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ESTIMATE FOR INITIAL MACLAURIN COEFFICIENTS OF SUBCLASS OF BI-UNIVALENT FUNCTIONS INVOLVING THE Q- DERIVATIVE OPERATOR

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Abstract. In this paper, estimates for second and third MacLaurin coefficients of a new subclass of analytic and bi-univalent functions in the open unit disk are determined, and certain special cases are also indicated.

1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The Koebe one-quarter theorem [3] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains the disk with the center in the origin and the radius $1/4$. Thus, every univalent function $f \in \mathcal{A}$ has an inverse $f^{-1} : f(\mathbb{D}) \to \mathbb{D}$, satisfying $f^{-1}(f(z)) = z$, $z \in \mathbb{D}$, and

$$f\left(f^{-1}(w)\right) = w, \ \left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right).$$

Moreover, it is easy to see that the inverse function has the series expansion of the form

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \dots, \ (w \in f(\mathbb{D}). \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{D}$, if both f and f^{-1} are univalent in $\mathbb{D}$, in the sense that f^{-1} has a univalent analytic continuation to $\mathbb{D}$, and we denote by σ this class of bi-univalent functions. For a brief history and interesting examples of functions in the class σ, see [11] (see also [2]). In fact, the aforecited work of

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Srivastava et al. [11] essentially revived the investigation of various subclasses of the bi-univalent function class σ in recent years; it was followed by such works as those by Frasin and Aouf [4], Goyal and Goswami [5], Xu et al.[12, 13] (see also the references cited in each of them).

In [9], the authors defined the classes of functions $\mathcal{P}_m(\beta)$ as follows:
Let $\mathcal{P}_m(\beta)$, with $m \geq 2$ and $0 \leq \beta < 1$, denote the class of univalent analytic functions p, normalized with $p(0) = 1$, and satisfying

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \beta}{1 - \beta} \right| \mathrm{d}\,\theta \leq m\pi,$$

where $z = re^{i\theta} \in \mathbb{D}$.

For $\beta = 0$, we denote $\mathcal{P}_m := \mathcal{P}_m(0)$. Paatero [8] showed that every function $p \in \mathcal{P}_m$ can be given by the Stieltjes integral representation

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{it}}{1 - ze^{it}}\, \mathrm{d}\,\mu(t), \tag{1.3}$$

where $\mu(t)$ is a real-valued function with bounded variation on $[0,2\pi]$, which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m\pi,\ m \geq 2. \tag{1.4}$$

Clearly, $\mathcal{P} := \mathcal{P}_2$ is the well-known class of *Carathéodory functions*, i.e. the normalized functions with positive real part in the open unit disk $\mathbb{D}$.

Quantum calculus is ordinary classical calculus without the notion of limits. It defines q-calculus and h-calculus. Here h ostensibly stands for Planck's constant, while q stands for quantum. Recently, the area of q-calculus has attracted the series attention of researchers. This great interest is due to its application in various branches of mathematics and physics. The application of q-calculus was initiated by Jackson [6, 7]. He was the first to develop q-integral and q-derivative in a systematic way. Later, geometrical interpretation of q-analysis has been recognized through studies on quantum groups. It also suggests a relation between integrable systems and q-analysis. A comprehensive study on applications of q-calculus in operator theory may be found in [1]. For a function $f \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, the q- derivative of function f is defined by (see [6, 7])

$$D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)},\ z \neq 0, \tag{1.5}$$

$D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (1.5), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{1.6}$$

where
$$[k]_q = \frac{1-q^k}{1-q} \tag{1.7}$$
As $q \to 1^-$, $[k]_q \to k$. For a function $g(z) = z^k$, we get
$$D_q f(z) = [k]_q z^{k-1}$$
$$lim_{q\to 1^-}(D_q(z^k)) = kz^{k-1} = g'(z)$$
where g' is the ordinary derivative.
By making use of the q-derivative of a function $f \in \mathcal{A}$, we introduce a new subclass of the function class σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass of the function class σ.

Definition 1.1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{BR}_\sigma^q(m;\beta;\lambda)$, with $m \geq 2,\ \lambda \geq 1$,
$q \in (0,1)$ and $0 \leq \beta < 1$, if the following conditions are satisfied*
$$(1-\lambda)\frac{f(z)}{z} + \lambda D_q f(z) \in \mathcal{P}_m(\beta),$$
$$(1-\lambda)\frac{g(w)}{w} + \lambda D_q g(w) \in \mathcal{P}_m(\beta),$$

where $g = f^{-1}$ is given by (1.2) and $z, w \in \mathbb{D}$.

2. Main results

In order to prove our main result for the functions $f \in \mathcal{BR}_\sigma^q(m;\beta;\lambda)$, we need the following lemma:

Lemma 2.1. *Let the function $\Phi(z) = 1 + \sum\limits_{n=1}^{\infty} h_n z^n$, $z \in \mathbb{D}$, such that $\Phi \in \mathcal{P}_m(\beta)$. Then,*
$$|h_n| \leq m(1-\beta),\ n \geq 1.$$

Proof. Proof of this lemma is straight forward, if we write
$\Phi(z) = (1-\beta)p(z) + \beta, p(z) = 1 + \sum\limits_{n=1}^{\infty} p_n z^n \in \mathcal{P}_m$
Then $\Phi(z) = 1 + (1-\beta)\sum\limits_{n=1}^{\infty} p_n z^n$
This gives
$$h_n = (1-\beta)p_n.$$
Using known result [10] for class P_m, we have our result. □

Theorem 2.1. *Let the function f given by (1.1) be in the class $\mathcal{BR}_\sigma^q(m;\beta;\lambda)$. Then*
$$|a_2| \leq \min\left\{\sqrt{\frac{m(1-\beta)}{1-\lambda+\lambda[3]_q}};\ \frac{m(1-\beta)}{1-\lambda+\lambda[2]_q}\right\},$$

$$|a_3| \leq \frac{m(1-\beta)}{1-\lambda+\lambda[3]_q},$$

and

$$\left|2a_2^2 - a_3\right| \leq \frac{m(1-\beta)}{1-\lambda+\lambda[3]_q}.$$

Proof. Since $\mathcal{BR}_\sigma^q(m;\beta;\lambda)$, from the Definition 1.1 we have

$$(1-\lambda)\frac{f(z)}{z} + \lambda D_q f(z) = \varphi(z), \tag{2.1}$$

and

$$(1-\lambda)\frac{g(w)}{w} + \lambda D_q g(w) = \psi(w), \tag{2.2}$$

where $\varphi,\ \psi \in \mathcal{P}_m(\beta)$ and $g = f^{-1}$ is given by (1.2). Using the fact that the functions φ and ψ have the following Taylor expansions

$$\varphi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots,\ z \in \mathbb{D}, \tag{2.3}$$

$$\psi(w) = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \dots,\ w \in \mathbb{D}, \tag{2.4}$$

and equating the coefficients in (2.1) and (2.2), from (1.2) we get

$$(1-\lambda+\lambda[2]_q)a_2 = c_1, \tag{2.5}$$

$$(1-\lambda+\lambda[3]_q)a_3 = c_2, \tag{2.6}$$

$$-(1-\lambda+\lambda[2]_q)a_2 = d_1, \tag{2.7}$$

and

$$(1-\lambda+\lambda[3]_q)\left(2a_2^2 - a_3\right) = d_2. \tag{2.8}$$

Since $\varphi,\ \psi \in \mathcal{P}_m(\beta)$, according to Lemma 2.1, we have:

$$|c_n| \leq m(1-\beta), \tag{2.9}$$

$$|d_n| \leq m(1-\beta), \tag{2.10}$$

for $n \geq 1$ and thus, from (2.6) and (2.8), by using the inequalities (2.9) and (2.10), we obtain

$$|a_2|^2 \leq \frac{|c_2| + |d_2|}{2(1-\lambda+\lambda[3]_q)} \leq \frac{m(1-\beta)}{(1-\lambda+\lambda[3]_q)},$$

which gives

$$|a_2| \leq \sqrt{\frac{m(1-\beta)}{1-\lambda+\lambda[3]_q)}}. \tag{2.11}$$

From (2.5), by using (2.9) we obtain immediately that

$$|a_2| = \left|\frac{c_1}{1-\lambda+\lambda[2]_q}\right| \leq \frac{m(1-\beta)}{1-\lambda+\lambda[2]_q},$$

and combining this with the inequality (2.11), the first inequality of the conclusion is proved. According to (2.6), from (2.9) we easily obtain

$$|a_3| = \left|\frac{c_2}{1-\lambda+\lambda[3]_q}\right| \leq \frac{m(1-\beta)}{1-\lambda+\lambda[3]_q},$$

and from (2.8), by using (2.9) and (2.10) we finally deduce

$$\left|2a_2^2 - a_3\right| = \left|\frac{d_2}{1-\lambda+\lambda[3]_q}\right| \leq \frac{m(1-\beta)}{1-\lambda+\lambda[3]_q},$$

which completes our proof. □

Setting $\lambda = 1$ in Theorem 2.1 we obtain the following result:

Corollary 2.1. *Let the function f given by (1.1) be in the class $\mathcal{BR}_\sigma^q(m;\beta;1)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{m(1-\beta)}{[3]_q}};\ \frac{m(1-\beta)}{[2]_q}\right\},$$

$$|a_3| \leq \frac{m(1-\beta)}{[3]_q},$$

and

$$\left|2a_2^2 - a_3\right| \leq \frac{m(1-\beta)}{[3]_q}.$$

Taking $q \to 1^-$ in Theorem 2.1, we obtain the following result:

Corollary 2.2. *Let the function f given by (1.1) be in the class $\mathcal{BR}_\sigma^{'}(m;\beta;\lambda)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{m(1-\beta)}{1+2\lambda}};\ \frac{m(1-\beta)}{1+\lambda}\right\}$$

,

$$|a_3| \leq \frac{m(1-\beta)}{1+2\lambda},$$

and

$$\left|2a_2^2 - a_3\right| \leq \frac{m(1-\beta)}{1+2\lambda}.$$

Setting $\beta = 0$ in Theorem 2.1 we obtain the following result:

Corollary 2.3. *Let the function f given by (1.1) be in the class $\mathcal{BR}_\sigma^q(m;0;\lambda)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{m}{1-\lambda+\lambda[3]_q}};\ \frac{m}{1-\lambda+\lambda[2]_q}\right\},$$

$$|a_3| \leq \frac{m}{1-\lambda+\lambda[3]_q},$$

and

$$\left|2a_2^2 - a_3\right| \leq \frac{m}{1-\lambda+\lambda[3]_q}.$$

Setting $\beta = 0, \lambda = 1$ in Theorem 2.1 we obtain the following result:

Corollary 2.4. *Let the function f given by (1.1) be in the class $\mathcal{BR}_\sigma^q(m;0;1)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{m}{[3]_q}};\ \frac{m}{[2]_q}\right\},$$

$$|a_3| \leq \frac{m}{[3]_q},$$

and

$$\left|2a_2^2 - a_3\right| \leq \frac{m}{[3]_q}.$$

Setting $\beta = 0, \lambda = 1, q \to 1^-$ in Theorem 2.1 we obtain the following result:

Corollary 2.5. *Let the function f given by (1.1) be in the class $\mathcal{BR}_\sigma^{'}(m;0;1)$. Then*

$$|a_2| \leq \sqrt{\frac{m}{3}},$$

$$|a_3| \leq \frac{m}{3},$$

and

$$\left|2a_2^2 - a_3\right| \leq \frac{m}{3}.$$

Competing interest. The authors declare that they have no competing interests.

Author's contribution. We further declare that all authors contribute equally.

References

[1] A. Aral, V. Gupta and R.P.Agrawal, Applications of q-calculus in Operator Theory, Springer, New York, USA, 2013.

[2] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, Stud. Univ. Babeş-Bolyai Math., **31**(1986), no. 2, 70–77.

[3] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften Series, 259, Springer Verlag, New York, 1983.

[4] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, Appl. Math. Lett., **24**(2011), no. 9, 1569–1573.

[5] S. P. Goyal and P. Goswami, Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives, J. Egyptian Math. Soc., **20**(3)(2012), 179–182.

[6] F. H. Jackson, On q-definite integrals, Quarterly J. Pure Appl. Math. **41** (1910), 193-203.

[7] F. H. Jackson, On q-functions and a certain difference operator, Transactions of the Royal Society of Edinburgh **46** (1908), 253-281.

[8] V. Paatero, Uber die konforme Abbildung von Gebieten deren Rander von beschrÃ¤nkter Drehung sind, Ann. Acad. Sci. Fenn. Ser. A33 No. 9 (1931).

[9] K. Padmanabhan and R. Parvatham, Properties of a class of functions with bounded boundary rotation, Ann. Polon. Math., 31(1975), 311–323.

[10] B. Pinchuk, Functions with bounded boundary rotation, Israel J. Math., **10**(1971), 7–16.

[11] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., **23**(2010), no. 10, 1188–1192.
[12] Q.-H. Xu, Y.-C. Gui and H.M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett.**25** (2012) 990-994.
[13] Q.-H. Xu, H.-G. Xiao and H.M. Srivastava, A certain general subclass of analytic and biunivalent functions and associated coefficient estimate problems, Appl. Math. Comput. **218** (2012) 11461165.

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