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ON THE GENERALIZED PERRIN AND CORDONNIER MATRICES

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Abstract. In the present paper, we study the associated polynomials of Perrin and Cordonnier numbers. We define generalized Perrin and Cordonnier matrices using these polynomials. We obtain the inverse of generalized Cordonnier matrices and give some relationships between generalized Perrin and Cordonnier matrices. In addition, we give a factorization of generalized Cordonnier matrices. Finally, we give some determinantal representation of associated polynomials Cordonnier numbers.

1. Introduction

There are several hundreds of papers on Fibonacci numbers and other recurrence related sequences published during the last 30 years. Perrin numbers and Cordonnier numbers are some of them. Perrin numbers and Cordonnier numbers are

$$P_n = P_{n-2} + P_{n-3} \text{ for } n > 3 \text{ and } P_1 = 0,\ P_2 = 2,\ P_3 = 3,$$
$$C_n = C_{n-2} + C_{n-3} \text{ for } n > 3 \text{ and } C_1 = 1,\ C_2 = 1,\ C_3 = 1,$$

respectively.

The characteristic equation associated with the Perrin and Cordonnier sequence is $x^3 - x - 1 = 0$ with roots $\alpha, \beta, \overline{\beta}$, in which $\alpha = \rho \approx 1,324718$, is called plastic number and

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \rho.$$

The plastic number is used in art and architecture. Richard Padovan studied on plastic number in Architecture and Mathematics in [20, 21]. Christopher Bartlett found a significant number of paintings with canvas sizes that have the aspect ratio of approximately 1.35. This ratio reminds Plastic number[1]. In [17] authors constructed the Plastic number in a heuristic way, explaining its relation to human

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perception in three-dimensional space through architectural style of Dom Hans van der Laan.

In [22], authors defined associated polynomials of Perrin and Cordonnier sequences as;

$$P_n(x) = x^2 P_{n-2}(x) + P_{n-3}(x) \text{ for } n > 3 \text{ and } P_1(x) = 0,\ P_2(x) = 2,\ P_3(x) = 3x,$$

$$C_n(x) = x^2 C_{n-2}(x) + C_{n-3}(x) \text{ for } n > 3 \text{ and } C_1(x) = 1,\ C_2(x) = x,\ C_3(x) = x^2,$$

respectively.

If we take $x = 1$ we obtain $P_n(x) = P_n$ and $C_n(x) = C_n$.

Yilmaz and Taskara [26] developed the matrix sequences that represent Padovan and Perrin numbers. Kaygisiz and Bozkurt [5] defined k sequences of generalized order-k Perrrin numbers. Kaygisiz and Sahin [9] defined generalized Van der Laan and Perrin Polynomials, and generalizations of Van der Laan and Perrin Numbers.

Many researchers have studied Linear Algebra of the some matrices. In [2] authors discussed the Linear Algebra of the Pascal Matrix, in [14] authors examined the linear algebra of the k-Fibonacci matrix and the symmetric k-Fibonacci matrix. In [12] authors studied on the Pell Matrix. Sahin [23] gave the (q, x, s)-Fibonacci and Lucas matrices, obtained the inverse of these matrices and give some factorization of these matrices. Lee et al. [13] defined Fibonacci matrices and gave the factorization of Fibonacci matrix and obtained inverse of this matrix.

In addition many researchers have studied matrix representations of number sequences. Yilmaz and Bozkurt [25] gave matrix representation of Perrin sequences. Kaygisiz and Sahin [10] calculated terms of associated polynomials of Perrin and Cordonnier numbers by using determinants and permanents of various Hessenberg matrices. More examples can be found in [3, 6, 7, 8, 11, 15, 16, 19, 23].

Lemma 1.1. *(Cf. Theorem of [3]) Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and define $\det(A_0) = 1$. Then, $\det(A_1) = a_{11}$ and for $n \geq 2$*

$$\det(A_n) = a_{n,n} \det(A_{n-1}) + \sum_{r=1}^{n-1} [(-1)^{n-r} a_{n,r} (\prod_{j=r}^{n-1} a_{j,j+1}) \det(A_{r-1})].$$

In this paper, first we define generalized Perrin and Cordonnier matrices using associated polynomials of Perrin and Cordonnier numbers. We obtain the inverse of generalized Cordonnier matrices with the aid of determinants of some Hessenberg matrices which obtained from a part of these matrices. We also give some relationships between generalized Perrin and Cordonnier matrices in this section. Secondly, we give a factorization of generalized Cordonnier matrices. In the last section we give some determinantal representation of associated polynomials Cordonnier numbers.

2. Generalized Perrin and Cordonnier matrices

Definition 2.1. Let n be any positive integer, the $n\times n$ lower triangular generalized Cordonnier matrix $\mathcal{C}_{n,x}=[c_{i,j}]_{i,j=1,2,\ldots,n}$ are defined by

$$c_{i,j}=\begin{cases} C_{i-j+1}(x), & \text{if } i-j\geqslant 0\\ 0, & \text{otherwise.}\end{cases} \tag{1}$$

For example,

$$\mathcal{C}_{4,x}=\begin{bmatrix} 1 & 0 & 0 & 0\\ x & 1 & 0 & 0\\ x^2 & x & 1 & 0\\ 1+x^3 & x^2 & x & 1\end{bmatrix}.$$

Definition 2.2. Let n be any positive integer, the $n\times n$ lower triangular generalized Perrin matrix $\mathcal{P}_{n,x}=[p_{i,j}]_{i,j=1,2,\ldots,n}$ are defined by

$$p_{i,j}=\begin{cases} P_{i-j+2}(x), & \text{if } i-j\geqslant 0\\ 0, & \text{otherwise.}\end{cases} \tag{2}$$

For example,

$$\mathcal{P}_{4,x}=\begin{bmatrix} 2 & 0 & 0 & 0\\ 3x & 2 & 0 & 0\\ 2x^2 & 3x & 2 & 0\\ 2+3x^3 & 2x^2 & 3x & 2\end{bmatrix}.$$

Definition 2.3. Let n be any positive integer, the $n\times n$ lower Hessenberg matrix sequence ${}_HC_{n,x}=[a_{i,j}]_{i,j=1,2,\ldots,n}$ are defined by

$$a_{i,j}=\begin{cases} C_{i-j+2}(x), & \text{if } i-j+1\geq 0\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

Lemma 2.4. *Let* $c_0(x)=1$, $c_1(x)=x$, $c_{n+1}(x)=xc_n(x)+\sum_{k=1}^{n}(-1)^{n-k+1}C_{n-k+3}(x)c_{k-1}(x)$. *Then,* $\det({}_HC_{n,x})=c_n(x)$ *for any positive integer* $n\geq 1$.

Proof. We proceed by induction on n. The result clearly holds for $n=1$. Now suppose that the result is true for all positive integers less than or equal to n. We prove it for $n+1$. In fact, by using Lemma 1.1 we have

$$\begin{aligned}\det({}_HC_{n+1,x}) &= x\det({}_HC_{n,x})+\sum_{i=1}^{n}\left[(-1)^{n+1-i}a_{n+1,i}\prod_{j=i}^{n}a_{j,j+1}\det({}_HC_{i-1,x})\right]\\ &= x\det({}_HC_{n,x})+\sum_{i=1}^{n}\left[(-1)^{n+1-i}C_{n-i+3}(x)\det({}_HC_{i-1,x})\right].\end{aligned}$$

From the hypothesis of induction, we obtain

$$\det({}_HC_{n+1,x})=xc_n(x)+\sum_{i=1}^{n}\left[(-1)^{n+1-i}C_{n-i+3}(x)c_{i-1}(x)\right].$$

Therefore, $\det({}_HC_{n,x}) = c_n(x)$ holds for all positive integers n. $\square$

Example 2.5. We obtain $c_3(x), c_4(x)$ by using Lemma 2.4.

$$\det\begin{bmatrix} x & 1 & 0 \\ x^2 & x & 1 \\ 1+x^3 & x^2 & x \end{bmatrix} = 1 = c_3(x), \det\begin{bmatrix} x & 1 & 0 & 0 \\ x^2 & x & 1 & 0 \\ 1+x^3 & x^2 & x & 1 \\ x+x^4 & 1+x^3 & x^2 & x \end{bmatrix} = x = c_4(x).$$

Corollary 2.6. *Let $({}_HC_{n,x})$ be the $n\times n$ Hessenberg matrix in (3). Then,* $\det({}_HC_{n,x}) = c_n(x) = x^{n-3}$ *for any positive integer* $n \geq 3$.

Proof. We proceed by induction on n. The result clearly holds for $n = 3$. Now suppose that the result is true for all positive integers less than or equal to n. We prove it for $n+1$. It is clear by the Laplace expansion of the last column that,

$$\begin{aligned} \det({}_HC_{n+1,x}) &= x\det(M_{n,n}) - \det(M_{n-1,n}) \\ &= x\det({}_HC_{n,x}) - \det(M_{n-1,n}) \\ &= xx^{n-3} - \det(M_{n-1,n}) \end{aligned}$$

and since nth row of $M_{n-1,n}$ is equal $x^2((n-1)\text{th row of } M_{n-1,n}) + ((n-2)\text{th row of } M_{n-1,n})$, $\det(M_{n-1,n}) = 0$, where $M_{i,j}$ is the (i,j) minor matrix of ${}_HC_{n,x}$. So we obtain

$$\det({}_HC_{n+1,x}) = x^{n-2}.$$

$\square$

Theorem 2.7. *Let n be any positive integer, $\mathcal{C}_{n,x}$ is $n\times n$ lower triangular generalized Cordonnier matrix in (1) and $({}_HC_{n,x})$ be the $n\times$ n Hessenberg matrix in (3). Then $(\mathcal{C}_{n,x})^{-1} = [c^{\shortmid}_{i,j}]$ is obtained by*

$$[c^{\shortmid}_{i,j}] = \begin{cases} (-1)^{i-j}\det({}_HC_{i-j,x}), & \text{if } i-j>0 \\ 1, & \text{if } i-j=0 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Note that it suffices to prove that $\mathcal{C}_{n,x}(\mathcal{C}_{n,x})^{-1} = I_n$. We take $\mathcal{C}_{n,x}(\mathcal{C}_{n,x})^{-1} = [a_{i,j}]_{1\leq i,j\leq n}$. It is obvious that $a_{i,j} = \sum_{k=0}^{n} c_{i,k}c^{\shortmid}_{k,j} = 0$ for $i-j<0$ and $a_{i,j} = \sum_{k=0}^{n} c_{i,k}c^{\shortmid}_{k,j} = c_{i,i}c^{\shortmid}_{i,i} = 1$ for $i=j$. For $i>j\geq 1$ we have

$$\begin{aligned} a_{i,j} = \sum_{k=0}^{n} c_{i,k}c^{\shortmid}_{k,j} &= \sum_{k=j}^{i} c_{i,k}c^{\shortmid}_{k,j} \\ &= C_{i-j+1}(x) - C_{i-j}(x)c_1(x) + \cdots + C_1(x)(-1)^{i-j}c_{i-j}(x) \end{aligned}$$

and we know $C_{i-j+1}(x) = \sum_{s=1}^{i-j}(-1)^{s+1}c_s(x)C_{i-j+1-s}(x)$ from definition of $c_n(x)$. Thus, we obtain $a_{i,j} = \sum_{k=0}^{n} c_{i,k}c^{\shortmid}_{k,j} = 0$ for $i>j\geq 1$ which implies that $\mathcal{C}_{n,x}(\mathcal{C}_{n,x})^{-1} = I_n$. $\square$

Now, we show a relation between the generalized Perrin and Cordonnier matrices. The $n \times n$ lower triangular matrix $\mathcal{T}_{n,x} := [r_{i,j}]$, $(1 \leq i, j \leq n)$ is defined by

$$r_{i,j} = \begin{cases} \sum_{k=0}^{i-j} (-1)^k P_{i-j-k+2}(x) c_k(x), & \text{if } i \geqslant j; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.8. *Let n be any positive integer, $\mathcal{P}_{n,x}$ and $\mathcal{C}_{n,x}$ are $n \times n$ lower triangular generalized Perrin and Cordonnier matrices, then*

$$\mathcal{T}_{n,x}\mathcal{C}_{n,x} = \mathcal{P}_{n,x}.$$

Proof. Note that it suffices to prove that $\mathcal{P}_{n,x}(\mathcal{C}_{n,x})^{-1} = \mathcal{T}_{n,x}$. It is obvious that $r_{i,j} = 0$ for $i - j > 0$. For $i \geq j \geq 1$ we have

$$\begin{aligned} \sum_{k=0}^{n} p_{i,k} c'_{k,j} &= \sum_{k=0}^{n} P_{i-k+2}(x)(-1)^{k-j} c_{k-j}(x) \\ &= \sum_{k=j}^{i} P_{i-k+2}(x)(-1)^{k-j} c_{k-j}(x) = \sum_{k=0}^{i-j} P_{i+j-k+2}(x)(-1)^k c_k(x) = r_{i,j} \end{aligned}$$

which implies that $\mathcal{P}_{n,x}(\mathcal{C}_{n,x})^{-1} = \mathcal{T}_{n,x}$, as desired. □

Example 2.9. We obtain relation between the generalized Perrin and Cordonnier matrices for n=5 by using Theorem 2.8.

$$\begin{aligned} \mathcal{T}_{5,x}\mathcal{C}_{5,x} &= \mathcal{P}_{5,x}. \\ & \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ x & 2 & 0 & 0 & 0 \\ -x^2 & x & 2 & 0 & 0 \\ x^3 & -x^2 & x & 2 & 0 \\ -x^4 & x^3 & -x^2 & x & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ x^2 & x & 1 & 0 & 0 \\ 1+x^3 & x^2 & x & 1 & 0 \\ x+x^4 & 1+x^3 & x^2 & x & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 3x & 2 & 0 & 0 & 0 \\ 2x^2 & 3x & 2 & 0 & 0 \\ 3x^3+2 & 2x^2 & 3x & 2 & 0 \\ 3x+2x^4 & 3x^3+2 & 2x^2 & 3x & 2 \end{bmatrix}. \end{aligned}$$

Now we give a companion matrix $Q_{n,x}$ as follows:

$$Q_{n,x} = \begin{bmatrix} 0 & 1 & & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & & 0 & 0 & 1 \\ (-1)^{n+1}c_n(x) & (-1)^n c_{n-1}(x) & \cdots & c_3(x) & 0 & x \end{bmatrix}_{n \times n}.$$

Theorem 2.10. *Let $m \leq n$ be the integers, then the last column of matrix $(Q_{n,x})^m$ is*

$$\begin{bmatrix} C_{m-n+2}(x) \\ \vdots \\ C_m(x) \\ C_{m+1}(x) \end{bmatrix}.$$

Proof. We proceed by induction on m. The result clearly holds for $m = 1$. Now suppose that the result is true for all positive integers less than or equal to m. We prove it for $m+1$. The last column of matrix $(Q_{n,x})^{m+1}$ is

$$\begin{bmatrix} 0 & 1 & & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 1 & 0 \\ 0 & 0 & & 0 & 0 & 1 \\ (-1)^{n+1}c_n(x) & (-1)^n c_{n-1}(x) & \cdots & c_3(x) & 0 & x \end{bmatrix} \begin{bmatrix} C_{m-n+2}(x) \\ \vdots \\ C_m(x) \\ C_{m+1}(x) \end{bmatrix}$$

$$= \begin{bmatrix} C_{m-n+3}(x) \\ \vdots \\ C_{m+1}(x) \\ (-1)^{n+1}c_n(x)C_{m-n+2}(x) + \cdots + c_1(x)C_{m+1}(x) \end{bmatrix} = \begin{bmatrix} C_{m-n+3}(x) \\ \vdots \\ C_{m+1}(x) \\ C_{m+2}(x) \end{bmatrix}.$$

□

2.1. **Factorizations of Generalized Cordonnier matrices.** The set of all n-square matrices is denoted by H_n. A matrix $H \in H_n$ of the form

$$H = \begin{bmatrix} H_{11} & 0 & \cdots & 0 \\ 0 & H_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & H_{kk} \end{bmatrix}$$

in which $H_{ii} \in H_{n_i}$ $(i = 1, 2, ..., k)$ and $\sum_{i=1}^{k} n_i = n$, is called block diagonal. Notationally, such a matrix is often indicated as $H = H_{11} \oplus H_{22} \oplus \cdots \oplus H_{kk}$; this is called the direct sum of the matrices $H_{11}, H_{22}, \cdots, H_{kk}$.

Lee et al. [13, 14] and Sahin[23] gave some factorization. Like these we consider factorization of $\mathcal{C}_{n,x}$. Let I_n be the identity matrix of order n. We define the matrices $\overline{\mathcal{C}}_{n,x} = [1] \oplus (\mathcal{C}_{n-1,x})$ and

$$D_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ c_1(x) & & & \\ \vdots & & I_{n+2} & \\ (-1)^{n+1}c_{n+2}(x) & & & \end{bmatrix}. \tag{4}$$

Lemma 2.11. $(\overline{\mathcal{C}}_{k,x})(D_{k-3}) = \mathcal{C}_{k,x}$ *for* $k \geq 3$.

Proof. We take $(\overline{\mathcal{C}}_{k,x}) = [\overline{c}_{i,j}]$, $(D_{k-3}) = [d_{i,j}]$ and $\mathcal{C}_{k,x} = [c_{i,j}]$ and obtain $\sum\limits_{s=1}^{k} \overline{c}_{i,s} d_{s,j}$ for $i, j = 1, 2, ..., k$. It is obvious from matrix product and definition of I_{n+2} that $c_{11} = 1$, $\sum\limits_{s=1}^{k} \overline{c}_{i,s} d_{s,j} = c_{i,j}$ for $i = 1, 2, ..., k$ and $j = 2, ..., k$. For $j = 1$,

$$c_{i,1} = \sum_{s=1}^{k} \overline{c}_{i,s} d_{s,1} = C_{i-1}(x)c_1(x) - \cdots (-1)^{k-1} C_1(x)c_{i-1}(x).$$

Using $c_{k-1}(x) = xc_{k-2}(x) - C_3(x)c_{k-3}(x) + \cdots + (-1)^{k-2}C_k(x)c_0(x)$ and $c_0(x) = C_1(x) = 1$, we obtain $C_k(x) = C_{k-1}(x)c_1(x) - \cdots + (-1)^{k-1}c_{k-1}(x)$. So using these last two equation the equation $c_{i,1} = C_{i-1}(x)c_1(x) - \cdots + (-1)^{i-1}C_1(x)c_{i-1}(x) = C_i(x)$ is obtained. □

Theorem 2.12. *Let* $n \geq 3$ *be any positive integer. Then* $\mathcal{C}_{n,x} = (I_{n-2} \oplus (D_{-1})) \cdots (I_1 \oplus (D_{n-4}))(D_{n-3})$.

Proof. From Lemma 2.11 and matrix product we obtain

$$\begin{aligned} \mathcal{C}_{n,x} &= (\overline{\mathcal{C}}_{n,x})(D_{n-3}) = [(I_1 \oplus \overline{\mathcal{C}}_{n-1,x})(I_1 \oplus (D_{n-4}))](D_{n-3}) \\ &= [(I_2 \oplus \overline{\mathcal{C}}_{n-2,x})(I_2 \oplus (D_{n-5}))](I_1 \oplus (D_{n-4}))(D_{n-3}) \\ &\vdots \\ &= (I_{n-3} \oplus \overline{\mathcal{C}}_{3,x})(I_{n-3} \oplus (D_0)) \cdots (I_1 \oplus (D_{n-4}))(D_{n-3}) \end{aligned}$$

and $(I_{n-3} \oplus \overline{\mathcal{C}}_{3,x}) = (I_{n-2} \oplus (D_{-1}))$. Thus, we obtain

$$\mathcal{C}_{n,x} = (I_{n-2} \oplus (D_{-1}))(I_{n-3} \oplus (D_0)) \cdots (I_1 \oplus (D_{n-4}))(D_{n-3}).$$

□

Example 2.13. We give a factorization for $\mathcal{C}_{5,x}$ by using Theorem 2.12:

$$\begin{aligned} &(I_3 \oplus D_{-1})(I_2 \oplus D_0)(I_1 \oplus D_1)D_2 \\ = &\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ -x & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ x^2 & x & 1 & 0 & 0 \\ x^3+1 & x^2 & x & 1 & 0 \\ x+x^4 & x^3+1 & x^2 & x & 1 \end{bmatrix}.$$

Lemma 2.14. *D_n are the $(n+3)\times(n+3)$ Hessenberg matrices in (4). Then*

$$(D_n)^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -c_1(x) & & & \\ \vdots & & I_{n+2} & \\ (-1)^{n+2}c_{n+2}(x) & & & \end{bmatrix}.$$

Proof. The proof is obvious that matrix product. □

Corollary 2.15. *Let $n \geq 3$ be any positive integer. Then $(\mathcal{C}_{n,x})^{-1} = (D_{n-3})^{-1}(I_1 \oplus (D_{n-4}))^{-1}\cdots(I_{n-2}\oplus(D_{-1}))^{-1}$.*

Proof. Proof is obvious that previous lemma and equations $(I_k \oplus (D_{n-k-3}))^{-1} = I_k \oplus (D_{n-k-3})^{-1}$. □

3. Determinantal representation of associated polynomials Cordonnier numbers

Sahin and Ramirez gave a method for determinantal representation of Convolved Lucas polynomials in [24]. Using similar method, we give determinantal representation of $C_n(x)$.

Theorem 3.1. *Let $n \geq 1$ be an integer, $C_n(x)$ be the nth associated polynomials Cordonnier numbers and ${}_{-}A_n^{(x)} = [a_{i,j}]_{i,j=1,2,\ldots,n}$ be an $n\times n$ Hessenberg matrix defined as*

$$a_{i,j} = \begin{cases} -1, & \text{if } i-j=-1; \\ (-1)^{i-j}c_{i-j+1}(x), & \text{if } i \geq j \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Then

$$\det({}_{-}A_n^{(x)}) = C_{n+1}(x).$$

Proof. We proceed by induction on m. The result clearly holds for $n=1$, $\det({}_{-}A_1^{(x)}) = x = C_2(x)$. Now suppose that the result is true for all positive integers less than or equal to $n-1$. We prove it for n.

$${}_{-}A_n^{(x)}\,[C_1(x)\ C_2(x)\ \cdots\ C_n(x)]^T = [0\ 0\ \cdots\ 0\ C_{n+1}(x)]^T$$

In fact, using Cramers rule we have

$$C_n(x) = \frac{\det({}_{-}A_{n-1}^{(x)})C_{n+1}(x)}{\det({}_{-}A_n^{(x)})}$$

$$\Rightarrow \quad C_{n+1}(x) = \frac{\det({}_{-}A_n^{(x)})C_n(x)}{\det({}_{-}A_{n-1}^{(x)})}$$

From the hypothesis of induction we obtain

$$\det({}_{-}A_n^{(x)}) = C_{n+1}(x).$$

Therefore, $\det({}_{-}A_n^{(x)}) = C_{n+1}(x)$ holds for all positive integers n. □

Example 3.2. We obtain the polynomial $C_7(x)$ by using Theorem 3.1.

$$\det \begin{bmatrix} x & -1 & 0 & 0 & 0 & 0 \\ 0 & x & -1 & 0 & 0 & 0 \\ 1 & 0 & x & -1 & 0 & 0 \\ -x & 1 & 0 & x & -1 & 0 \\ x^2 & -x & 1 & 0 & x & -1 \\ -x^3 & x^2 & -x & 1 & 0 & x \end{bmatrix} = 2x^3 + x^6 + 1.$$

Corollary 3.3. *Let $m \leq n$ be the integers, e_n is nth row of the identity matrix I_n. Then*

$$e_n({}_{-}A_n^{(x)})(Q_{n,x})^m e_n^T = C_{m+2}(x).$$

Proof. Proof is obvious from matrix product, Theorem 2.10 and equation

$${}_{-}A_n^{(x)} \left[C_{m-n+2}(x) \ \cdots \ C_{m+1}(x)\right]^T = \left[0 \ 0 \ \cdots \ 0 \ C_{n+2}(x)\right]^T.$$

□

Example 3.4. We obtain the polynomial $C_7(x)$ by using Corollary 3.3.

$$e_6({}_{-}A_6^{(x)})(Q_{6,x})^5 e_6^T = 2x^3 + x^6 + 1 = C_7(x)$$

Theorem 3.5. *Let $n \geq 1$ be an integer, $C_n(x)$ be the nth associated polynomials Cordonnier numbers and ${}_{+}B_n^{(x)} = [b_{s,t}]_{s,t=1,2,\ldots,n}$ be an $n \times n$ Hessenberg matrix defined as*

$$b_{s,t} = \begin{cases} i, & \text{if } s-t=-1; \\ (i)^{s-t}c_{s-t+1}, & \text{if } s \geq t \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\det({}_{+}B_n^{(x)}) = C_{n+1}(x).$$

Proof. If we multiply the kth column by $(-1)(-i)^k$ and the jth row by $(-1)i^j$ of the matrix ${}_{-}A_n^{(x)}$, where $i = \sqrt{-1}$, then the determinant is not altered. Therefore we get the desired result. □

Example 3.6. We obtain the polynomial $C_6(x)$ by using Theorem 3.5.

$$\det\begin{bmatrix} x & i & 0 & 0 & 0 \\ 0 & x & i & 0 & 0 \\ -1 & 0 & x & i & 0 \\ ix & -1 & 0 & x & i \\ x^2 & ix & -1 & 0 & x \end{bmatrix} = 2x^2 + x^5$$

The permanent of a n-square matrix is defined by $\mathrm{per}A = \sum_{\sigma\in S_n}\prod_{i=1}^{n} a_{i\sigma(i)}$, where the summation extends over all permutations σ of the symmetric group S_n (cf. [18]). There is a relation between permanent and determinant of a Hessenberg matrix (cf. [4, 7]). Then it is clear the following corollary.

Corollary 3.7. *Let $n \geq 1$ be an integer, $C_n(x)$ be the nth associated polynomials Cordonnier numbers, ${}_{+}A_n^{(x)} = [u_{s,t}]_{s,t=1,2,\ldots,n}$ and ${}_{-}B_n^{(x)} = [v_{s,t}]_{s,t=1,2,\ldots,n}$ be the $n\times n$ Hessenberg matrices defined as*

$$u_{s,t} = \begin{cases} 1, & \text{if } s-t=-1; \\ (-1)^{s-t}c_{s-t+1}, & \text{if } s \geq t \\ 0, & \text{otherwise} \end{cases} \quad \text{and } v_{s,t} = \begin{cases} -i, & \text{if } s-t=-1; \\ (i)^{s-t}c_{s-t+1}, & \text{if } s \geq t \\ 0, & \text{otherwise} \end{cases}.$$

Where $i=\sqrt{-1}$. Then $per({}_{+}A_n^{(x)}) = per({}_{-}B_n^{(x)}) = C_{n+1}(x)$.

Corollary 3.8. *Let $n \geq 1$ be an integer, ${}_{-}A_n^{(x)}$ be the $n\times n$ Hessenberg matrix in (5), $C_n(x)$ is the nth associated polynomials Cordonnier numbers and*

$$\widehat{{}_{-}A_n^{(x)}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & & & \vdots \\ & -({}_{-}A_{n-1}^{(x)}) & & 0 \\ & & & 1 \end{bmatrix}.$$

Then,

$$(\widehat{{}_{-}A_n^{(x)}})^{-1} = \mathcal{C}_{n,x}.$$

Proof. Proof is obvious from Theorem 2.7. □

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