Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.
Volume 66, Number 1, Pages 179–186 (2017)
DOI: 10.1501/Commua1_0000000787
ISSN 1303–5991

VECTOR-VALUED CESÀRO SUMMABLE GENERALIZED LORENTZ SEQUENCE SPACE

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Abstract. The main purpose of this paper is to introduce Cesàro summable generalized Lorentz sequence space $C_1[d(v,p)]$. We study some topologic properties of this space and obtain some inclusion relations.

1. Introduction

Throughout this work, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of positive integers, real numbers and complex numbers, respectively. For some properties of sequences, we refer to [4, 8].

For $1 \leq p < \infty$, the Cesàro sequence space is defined by

$$Ces_p = \left\{ x \in w : \sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^p < \infty \right\},$$

equipped with norm

$$\|x\| = \left(\sum_{j=1}^{\infty} \left(\frac{1}{j} \sum_{i=1}^{j} |x(i)| \right)^p \right)^{\frac{1}{p}}.$$

This space was first introduced by Shiue [14]. It is very useful in the theory of matrix operators and others. Later, many authors studied this space [see 1, 5, 11, 13].

Let $(E, \|\cdot\|)$ be a Banach space. The Lorentz sequence space $l(p,q,E)$ (*or* $l_{p,q}(E)$) for $1 \leq p, q \leq \infty$ is the collection of all sequences $\{a_i\} \in c_0(E)$ such that

$$\|\{a_i\}\|_{p,q} = \begin{cases} \left(\sum_{i=1}^{\infty} i^{q/p-1} \left\| a_{\phi(i)} \right\|^q \right)^{1/q} & for\ 1 \leq p < \infty,\ 1 \leq q < \infty \\ \sup_i i^{1/p} \left\| a_{\phi(i)} \right\| & for\ 1 \leq p \leq \infty,\ q = \infty \end{cases}$$

Received by the editors: March 18, 2016, Accepted: Aug 14, 2016.
2010 *Mathematics Subject Classification.* 40A05, 40H05, 46A45, 46E30.
Key words and phrases. Lorentz sequence space, Cesàro summable, vector-valued space.

is finite, where $\left\{\left\|a_{\phi(i)}\right\|\right\}$ is non-increasing rearrangement of $\{\|a_i\|\}$ (We can interpret that the decreasing rearrangement $\left\{\left\|a_{\phi(i)}\right\|\right\}$ is obtained by rearranging $\{\|a_i\|\}$ in decreasing order). This space was introduced by Miyazaki in [9] and examined comprehensively by Kato in [3] (see also [6, 7]).

A weight sequence $v = \{v(i)\}$ is a positive decreasing sequence such that $v(1) = 1$, $\lim_{i\to\infty} v(i) = 0$ and $\lim_{i\to\infty} V(i) = \infty$, where $V(i) = \sum\limits_{n=1}^{i} v(n)$ for every $i \in \mathbb{N}$. Popa [12] defined the generalized Lorentz sequence space $d(v, p)$ for $0 < p < \infty$ as follows

$$d(v,p) = \left\{ x = \{x_i\} \in w : \|x\|_{v,p} = \sup_{\pi} \left(\sum_{i=1}^{\infty} \left|x_{\pi(i)}\right|^p v(i) \right)^{1/p} < \infty \right\},$$

where π ranges over all permutations of the positive integers and $v = \{v(i)\}$ is a weight sequence. It is know that $d(v, p) \subset c_0$ and hence for each $x \in d(v, p)$ there exists a non-increasing rearrangement $\{x^*\} = \{x_i^*\}$ of x and

$$\|x\|_{v,p} = \left(\sum_{n=1}^{\infty} |x_i^*|^p v(i) \right)^{\frac{1}{p}}$$

(see [10, 12]).

Let $(X, \|\cdot\|)$ be a Banach space and $v = \{v(k)\}$ be a weight sequence. We introduce the vector-valued Cesáro summable generalized Lorentz sequence space $C_1[d(v, p)]$ for $0 < p < \infty$. The space $C_1[d(v, p)]$ is the collection of all $X-$valued $0-$sequences $\{x_n\}$ ($\{x_n\} \in c_0\{X\}$) such that

$$\left(\sum_{k=1}^{\infty} \left[\frac{1}{k} \sum_{n=1}^{k} \left\|x_{\phi(n)}\right\| \right]^p v(k) \right)^{\frac{1}{p}}$$

is finite, where $\left\{\left\|x_{\phi(n)}\right\|\right\}$ is non-increasing rearrangement of $\{\|x_n\|\}$.

We shall need the following lemmas.

Lemma 1. *(Hardy, Littlewood and Pólya [2]). Let $\{a_i\}_{1\le i\le n}$ and $\{b_i\}_{1\le i\le n}$ be two sequences of positive numbers. Then we have*

$$\sum_i a_i^* \cdot^* b_i \le \sum_i a_i \cdot b_i \le \sum_i a_i^* \cdot b_i^*,$$

where $\{a_i^*\}$ is the non-increasing rearrangements of sequence $\{a_i\}_{1\le i\le n}$ and $\{b_i^*\}$ and $\{^*b_i\}$ are the non-increasing and non-decreasing rearrangements of sequence $\{b_i\}_{1\le i\le n}$, respectively.

Lemma 2. *(Kato [3]) Let $\left\{x_i^{(\mu)}\right\}$ be an $X-$valued double sequence such that $\lim_{i\to\infty} x_i^{(\mu)} = 0$ for each $\mu \in \mathbb{N}$ and let $\{x_i\}$ be an $X-$valued sequence such that*

$\lim_{\mu\to\infty} x_i^{(\mu)} = x_i$ *(uniformly in i). Then* $\lim_{i\to\infty} x_i = 0$ *and for each* $i \in \mathbb{N}$

$$\left\|x_{\phi(i)}\right\| \leq \varliminf_{\mu\to\infty} \left\|x^{(\mu)}_{\phi_\mu(i)}\right\|,$$

where $\left\{\left\|x_{\phi(i)}\right\|\right\}$ *and* $\left\{\left\|x^{(\mu)}_{\phi_\mu(i)}\right\|\right\}_i$ *are the non-increasing rearrangements of* $\{\|x_i\|\}$ *and* $\left\{\left\|x_i^{(\mu)}\right\|\right\}_i$*, respectively.*

2. MAIN RESULTS

Theorem 1. *The space* $C_1\left[d(v,p)\right]$ *for* $0 < p < \infty$ *is a linear space over the field* $K = \mathbb{R}$ *or* $\mathbb{C}$.

Proof. Let $x, y \in C_1\left[d(v,p)\right]$. Since v is non-increasing, the non-increasing rearrangements of v is itself. Thus, using the inequality $\sum_i a_i \cdot b_i \leq \sum_i a_i^* \cdot b_i^*$ from Lemma 1, we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x_{\psi(n)}+y_{\psi(n)}\right\|\right]^p v(k) &\leq \sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left(\left\|x_{\psi(n)}\right\|+\left\|y_{\psi(n)}\right\|\right)\right]^p v(k)\\
&\leq D\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x_{\psi(n)}\right\|\right]^p v(k)\\
&\quad +D\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|y_{\psi(n)}\right\|\right]^p v(k)\\
&\leq D\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x_{\phi(n)}\right\|\right]^p v(k)\\
&\quad +D\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|y_{\eta(n)}\right\|\right]^p v(k)\\
&< \infty,
\end{aligned}$$

where $D = \max\left\{1, 2^{p-1}\right\}$. Here $\left\{\left\|x_{\phi(n)}\right\|\right\}$, $\left\{\left\|y_{\eta(n)}\right\|\right\}$ and $\left\{\left\|x_{\psi(n)}+y_{\psi(n)}\right\|\right\}$ denote the non-increasing rearrangements of the sequences $\{\|x_n\|\}$, $\{\|y_n\|\}$ and $\{\|x_n + y_n\|\}$, respectively. Let $\alpha \in K$. Hence we get

$$\begin{aligned}
\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|\alpha x_{\phi(n)}\right\|\right]^p v(k) &= \sum_{k=1}^{\infty}\left[\frac{|\alpha|}{k}\sum_{n=1}^{k}\left\|x_{\phi(n)}\right\|\right]^p v(k)\\
&= |\alpha|^p\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x_{\phi(n)}\right\|\right]^p v(k)\\
&< \infty.
\end{aligned}$$

This shows that $x+y\in C_1\left[d(v,p)\right]$, $\alpha x\in C_1\left[d(v,p)\right]$ and so $C_1\left[d(v,p)\right]$ is a linear space. □

Theorem 2. *The space $C_1\left[d(v,p)\right]$ for $1\le p<\infty$ is normed space with the norm*

$$\|x\|_{C,v,p}=\left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x_{\phi(n)}\right\|\right]^{p}v(k)\right)^{\frac{1}{p}},$$

where $\left\{\left\|x_{\phi(n)}\right\|\right\}$ denotes the non-increasing rearrangements of $\{\|x_n\|\}$.

Proof. It is clear that $\|0\|_{C,v,p}=0$. Let $\|x\|_{C,v,p}=0$. Then we have $\frac{1}{k}\sum\limits_{n=1}^{k}\left\|x_{\phi(n)}\right\|=0$ for all $k\in\mathbb{N}$. Hence we get $\left\|x_{\phi(n)}\right\|=0$ for all $n\in\mathbb{N}$ and so $x=0$.

Let $x,y\in C_1\left[d(v,p)\right]$. Since weight sequence v is decreasing, the non-increasing rearrangements of v is itself. Thus, using the inequality $\sum\limits_{i}a_i\cdot b_i\le\sum\limits_{i}a_i^*\cdot b_i^*$ from Lemma 1, we have

$$\begin{aligned}\|x+y\|_{C,v,p} &= \left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x_{\psi(n)}+y_{\psi(n)}\right\|\right]^{p}v(k)\right)^{\frac{1}{p}}\\ &\le \left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x_{\psi(n)}\right\|\right]^{p}v(k)\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|y_{\psi(n)}\right\|\right]^{p}v(k)\right)^{\frac{1}{p}}\\ &\le \left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x_{\phi(n)}\right\|\right]^{p}v(k)\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|y_{\eta(n)}\right\|\right]^{p}v(k)\right)^{\frac{1}{p}}\\ &= \|x\|_{C,v,p}+\|y\|_{C,v,p}\ ,\end{aligned}$$

where $\left\{\left\|x_{\phi(n)}\right\|\right\}$, $\left\{\left\|y_{\eta(n)}\right\|\right\}$ and $\left\{\left\|x_{\psi(n)}+y_{\psi(n)}\right\|\right\}$ denote the non-increasing rearrangements of $\{\|x_n\|\}$, $\{\|y_n\|\}$ and $\{\|x_n+y_n\|\}$, respectively.

Let λ be an element in K and let x be a vector in $C_1\left[d(v,p)\right]$. Hence we have

$$\begin{aligned}\|\lambda x\|_{C,v,p} &= \left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|\lambda x_{\phi(n)}\right\|\right]^{p}v(k)\right)^{\frac{1}{p}}\\ &= |\lambda|\left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x_{\phi(n)}\right\|\right]^{p}v(k)\right)^{\frac{1}{p}}\\ &= |\lambda|\,\|x\|_{C,v,p}\,.\end{aligned}$$

□

Theorem 3. *The space $C_1\left[d(v,p)\right]$ for $1\le p<\infty$ is complete with respect to its norm.*

Proof. Let $\left\{x^{(s)}\right\}$ be an arbitrary Cauchy sequence in $C_1\left[d(v,p)\right]$ with $x^{(s)} = \left\{x_n^{(s)}\right\}_{n=1}^{\infty}$ for all $s \in \mathbb{N}$. Then we have

$$\lim_{s,t\to\infty}\left\|x^{(s)}-x^{(t)}\right\|_{C,v,p} = \lim_{s,t\to\infty}\left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x^{(s)}_{\pi_{s,t}(n)}-x^{(t)}_{\pi_{s,t}(n)}\right\|\right]^p v(k)\right)^{\frac{1}{p}} = 0, \tag{1}$$

where $\left\{\left\|x^{(s)}_{\pi_{s,t}(n)}-x^{(t)}_{\pi_{s,t}(n)}\right\|\right\}$ denotes the non-increasing rearrangement of $\left\{\left\|x_n^{(s)}-x_n^{(t)}\right\|\right\}$. Hence we obtain $\lim_{s,t\to\infty}\left\|x^{(s)}_{\pi_{s,t}(n)}-x^{(t)}_{\pi_{s,t}(n)}\right\| = 0$ for each $n\in\mathbb{N}$ and so $\left\{x_n^{(s)}\right\}$, for a fixed $n \in \mathbb{N}$, is a Cauchy sequence in X.

Then, there exists $x_n \in X$ such that $x_n^{(s)} \to x_n$ as $s \to \infty$. Let $x = \{x_n\}$. Since $\lim_{n\to\infty} x_n^{(s)} = 0$ for each $s \in \mathbb{N}$, by Lemma 2 we have $\lim_{n\to\infty} x_n = 0$. Therefore we can choose the non-increasing rearrangement $\left\{\left\|x_{\pi_t(n)}-x^{(t)}_{\pi_t(n)}\right\|\right\}_n$ of $\left\{\left\|x_n-x_n^{(t)}\right\|\right\}_n$. Also, for an arbitrary $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x^{(s)}_{\pi_{s,t}(n)}-x^{(t)}_{\pi_{s,t}(n)}\right\|\right]^p v(k)\right)^{\frac{1}{p}} < \varepsilon \tag{2}$$

for $s,t > N$. Let t be an arbitrary positive integer with $t > N$ and fixed. If we put

$$y_n^{(s)} = x_n^{(s)} - x_n^{(t)} \quad \text{and} \quad y_n = x_n - x_n^{(t)}\ ,$$

then we have

$$\lim_{n\to\infty} y_n^{(s)} = 0 \text{ for each } s\in\mathbb{N} \text{ and } \lim_{s\to\infty} y_n^{(s)} = y_n \text{ (uniformly in } n).$$

Thus by Lemma 2 we get

$$\left\|y_{\phi(n)}\right\| \le \varliminf_{s\to\infty}\left\|y^{(s)}_{\phi_s(n)}\right\|$$

for each $n \in \mathbb{N}$, that is,

$$\left\|x_{\pi_t(n)}-x^{(t)}_{\pi_t(n)}\right\| \le \varliminf_{s\to\infty}\left\|x^{(s)}_{\pi_{s,t}(n)}-x^{(t)}_{\pi_{s,t}(n)}\right\| \tag{3}$$

for each $n \in \mathbb{N}$. Hence, by (2), (3) we get

$$
\begin{aligned}
\left\|x-x^{(t)}\right\|_{C,v,p} &= \left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x_{\pi_t(n)}-x^{(t)}_{\pi_t(n)}\right\|\right]^p v(k)\right)^{\frac{1}{p}} \\
&\le \left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\lim_{s\to\infty}\left\|x^{(s)}_{\pi_{s,t}(n)}-x^{(t)}_{\pi_{s,t}(n)}\right\|\right]^p v(k)\right)^{\frac{1}{p}} \\
&= \lim_{s\to\infty}\left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x^{(s)}_{\pi_{s,t}(n)}-x^{(t)}_{\pi_{s,t}(n)}\right\|\right]^p v(k)\right)^{\frac{1}{p}} \\
&< \varepsilon.
\end{aligned}
$$

Also, since $C_1\left[d(v,p)\right]$ is a linear space we have $\{x_n\} = \left\{x_n - x_n^{(N)}\right\} + \left\{x_n^{(N)}\right\} \in C_1\left[d(v,p)\right]$. Hence the space $C_1\left[d(v,p)\right]$ is complete with respect to its norm. □

Theorem 4. *Let $1 < p < \infty$. Then, the inclusion $d(v,p) \subset C_1\left[d(v,p)\right]$ holds.*

Proof. Let $x \in d(v,p)$. Then there exists $T > 0$ such that

$$
\lim_{m\to\infty}\left(\sum_{n=1}^{m}\left\|x_{\phi(n)}\right\|^p v(n)\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty}\left\|x_{\phi(n)}\right\|^p v(n)\right)^{\frac{1}{p}} \le T < \infty,
$$

where $\left\{\left\|x_{\phi(n)}\right\|\right\}$ denotes the non-increasing rearrangements of $\{\|x_n\|\}$. Since $\sum\limits_{k=1}^{\infty}\frac{1}{k^p} < \infty$ for $1 < p < \infty$ and v is decreasing, we get

$$
\begin{aligned}
\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\left\|x_{\phi(n)}\right\|\right]^p v(k) &= \sum_{k=1}^{\infty}\frac{1}{k^p}\left[\sum_{n=1}^{k}\left\|x_{\phi(n)}\right\|\right]^p v(k) \\
&\le \max\left\{1,2^{p-1}\right\}\sum_{k=1}^{\infty}\frac{1}{k^p}\left[\sum_{n=1}^{k}\left\|x_{\phi(n)}\right\|^p v(n)\right] \\
&\le T\cdot\max\left\{1,2^{p-1}\right\}\sum_{k=1}^{\infty}\frac{1}{k^p} \\
&< \infty.
\end{aligned}
$$

This completes the proof. □

Theorem 5. *If $1 \le p < q < \infty$, then $C_1\left[d(v,p)\right] \subset C_1\left[d(v,q)\right]$.*

Proof. Let $x \in C_1[d(v,p)]$ and let $\{\|x_{\phi(n)}\|\}$ denotes the non-increasing rearrangement of $\{\|x_n\|\}$. Since $v(k)$ is decreasing we have

$$\begin{aligned}\left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\|x_{\phi(n)}\|\right]^p v(k)\right)^{\frac{1}{p}} &\geq \left(\sum_{k=1}^{m}\left[\frac{1}{k}\sum_{n=1}^{k}\|x_{\phi(n)}\|\right]^p v(k)\right)^{\frac{1}{p}} \\ &\geq \|x_{\phi(m)}\|\left(\sum_{k=1}^{m} v(k)\right)^{\frac{1}{p}} \\ &\geq \|x_{\phi(m)}\|\,(v(m))^{\frac{1}{p}}\, m^{\frac{1}{p}}\end{aligned}$$

for every $m \in \mathbb{N}$. Hence we get

$$\begin{aligned}\|x_{\phi(m)}\| &\leq (v(m))^{-\frac{1}{p}}\, m^{-\frac{1}{p}}\left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\|x_{\phi(n)}\|\right]^p v(k)\right)^{\frac{1}{p}} \\ &\leq (v(m))^{-\frac{1}{p}}\,\|x\|_{C,v,p}\end{aligned}$$

for every $m \in \mathbb{N}$. Thus

$$\begin{aligned}\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\|x_{\phi(n)}\|\right]^q v(k) &= \sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\|x_{\phi(n)}\|\right]^{q-p}\left[\frac{1}{k}\sum_{n=1}^{k}\|x_{\phi(n)}\|\right]^p v(k) \\ &\leq \sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}(v(n))^{-\frac{1}{p}}\|x\|_{C,v,p}\right]^{q-p}\left[\frac{1}{k}\sum_{n=1}^{k}\|x_{\phi(n)}\|\right]^p v(k) \\ &\leq \left((v(n))^{-\frac{1}{p}}\|x\|_{C,v,p}\right)^{q-p}\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\|x_{\phi(n)}\|\right]^p v(k) \\ &< \infty.\end{aligned}$$

This implies that $x \in C_1[d(v,q)]$. □

Comment. If we put $\triangle^m x$ instead of x, where $m \in \mathbb{N}$ and $\triangle^0 x_k = \{x_k\}$, $\triangle x_k = x_k - x_{k+1}$, $\triangle^m x_k = \triangle^{m-1}x_k - \triangle^{m-1}x_{k+1} = \sum\limits_{v=1}^{m}(-1)^v\binom{m}{v}x_{k+v}$ for all $k \in \mathbb{N}$ in the definition of $C_1[d(v,p)]$, we obtain Cesàro summable generalized Lorentz difference sequence space $C_1[d(v,\triangle^m,p)]$ of order m. It can be shown that the sequence space $C_1[d(v,\triangle^m,p)]$ is a Banach space with norm

$$\|x\|_{C,v,\triangle^m,p} = \sum_{k=1}^{m}\|x_{\phi(k)}\| + \left(\sum_{k=1}^{\infty}\left[\frac{1}{k}\sum_{n=1}^{k}\|\triangle^m x_{\phi(n)}\|\right]^p v(k)\right)^{\frac{1}{p}},$$

where $\{\|\triangle^m x_{\phi(n)}\|\}$ denotes the non-increasing rearrangements of $\{\|\triangle^m x_n\|\}$, and properties in this work.

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