y_{n+1}-p\|^2 \\
&\leq \left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\|y_n-p\|^2+\alpha_{(r-1)n}\|z_{n,r-1}-p\|^2 \qquad (3.4)\\
&\quad +\beta_{(r-1)n}\left\|u_{(r-1)n}-p\right\|^2-\alpha_{(r-1)n}\left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\varphi\left(\|y_n-z_{n,2}\|\right)\\
&\leq \left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\|y_n-p\|^2+\alpha_{(r-1)n}d\left(z_{n,r-1},T_2\left(p\right)\right)^2\\
&\quad +\beta_{(r-1)n}\left\|u_{(r-1)n}-p\right\|^2-\alpha_{(r-1)n}\left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\varphi\left(\|y_n-z_{n,2}\|\right)\\
&\leq \left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\|y_n-p\|^2+\alpha_{(r-1)n}H\left(T_{r-1}\left(y_n\right),T_{r-1}\left(p\right)\right)^2\\
&\quad +\beta_{(r-1)n}\left\|u_{(r-1)n}-p\right\|^2-\alpha_{(r-1)n}\left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\varphi\left(\|y_n-z_{n,2}\|\right)\\
&\leq \left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\|y_n-p\|^2+\alpha_{(r-1)n}\|y_n-p\|^2\\
&\quad +\beta_{(r-1)n}\left\|u_{(r-1)n}-p\right\|^2-\alpha_{(r-1)n}\left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\varphi\left(\|y_n-z_{n,2}\|\right)\\
&\leq \left(1-\beta_{(r-1)n}\right)\|y_n-p\|^2+\beta_{(r-1)n}\eta\\
&\quad -\alpha_{(r-1)n}\left(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\right)\varphi\left(\|y_n-z_{n,2}\|\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_n-p\|^2 &\leq (1-\alpha_{rn}-\beta_{rn})\|x_n-p\|^2+\alpha_{rn}\|z_{n,r}-p\|^2+\beta_{rn}\|u_{rn}-p\|^2 \qquad (3.5)\\
&\quad -\alpha_{rn}(1-\alpha_{rn}-\beta_{rn})\varphi(\|x_n-z_{n,r}\|)\\
&\leq (1-\alpha_{rn}-\beta_{rn})\|x_n-p\|^2+\alpha_{rn}d(z_{n,r},T_r(p))^2+\beta_{rn}\|u_{rn}-p\|^2\\
&\quad -\alpha_{rn}(1-\alpha_{rn}-\beta_{rn})\varphi(\|x_n-z_{n,r}\|)\\
&\leq (1-\alpha_{rn}-\beta_{rn})\|x_n-p\|^2+\alpha_{rn}H(T_r(x_n),T_r(p))^2\\
&\quad +\beta_{rn}\|u_{rn}-p\|^2-\alpha_{rn}(1-\alpha_{rn}-\beta_{rn})\varphi(\|x_n-z_{n,r}\|)\\
&\leq (1-\alpha_{rn}-\beta_{rn})\|x_n-p\|^2+\alpha_{rn}\|x_n-p\|^2+\beta_{rn}\eta\\
&\quad -\alpha_{rn}(1-\alpha_{rn}-\beta_{rn})\varphi(\|x_n-z_{n,r}\|)\\
&\leq \|x_n-p\|^2+\beta_{rn}\eta-\alpha_{rn}(1-\alpha_{rn}-\beta_{rn})\varphi(\|x_n-z_{n,r}\|).
\end{aligned}
$$

By using (3.2), (3.3), (3.4) and (3.5), we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|x_n-p\|^2+\beta_{1n}\eta+\beta_{2n}\eta+...+\beta_{(r-1)n}+\beta_{rn}\eta\\
&\quad -\prod_{i=1}^{r}\alpha_{in}\left[\sum_{i=1}^{r}(1-\alpha_{in}-\beta_{in})\varphi\left(\left\|y_{n+r-(i+1)}-z_{n,i}\right\|\right)\right].
\end{aligned}
$$

From the condition $\alpha_{in}+\beta_{in}\in[a,b]\subset(0,1)$ for $i=1,2,...,r$, we obtain

$$\begin{aligned} & a^r\sum_{i=1}^{r}(1-b)\,\varphi\left(\left\|y_{n+r-(i+1)}-z_{n,i}\right\|\right) \\ \leq\ & \prod_{i=1}^{r}\alpha_{in}\left[\sum_{i=1}^{r}(1-\alpha_{in}-\beta_{in})\,\varphi\left(\left\|y_{n+r-(i+1)}-z_{n,i}\right\|\right)\right] \\ \leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2+\Big(\beta_{1n}+\beta_{2n}+...+\beta_{(r-1)n}+\beta_{rn}\Big)\,\eta. \end{aligned}$$

This implies that

$$\begin{aligned} & \sum_{n=1}^{\infty}\left[a^r\sum_{i=1}^{r}(1-b)\,\varphi\left(\left\|y_{n+r-(i+1)}-z_{n,i}\right\|\right)\right] \\ \leq\ & \|x_1-p\|^2+\sum_{n=1}^{\infty}\Big(\beta_{1n}+\beta_{2n}+...+\beta_{(r-1)n}+\beta_{rn}\Big)\,\eta<\infty \end{aligned}$$

from which it follows that $\lim_{n\to\infty}\varphi\left(\left\|y_{n+r-(i+1)}-z_{n,i}\right\|\right)=0$. Since φ is continuous at 0 and is strictly increasing, we have

$$\lim_{n\to\infty}\left\|y_{n+r-(i+1)}-z_{n,i}\right\|=0. \tag{3.6}$$

Hence for $i=1,2,...,r$, we have

$$\lim_{n\to\infty}\|y_{n+r-2}-z_{n,1}\|=\lim_{n\to\infty}\|y_{n+r-3}-z_{n,2}\|=...=\lim_{n\to\infty}\|x_n-z_{n,r}\|=0. \tag{3.7}$$

Also, using (1.6), (3.7) and $\sum_{n=1}^{\infty}\beta_{in}<\infty$ for each i, we have

$$\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}\left(\alpha_{rn}\|z_{n,r}-x_n\|+\beta_{rn}\|u_{rn}-x_n\|\right)=0, \tag{3.8}$$

$$\begin{aligned} \lim_{n\to\infty}\|y_{n+1}-x_n\| &= \lim_{n\to\infty}\Big(\Big(1-\alpha_{(r-1)n}-\beta_{(r-1)n}\Big)\|y_n-x_n\| \\ &\quad +\alpha_{(r-1)n}\|z_{n,r-1}-x_n\|+\beta_{(r-1)n}\left\|u_{(r-1)n}-x_n\right\|\Big) \\ &= 0 \\ &\ \vdots \\ \lim_{n\to\infty}\|x_{n+1}-x_n\| &= \lim_{n\to\infty}\left(\alpha_{rn}\|z_{n,r}-x_n\|+\beta_{rn}\|u_{rn}-x_n\|\right)=0. \end{aligned}$$

From (1.6), we obtain

$$\|x_{n+1}-z_{n,1}\|=(1-\alpha_{1n}-\beta_{1n})\|y_{n+r-2}-z_{n,1}\|+\beta_{1n}\|u_{1n}-z_{n,1}\|.$$

It follows from (3.7) and $\sum_{n=1}^{\infty}\beta_{in}<\infty$ for each i that

$$\lim_{n\to\infty}\|x_{n+1}-z_{n,1}\|=0.$$

From the triangle inequality, we have

$$\|x_n-z_{n,1}\|\leq\|x_n-x_{n+1}\|+\|x_{n+1}-z_{n,1}\|.$$

Taking the limit of both sides of this inequality and using (3.8), we have

$$\lim_{n\to\infty} \|x_n - z_{n,1}\| = 0.$$

Again, by the triangle inequality, we obtain for each $i = 1, 2, ..., r$

$$\|x_n - z_{n,i}\| \le \left\|x_n - y_{n+r-(i+1)}\right\| + \left\|y_{n+r-(i+1)} - z_{n,i}\right\|.$$

Similarly, for $i = 1, 2, ..., r$

$$\lim_{n\to\infty} \|x_n - z_{n,i}\| = 0. \tag{3.10}$$

Hence, it follows from Lemma 1, (3.7), (3.8) and (3.10) that

$$\begin{aligned} d\left(x_n, T_1\left(x_n\right)\right) &\le d\left(x_n, T_1\left(y_{n+r-2}\right)\right) + H\left(T_1\left(y_{n+r-2}\right), T_1\left(x_n\right)\right) \\ &\le d\left(x_n, T_1\left(y_{n+r-2}\right)\right) + 2d\left(y_{n+r-2}, T_1\left(y_{n+r-2}\right)\right) + \|y_{n+r-2} - x_n\| \\ &\le \|x_n - z_{n,1}\| + 2\|y_{n+r-2} - z_{n,1}\| + \|y_{n+r-2} - x_n\| \to 0 \text{ as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned} d\left(x_n, T_2\left(x_n\right)\right) &\le d\left(x_n, T_2\left(y_{n+r-3}\right)\right) + H\left(T_2\left(y_{n+r-3}\right), T_2\left(x_n\right)\right) \\ &\le d\left(x_n, T_2\left(y_{n+r-3}\right)\right) + 2d\left(y_{n+r-3}, T_2\left(y_{n+r-3}\right)\right) + \|y_{n+r-3} - x_n\| \\ &\le \|x_n - z_{n,2}\| + 2\|y_{n+r-3} - z_{n,2}\| + \|y_{n+r-3} - x_n\| \text{ as } n \to \infty. \end{aligned}$$

In a similar way, for each $i = 1, 2, ..., r$ we obtain that

$$\lim_{n\to\infty} d\left(x_n, T_i\left(x_n\right)\right) = 0.$$

Step 2. We now show that $\{x_n\}$ converges strongly to $q \in \mathcal{F}$.

From Step 1, we know that $\lim_{n\to\infty} d\left(x_n, T_i\left(x_n\right)\right) = 0$. Since the condition (II), $\lim_{n\to\infty} d\left(x_n, \mathcal{F}\right) = 0$. Therefore, we can choose a sequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence p_k in $\mathcal{F}$ such that for all $k \in \mathbb{N}$

$$\|x_{n_k} - p_k\| < \frac{1}{2^k}.$$

From (3.1), we have the following inequality for all $p \in \mathcal{F}$,

$$\begin{aligned} \left\|x_{n_{k+1}} - p\right\| &\le \left\|x_{n_{k+1}-1} - p\right\| + \mu_{n_{k+1}-1} \\ &\le \left\|x_{n_{k+1}-2} - p\right\| + \mu_{n_{k+1}-2} + \mu_{n_{k+1}-1} \\ &\vdots \\ &\le \|x_{n_k} - p\| + \sum_{l=1}^{n_{k+1}-n_k-1} \mu_{n_k+l} \end{aligned}$$

which implies that

$$\begin{aligned}\left\|x_{n_{k+1}}-p\right\| &\leq \left\|x_{n_k}-p_k\right\|+\sum_{l=1}^{n_{k+1}-n_k-1}\mu_{n_k+l}\\ &< \frac{1}{2^k}+\sum_{l=1}^{n_{k+1}-n_k-1}\mu_{n_k+l}.\end{aligned}$$

Now, we will show that $\{p_k\}$ is a Cauchy sequence in E. Note that

$$\begin{aligned}\left\|p_{k+1}-p_k\right\| &\leq \left\|p_{k+1}-x_{n_{k+1}}\right\|+\left\|x_{n_{k+1}}-p_k\right\|\\ &< \frac{1}{2^{k+1}}+\frac{1}{2^k}+\sum_{l=1}^{n_{k+1}-n_k-1}\mu_{n_k+l}\\ &< \frac{1}{2^{k-1}}+\sum_{l=1}^{n_{k+1}-n_k-1}\mu_{n_k+l}.\end{aligned}$$

Thus $\{p_k\}$ is a Cauchy sequence in E. Since E is complete, this sequence is convergent. Let $\lim_{n\to\infty} p_k = q$. We need to show that $q \in \mathcal{F}$. Since for $i = 1, 2, ..., r$

$$d\left(p_k, T_i\left(q\right)\right) \leq H\left(T_i\left(p_k\right), T_i\left(q\right)\right) \leq \left\|p_k - q\right\|$$

and $p_k \to q$ as $k \to \infty$, it follows that $d\left(q, T_i\left(q\right)\right) = 0$ for $i = 1, 2, ..., r$. Hence $q \in \mathcal{F}$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty}\left\|x_n - q\right\|$ exists, we conclude that $\{x_n\}$ converges strongly to q. □

Since the condition (II) is weaker than the compactness of K and the semicompactness of the multivalued mappings $\{T_i : i = 1, 2, ..., r\}$, therefore we already have the following theorem.

Theorem 2. *Let E be a nonempty, closed and convex subset of a uniformly convex Banach space X. Let $T_i : E \longrightarrow CB(E)$, $(i = 1, 2, ..., r)$ be a finite family of multivalued mappings with nonempty convex-values and satisfying the condition (C). Assume that $\mathcal{F} = \cap_{i=1}^{r} F\left(T_i\right) \neq \emptyset$ and $T_i\left(p\right) = \{p\}, (i = 1, 2, ..., r)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be defined in (1.6), and $\alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., r$ and $\sum_{n=1}^{\infty}\beta_{in} < \infty$ for each i. Assume that either E is compact or one of the multivalued mappings $\{T_i : i = 1, 2, ..., r\}$ is semicompact. Then $\{x_n\}$ converges strongly to a common fixed point of T_i for $i = 1, 2, ..., r$.*

Proof. As in the proof of Theorem 1, we have $\lim_{n\to\infty} d\left(x_n, T_i\left(x_n\right)\right) = 0$ for each i. We assume that either E is compact or one of the multivalued mappings $\{T_i : i = 1, 2, ..., r\}$ is semicompact. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$

such that $\lim_{k\to\infty} x_{n_k} = z$ for some $z \in E$. By Lemma 1, we have for $i = 1, 2, ..., r$

$$\begin{aligned} d\left(z, T_i\left(z\right)\right) &\leq \left\|z - x_{n_k}\right\| + d\left(x_{n_k}, T_i\left(z\right)\right) \\ &\leq \left\|z - x_{n_k}\right\| + d\left(x_{n_k}, T_i\left(x_{n_k}\right)\right) + H\left(T_i\left(x_{n_k}\right), T_i\left(z\right)\right) \\ &\leq 3d\left(x_{n_k}, T_i\left(x_{n_k}\right)\right) + 2\left\|z - x_{n_k}\right\| \to 0 \text{ as } k \to \infty, \end{aligned}$$

this implies that $z \in \mathcal{F}$. Since $\{x_{n_k}\}$ converges strongly to z and the limit $\lim_{n\to\infty} \|x_n - z\|$ exists (as in the proof Theorem 1), it follows that $\{x_n\}$ converges strongly to z. □

Theorem 3. *Let E be a nonempty, closed and convex subset of a uniformly convex Banach space X with the Opial property. Let $T_i : E \longrightarrow CB(E)$, $(i = 1, 2, ..., r)$ be a finite family of multivalued mappings with nonempty convex-values and satisfying the condition (C). Assume that $\mathcal{F} = \cap_{i=1}^{k} F\left(T_i\right) \neq \emptyset$ and $T_i\left(p\right) = \{p\}, (i = 1, 2, ..., r)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be defined in (1.6), and $\alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., r$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each i. Then $\{x_n\}$ converges weakly to a common fixed point of T_i for $i = 1, 2, ..., r$.*

Proof. It follow from Lemma 6 and Theorem 1 that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d\left(x_n, T_i\left(x_n\right)\right) = 0$ for each i. Since a uniformly convex Banach space is reflexive, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup q$ as $n_k \to \infty$ for some $q \in X$. We will show that $q \in \mathcal{F}$. By Lemma 2, $I - T_i$ is demi-closed at zero for each i. Hence from $\lim_{n\to\infty} d\left(x_n, T_i\left(x_n\right)\right) = 0$, $q \in F(T_i)$. By the arbitrariness of $i \geq 1$, we have $q \in \mathcal{F}$.

If there exists another subsequence $\{x_{n_l}\} \subset \{x_n\}$ such that $x_{n_l} \rightharpoonup q^* \in E$ and $q \neq q^*$. As in the proof above, we can also prove that $q^* \in \mathcal{F}$. So by Lemma 6, $\lim_{n\to\infty} \|x_n - w\|$ and $\lim_{n\to\infty} \|x_n - z\|$ exist. Then by using Opial's property,

$$\begin{aligned} \lim_{n\to\infty} \|x_n - w\| &= \lim_{n_k\to\infty} \|x_{n_k} - w\| \\ &< \lim_{n_k\to\infty} \|x_{n_k} - z\| = \lim_{n\to\infty} \|x_n - z\| \\ &= \lim_{n_l\to\infty} \|x_{n_l} - z\| < \lim_{n_l\to\infty} \|x_{n_l} - w\| \\ &= \lim_{n\to\infty} \|x_n - w\| \end{aligned}$$

which is a contradiction. Therefore $\{x_n\}$ converges weakly to a common fixed point of T_i for $i = 1, 2, ..., r$. □

From Lemma 3, we know that if T is a multivalued nonexpansive mapping, then T satisfies the condition (C). So we have the following results:

Corollary 1. *Let E be a nonempty, closed and convex subset of a uniformly convex Banach space X. Let $T_i : E \longrightarrow CB(E)$, $(i = 1, 2, ..., r)$ be a finite family of multivalued nonexpansive mappings. Assume that $\mathcal{F} = \cap_{i=1}^{r} F\left(T_i\right) \neq \emptyset$ and $T_i\left(p\right) = \{p\}, (i = 1, 2, ..., r)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be defined in (1.6), and $\alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., r$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each i. Assume that T_i*

$(i = 1, 2, ..., r)$ satisfying the condition (II). Then $\{x_n\}$ converges strongly to a common fixed point of T_i for $i = 1, 2, ..., r$.

Corollary 2. *Let E be a nonempty, closed and convex subset of a uniformly convex Banach space X. Let $T_i : E \longrightarrow CB(E)$, $(i = 1, 2, ..., r)$ be a finite family of multivalued nonexpansive mappings. Assume that $\mathcal{F} = \cap_{i=1}^{r} F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, ..., r)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be defined in (1.6), and $\alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., r$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each i. Assume that either E is compact or one of the multivalued mappings $\{T_i : i = 1, 2, ..., r\}$ is semicompact. Then $\{x_n\}$ converges strongly to a common fixed point of T_i for $i = 1, 2, ..., r$.*

Corollary 3. *Let E be a nonempty, closed and convex subset of a uniformly convex Banach space X with the Opial property. Let $T_i : E \longrightarrow K(E)$, $(i = 1, 2, ..., r)$ be a finite family of multivalued nonexpansive mappings. Assume that $\mathcal{F} = \cap_{i=1}^{r} F(T_i) \neq \emptyset$ and $T_i(p) = \{p\}, (i = 1, 2, ..., r)$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be defined in (1.6), and $\alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., r$ and $\sum_{n=1}^{\infty} \beta_{in} < \infty$ for each i. Then $\{x_n\}$ converges weakly to a common fixed point of T_i for $i = 1, 2, ..., r$.*

Acknowledgement 1. *The author would like to express their thanks to the Reviewers and the Editors for their helpful suggestions and advices.*

References

[1] A. Abkar, M. Eslamian, Fixed point theorems for Suzuki generalized nonexpansive multivalued mappings in Banach spaces, Fixed Point Theory and Applications. 2010, Article ID 457935, 10 pp (2010).

[2] S.S. Chang, Y.K. Tang, L. W. Y.G. Xu, Y.H Zhao, G. Wang, Convergence theorems for some multi-valued generalized nonexpansive mappings, Fixed Point Theory and Applications. 2014, 2014:33.

[3] Y.J. Cho, H. Zhou, G. Gou, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Comput. Math. Appl., 47 (2004) 707-717.

[4] E.L. Dozo, Multivalued nonexpansive mappings and Opial's condition, Proc. Amer. Math. Soc., 38 (1973) 286-292.

[5] M. Eslamian, A. Abkar, One- step iterative process for a finite family of multivalued mappings, Math. Comput. Modell., 54 (2011) 105-111.

[6] M. Eslamian, S. Homaeipour, Strong convergence of three-step iterative process with erors for three multivalued mappings, arXiv.1105.2149v1 [math.FA] (2011).

[7] S. H. Khan, M Abbas, S. Ali, Fixed points of multivalued quasi-nonexpansive mappings using a faster iterative process, Acta Mathematica Sinica, English Series, July 2014, Vol. 30, Issue 7, pp1231-1241.

[8] S. H. Khan, I. Yildirim, Fixed points of multivalued nonexpansive mappings in Banach spaces, Fixed Point Theory and Applications 2012, 2012:73

[9] J. T. Markin, Continuous dependence of fixed point sets, Proc. Amer. Math. Soc., 38 (1973) 545-547.

[10] S. B. Nadler, Jr., Multivalued contraction mappings, Pacific J. Math., 30 (1969) 475-488.

[11] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc., 73 (1967) 591–597.

[12] B. Panyanak, Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces, Comp. Math. Appl., 54 (2007) 872-877.

[13] K. P. R. Sastry, G. V. R. Babu, Convergence of Ishikawa iterates for a multivalued mapping with a fixed point, Czechoslovak Math. J., 55 (2005) 817-826.
[14] Y. Song, Y. J. Cho, Some notes on Ishikawa iteration for mul-tivalued mappings. Bull Korean Math Soc 48(3):575–584(2011). doi:10.4134/BKMS.2011.48.3.575
[15] Y. Song, H. Wang, Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces" [Comp. Math. Appl., 54 (2007) 872-877]. Comp. Math. Appl., 55 (2008) 2999-3002.
[16] N. Shahzad, H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, Nonlinear Anal. 71 (2009) 838-844.
[17] T. Suzuki, Fixed point thoerems and convergence theorems for some generelized nonexpansive mappings, J. Math. Anal. Appl. 340 (2008) 1088-1095.
[18] K. K. Tan, H. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301-308.
[19] I. Yildirim, M. Ozdemir, Approximating Common Fixed Points of Asymptotically Quasi-Nonexpansive Mappings by a New Iterative Process, Arab J Sci Eng (2011) 36:393–403.
[20] I. Yildirim, M. Ozdemir, A new iterative process for common fixed points of finite families of non-self-asymptotically non-expansive mappings, Nonlinear Analysis, 71 (2009) 991-999.

Current address: Department of Mathematics, Ataturk University, Erzurum 25240, Turkey
E-mail address: isayildirim@atauni.edu.tr