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CONVERGENCE OF SOLUTIONS OF AN IMPULSIVE DIFFERENTIAL SYSTEM WITH A PIECEWISE CONSTANT ARGUMENT

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Abstract. We prove the existence and uniqueness of the solutions of an impulsive differential system with a piecewise constant argument. Moreover, we obtain sufficient conditions for the convergence of these solutions and then prove that the limits of the solutions can be calculated by a formula.

1. Introduction

The problem on asymptotic constancy for delay differential equations, difference equations, impulsive delay equations and impulsive equations with piecewise constant arguments has been dealt with by many authors. Now, let us give a quick overview on the existing literature of this subject.

Atkinson and Haddock [1] developed conditions which ensure that all solutions of certain retarded functional differential equations were asymptotically constant as $t \to \infty$. Bastinec *et.al* [2] considered the linear homogeneous differential equation with delay and they proved explicit tests for convergence of all its solutions. Diblik [11] established a criterion of asymptotic convergence of all solutions of a nonlinear scalar differential equation with delay corresponding to the initial point. Bereketoglu and Pituk [9] gave sufficient conditions for the asymptotic constancy of solutions of nonhomogeneous linear delay differential equations with unbounded delay and they also computed the limits of solutions in terms of the initial conditions and a special matrix solution of the corresponding adjoint equation. In [12] Diblik and Ruzickova studied the asymptotic behavior of the solutions of the first order differential equation containing two delays. Bereketoglu and Karakoc [4] obtained sufficient conditions for the asymptotic constancy of solutions for an impulsive differential equations. Sufficient conditions for the asymptotic constancy and asymptotic convergence of solutions of an initial value problem for impulsive

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linear delay differential equations were presented in [15] by Karakoc and Bereketoglu. Bereketoglu and Huseynov in [3] gave sufficient conditions for the asymptotic constancy of the solutions of a linear system of difference equations with delays. Berezansky *et.al.* [10] investigated the asymptotic convergence of the solutions of a discrete equation with two delays in the critical case. Györi *et.al.* derived sufficient conditions for the convergence of solutions of a nonhomogeneous linear system of impulsive delay differential equations and a limit formula in [13]. In [5] Bereketoglu and Karakoc obtained sufficient conditions for the asymptotic constancy of the solutions of a system of nonhomogeneous linear impulsive pantograph equations. In [16], [7], [8] and [6] authors considered the asymptotic constancy of different types of impulsive differential equations with piecewise constant arguments and formulated the limit value of the solutions in terms of the initial condition and the solution of the integral equation for each type of equations.

In the view of the our experiences, we aim to extend the results obtained in [8] to a impulsive differential equations system with a piecewise constant argument.

In this paper, we consider the following initial value problem (IVP) which consists of a non-homogeneous linear impulsive differential system with a piecewise constant argument

$$X'(t) = A(t)\left(X(t) - X(\lfloor t \rfloor)\right) + F(t), \quad t \neq n \in \mathbb{Z}^+,\ t \geq 0, \tag{1}$$

$$\Delta X(n) = B(n)\, X(n) + D(n),\ n \in \mathbb{Z}^+, \tag{2}$$

with an initial condition

$$X(0) = X^0, \tag{3}$$

where $\mathbb{Z}^+ = \{1, 2, \ldots\}, A : [0, \infty) \to \mathbb{R}^{k \times k}$ is a continuous matrix function, $F : [0, \infty) \to \mathbb{R}^k$ is a continuous vector function, $B : \mathbb{Z}^+ \to \mathbb{R}^{k \times k}$ is a continuous matrix function such that $det(I - B(n+1)) \neq 0$ where I is the $k \times k$ identity matrix, $D : \mathbb{Z}^+ \to \mathbb{R}^k$ is a continuous vector function, $\Delta X(n) = X(n^+) - X(n^-)$ such that $X(n^+) = \lim_{t \to n^+} X(t)$ and $X(n^-) = \lim_{t \to n^-} X(t)$, $\lfloor . \rfloor$ denotes the greatest integer function and $X^0 \in \mathbb{R}^k$.

Throughout this paper, the norm $\|.\|$ of a vector is the sum of the absolute values of its elements and the corresponding matrix norm is given by $\|A\| = \max\limits_{1 \leq j \leq k} \{\sum\limits_{i=1}^{k} |a_{ij}|\}$ where $A = (a_{ij})$ is a $k \times k$ matrix.

2. Existence and Uniqueness

In this section we give a theorem which insures that the system (1)-(3) has a unique solution, but first of all, let us give definitions for the solution and set of piecewise right continuous functions:

Definition 1. *A function* $X(t)$ *defined on* $[0, \infty)$ *is said to be a solution of the initial value problem* (1)-(3) *if it satisfies the following conditions:*

$\mathbf{D_1}$. *$X : [0,\infty) \to \mathbb{R}^k$ is continuous with the possible exception of the points $t = n \in \mathbb{Z}^+$,*

$\mathbf{D_2}$. *$X(t)$ is right continuous and has left-hand limits at the points $t = n \in \mathbb{Z}^+$,*

$\mathbf{D_3}$. *$X'(t)$ exists for every $t \in [0,\infty)$ with the possible exception of the points $t = n \in \mathbb{Z}^+$ where one-sided derivatives exist,*

$\mathbf{D_4}$. *$X(t)$ satisfies (1) for any $t \in (0,\infty)$ with the possible exception of the points $t = n \in \mathbb{Z}^+$,*

$\mathbf{D_5}$. *$X(t)$ satisfies (2) for every $t = n \in \mathbb{Z}^+$,*

$\mathbf{D_6}$. *$X(0) = X^0$.*

Definition 2. *If $\varphi : [0,\infty) \to \mathbb{R}^{k\times k}$ is continuous for $t \in [0,\infty), t \neq n \in \mathbb{Z}^+$ and right continuous at the points $t = n \in \mathbb{Z}^+$, then the set of such kind of functions is called the set of piecewise right continuous functions and is denoted by $\mathcal{PRC}\left([0,\infty), \mathbb{R}^{k\times k}\right)$.*

Theorem 1. *The initial value problem (1)-(3) has a unique solution.*

Proof. Since $\lfloor t \rfloor = 0$ for $0 \le t < 1$, (1) can be written as

$$X'(t) = A(t)X(t) - A(t)X(0) + F(t)$$

or

$$X'(t) = A(t)X(t) - A(t)X^0 + F(t) \tag{4}$$

where X^0 is the initial condition given in (3). Since $A(t)$ and $F(t)$ are continuous functions, non-homogeneous ordinary differential equations system (4) has a unique solution and this solution is given by

$$X(t) = \Phi(t)\Phi^{-1}(0)X^0 + \int_0^t \Phi(t)\Phi^{-1}(s)(-A(s)X^0 + F(s))\,ds \tag{5}$$

where $\Phi(t)$ is the fundamental matrix of the homogeneous system

$$X'(t) = A(t)X(t).$$

Let us denote the solution (5) as $X_0(t)$ since it is defined on the interval $0 \le t < 1$. On the other hand, let $X_1(t)$ be the solution of Eq.(1) on the interval $[1, 2)$. Then $X_1(t)$ is

$$X_1(t) = \Phi(t)\Phi^{-1}(1)X_1(1) + \int_1^t \Phi(t)\Phi^{-1}(s)(-A(s)X_1(1) + F(s))\,ds. \tag{6}$$

Now we use the impulse condition (2) at the point $t = 1$:
Substituting $t = 1$ in (2), we get

$$\Delta X(1) = X(1^+) - X(1^-) = B(1)X(1) + D(1).$$

Since the solution of (1) is right continuous at integer points, we have

$$\begin{aligned} X_1(1) - X_0(1) &= B(1)X_1(1) + D(1) \\ X_0(1) &= (I - B(1))X_1(1) - D(1). \end{aligned} \tag{7}$$

Considering (5) and (6) in (7) yields us

$$X_1(1) = (I - B(1))^{-1}\left(\Phi(1)\Phi^{-1}(0) - \int_0^1 \Phi(1)\Phi^{-1}(s)A(s)\,ds\right)X^0$$
$$+ (I - B(1))^{-1}\left(\int_0^1 \Phi(1)\Phi^{-1}(s)F(s)\,ds + D(1)\right). \tag{8}$$

Hence we can find $X_1(1)$ in terms of the given impulse and initial conditions. Writing (8) in (6) gives us the solution $X_1(t)$ of Eq.(1) on $[1, 2)$.
Moreover, the solution $X_2(t)$ of Eq.(1) on the interval $[2, 3)$ is given by

$$X_2(t) = \Phi(t)\Phi^{-1}(2)X_2(2) + \int_2^t \Phi(t)\Phi^{-1}(s)(-A(s)X_2(2) + F(s))\,ds$$

and using impulse condition at $t = 2$, we obtain

$$X_1(2) = (I - B(2))X_2(2) - D(2).$$

So again we can find $X_2(2)$ in terms of the given initial and impulse conditions.
Following this method, we find the solution $X_n(t)$ of Eq.(1) on the interval $[n, n+1)$ as

$$X_n(t) = \Phi(t)\Phi^{-1}(n)X_n(n) + \int_n^t \Phi(t)\Phi^{-1}(s)(-A(s)X_n(n) + F(s))\,ds. \tag{9}$$

Then considering the impulse condition (2) at the point $t = n + 1$ yields the nonhomogeneous difference equation system

$$Z_{n+1} = M(n)Z_n + N(n), n \geq 0, \tag{10}$$

where $Z_n = X_n(n)$,

$$M(n) = (I - B(n+1))^{-1}\left(\Phi(n+1)\Phi^{-1}(n) - \int_n^{n+1} \Phi(n+1)\Phi^{-1}(s)A(s)\,ds\right), \tag{11}$$

and

$$N(n) = (I - B(n+1))^{-1}\left(\int_n^{n+1} \Phi(n+1)\Phi^{-1}(s)F(s)\,ds + D(n+1)\right). \tag{12}$$

The difference system (10) with the condition $Z_0 = X^0$ has a unique solution. Writing this unique solution in (9) gives the unique solution of (1)-(3) on the interval $[n, n+1)$. So by taking into account $\lfloor t \rfloor = n$, we obtain the unique solution of (1)-(3) on the interval $[0, \infty)$. □

In the rest of the paper, assume that there is a constant $L > 0$ such that

$$\|Z_n\| \leq L,\ n \geq 0, \tag{13}$$

where Z_n is the solution of (10).

Remark 1. *Note that a straightforward verification shows that the solution of the initial value problem* (1)-(3) *satisfies the integral equation*

$$X(t) = X^0 + \int_0^t A(s)\,X(s)\,ds - \int_0^t A(s)\,X(\lfloor s \rfloor)\,ds + \int_0^t F(s)\,ds + \sum_{i=1}^{\lfloor t \rfloor} B(i)\,X(i) + \sum_{i=1}^{\lfloor t \rfloor} D(i)\,. \quad (14)$$

Proof. Taking the integral of both sides of Eq.(1) from 0 to t gives us

$$\int_0^t X'(s)\,ds = \int_0^t A(s)\,X(s)\,ds - \int_0^t A(s)\,X(\lfloor s \rfloor)\,ds + \int_0^t F(s)\,ds. \quad (15)$$

On the other hand, the left side of the Eq.(15) can be re-written as follows

$$\begin{aligned}\int_0^t X'(s)\,ds &= \int_{0^+}^{1^-} X'(s)\,ds + \int_{1^+}^{2^-} X'(s)\,ds + \int_{2^+}^{3^-} X'(s)\,ds + \dots \int_{n^+}^{t} X'(s)\,ds \\ &= X(1^-) - X(0^+) + X(2^-) - X(1^+) + X(3^-) - X(2^+) \\ &+ \dots + X(t) - X(n^+) \\ &= X(t) - X(0^+) - \{[X(1^+) - X(1^-)] + [X(2^+) - X(2^-)] \\ &+ \dots + [X(n^+) - X(n^-)]\} \\ &= X(t) - X(0^+) - \{\Delta X(1) + \Delta X(2) + \dots + \Delta X(n)\}. \quad (16)\end{aligned}$$

Since X is right continuous at $t = 0$, Eq. (16) is written as

$$\begin{aligned}\int_0^t X'(s)\,ds &= X(t) - X(0) - \sum_{i=1}^{n} \Delta X(i) \\ &= X(t) - X(0) - \sum_{i=1}^{n} (B(i)X(i) + D(i)) \\ &= X(t) - X(0) - \sum_{i=1}^{\lfloor t \rfloor} B(i)X(i) - \sum_{i=1}^{\lfloor t \rfloor} D(i). \quad (17)\end{aligned}$$

Writing (17) in (15) gives us the formula (14). □

We will use this formula in the proof of our results in the next section.

3. Main Results

In this part, it is shown that the IVP (1)-(3) tends to a constant vector as $t \to \infty$, and then the limit value of the solution of (1)-(3) is computed when $B(n) = 0$.

Theorem 2. *Assume that* K_1, K_2, L_1 *and* L_2 *are real positive constants such that*

$$(i) \int_0^\infty \|A(s)\|\, ds \le K_1 < \infty, \qquad (ii) \int_0^\infty \|F(s)\|\, ds \le K_2 < \infty,$$

$$(iii) \prod_{i=1}^\infty (1 + \|B(i)\|) \le L_1 < \infty, \qquad (iv) \sum_{i=1}^\infty \|D(i)\| \le L_2 < \infty.$$

Then, the solution of the IVP (1)-(3) *tends to a constant vector as* $t \to \infty$.

Proof. For the proof of Theorem 2, we need the following Samoilenko and Perestyuk's well-known lemma [17] and Theorem 7.4.6 in [14]:

Lemma 1. *Let a non-negative piecewise continuous function* $u(t)$ *satisfy the inequality*

$$u(t) \le c + \int_{t_0}^t v(s)\, u(s)\, ds + \sum_{t_0 \le \tau_i < t} \beta_i u(\tau_i), \ t \ge t_0,$$

where $c \ge 0$, $\beta_i \ge 0$, $v(s) > 0$, τ_i *are the first kind discontinuity points of the function* $u(t)$. *Then the following estimate holds for the function* $u(t)$

$$u(t) \le c \prod_{t_0 \le \tau_i < t} (1 + \beta_i) \exp\left(\int_{t_0}^t v(s)\, ds \right), \ t \ge t_0.$$

Theorem 3. *(Theorem 7.4.6 in* [14]*) The infinite product* $\prod_{k=1}^\infty (1 + c_k)$ *converges absolutely if and only if the infinite series* $\sum_{k=1}^\infty c_k$ *converges absolutely.*

Now we can start the proof of Theorem 2:

Let $X(t)$ be the solution of the IVP (1)-(3). So the integral equation (14) is satisfied and we have

$$\|X(t)\| \le \left\|X^0\right\| + \int_0^t \|A(s)\| \|X(s)\|\, ds + \int_0^t \|A(s)\| \|X(\lfloor s \rfloor)\|\, ds + \int_0^t \|F(s)\|\, ds$$
$$+ \sum_{i=1}^{\lfloor t \rfloor} \|B(i)\| \|X(i)\| + \sum_{i=1}^{\lfloor t \rfloor} \|D(i)\|.$$

Since $0 \leq s \leq t$, the function $X(\lfloor s \rfloor)$ corresponds to the solution of the difference equation (10). Hence considering (13) with the assumptions $(i), (ii), (iv)$, we get

$$\|X(t)\| \leq c + \int_0^t \|A(s)\| \|X(s)\| \, ds + \sum_{i=1}^{\lfloor t \rfloor} \|B(i)\| \|X(i)\| \tag{18}$$

where $c = \|X^0\| + LK_1 + K_2 + L_2$. Applying Lemma 1 to (18) yields

$$\|X(t)\| \leq c \prod_{i=0}^{\lfloor t \rfloor} (1 + \|B(i)\|) \exp\left(\int_0^t \|A(s)\| \, ds \right)$$
$$\leq c \prod_{i=0}^{\infty} (1 + \|B(i)\|) \exp\left(\int_0^\infty \|A(s)\| \, ds \right).$$

So, considering (i) and (iii) in the last inequality gives us that $X(t)$ is bounded, that is

$$\|X(t)\| \leq M, \; t \geq 0, \tag{19}$$

where $M = cL_1 e^{K_1}$.

On the other hand, from the integral equation (14) it can be written that

$$\|X(t) - X(s)\| \leq \int_s^t \|A(u)\| \|X(u)\| \, du + \int_s^t \|A(u)\| \|X([u])\| \, du + \int_s^t \|F(u)\| \, du + \sum_{i=\lfloor s \rfloor + 1}^{\lfloor t \rfloor} \|B(i)\| \|X(i)\| + \sum_{i=\lfloor s \rfloor + 1}^{\lfloor t \rfloor} \|D(i)\|, \tag{20}$$

for $0 \leq s \leq t < \infty$. Using the boundedness of $X(n)$ and $X(t)$ which are given in (13) and (19), we obtain

$$\|X(t) - X(s)\| \leq (M + L) \int_s^\infty \|A(u)\| \, du + \int_s^\infty \|F(u)\| \, du + L \sum_{i=\lfloor s \rfloor + 1}^{\infty} \|B(i)\| + \sum_{i=\lfloor s \rfloor + 1}^{\infty} \|D(i)\|. \tag{21}$$

Here we note that condition (iii) implies

$$\sum_{i=1}^{\infty} \|B(i)\| < \infty \tag{22}$$

from Theorem 3. So considering (22) with the conditions $(i), (ii), (iv)$ in (21), it is easy to see that

$$\lim_{s \to \infty} \|X(t) - X(s)\| = 0.$$

By Cauchy convergence criterion, we get $\lim_{t\to\infty} X(t) \in \mathbb{R}^k$. □

Now let us take $B(n) = 0$ in (2). In this case, the IVP (1)-(3) reduces to

$$X'(t) = A(t)(X(t) - X(\lfloor t \rfloor)) + F(t), \quad t \neq n \in \mathbb{Z}^+, \ t \geq 0, \tag{23}$$

$$\Delta X(n) = D(n), \ n \in \mathbb{Z}^+, \tag{24}$$

$$X(0) = X^0. \tag{25}$$

Theorem 4. *If*

$$\int_t^{\lfloor t+1 \rfloor} \|A(s)\| \, ds \leq \rho < 1, \tag{26}$$

then there is a unique bounded matrix function $Y \in \mathcal{PRC}\left([0,\infty), \mathbb{R}^{k\times k}\right)$ *such that the equation*

$$Y(t) = I + \int_t^{\lfloor t+1 \rfloor} Y(s) A(s) \, ds, \ t \geq 0 \tag{27}$$

holds.

Proof. Consider the space

$$B = \left\{ Y \in \mathcal{PRC}\left([0,\infty), \mathbb{R}^{k\times k}\right) : \|Y\|_B \leq \lambda, \ \lambda \geq \frac{1}{1-\rho} \right\}$$

where

$$\|Y\|_B = \sup_{t\geq 0} \|Y(t)\|, \ Y \in B.$$

For $Y \in B$ and $t \geq 0$, let us define

$$TY(t) = I + \int_t^{\lfloor t+1 \rfloor} Y(s) A(s) \, ds. \tag{28}$$

It can be easily seen that

$$TY\left(t_*^+\right) = TY\left(t_*^-\right) = TY(t_*), \ t_* \in (n, n+1),$$

$$TY\left(n^+\right) = \lim_{t\to n^+} TY(t) = TY(n), \ n \in \mathbb{Z}^+,$$

$$TY\left(n^-\right) = \lim_{t\to n^-} TY(t) = I, \ n \in \mathbb{Z}^+.$$

So $TY \in \mathcal{PRC}\left([0,\infty), \mathbb{R}^{k\times k}\right)$. Moreover, taking the norm of both sides of (28) yields that

$$\|TY\|_B \leq 1 + \|Y\|_B \left(\int_t^{\lfloor t+1 \rfloor} \|A(s)\| \, ds \right). \tag{29}$$

Considering (26) in (29) gives us that

$$\|TY\|_B \leq 1 + \rho \|Y\|_B \leq \lambda.$$

Hence T maps B into itself.

On the other hand, for any Y and $Z \in B$

$$\|TY - TZ\|_B \leq \rho \|Y - Z\|_B .$$

Since $\rho < 1, T : B \to B$ is a contraction. Therefore, by the well known Banach fixed point theorem, there is a unique piecewise right continuous and bounded solution of Eq.(27). □

Lemma 2. *If* (26) *is true, then the solution* Y *of the integral equation* (27) *satisfies the equation*

$$\begin{cases} Y'(t) = -Y(t)\, A(t), \; t \neq n, \; t \geq 0, \\ \Delta Y(n) = \int\limits_{n}^{n+1} Y(s)\, A(s)\, ds, \; n \in \mathbb{Z}^+. \end{cases} \tag{30}$$

Proof. Taking the derivative of (27) for $t \in (n, n+1)$, $n \in \mathbb{Z}^+$, we obtain

$$Y'(t) = -Y(t)\, A(t).$$

Moreover, we calculate $\Delta Y(n)$ as

$$\begin{aligned} \Delta Y(n) &= Y\left(n^+\right) - Y\left(n^-\right) \\ &= I + \int\limits_{n}^{n+1} Y(s)\, A(s)\, ds - I \\ &= \int\limits_{n}^{n+1} Y(s)\, A(s)\, ds. \end{aligned}$$

So we obtain the Eq.(30). □

Now, for $t \geq 0$ let us define the function

$$C(t) = Y(t)\, X(t) - \int\limits_{t}^{\lfloor t+1 \rfloor} Y(s)\, A(s)\, X(\lfloor s \rfloor)\, ds \tag{31}$$

where Y is the solution of Eq.(27) and X is the solution of (23)-(25).

Lemma 3. *If* (26) *is satisfied, then*

$$C(t) = C(0) + \int\limits_{0}^{t} Y(s)\, F(s)\, ds + \sum_{i=1}^{\lfloor t \rfloor} D(i). \tag{32}$$

Proof. For the proof it is enough to show that $C(t)$ defined by (31) satisfies the equation

$$\begin{cases} C'(t) = Y(t)\,F(t)\,,\ t \neq n,\ t \geq 0, \\ \Delta C(n) = D(n)\,,\ n \in \mathbb{Z}^{+}, \end{cases} \tag{33}$$

because taking the integral of both sides of the (33) from 0 to t gives us (32) as in Remark 1. Now, let us obtain (33):
For $t \in (n, n+1)$, (31) is reduced to

$$C(t) = Y(t)\,X(t) - \left(\int_{t}^{n+1} Y(s)\,A(s)\,ds \right) X(n)\,. \tag{34}$$

Differentiating (34) and considering (30) and (23) yields

$$\begin{aligned} C'(t) &= Y'(t)\,X(t) + Y(t)\,X'(t) + Y(t)\,A(t)\,X(n) \\ &= -Y(t)\,A(t)\,X(t) + Y(t)\,\{A(t)\,(X(t) - X(n)) + F(t)\} + Y(t)\,A(t)\,X(n) \\ &= Y(t)\,F(t)\,. \end{aligned}$$

So the first part of the Eq.(33) is obtained.
On the other hand, we need $C(n^{+})$ and $C(n^{-})$ to compute $\Delta C(n)$:

$$C\left(n^{+}\right) = \lim_{t \to n^{+}} C(t) = Y(n)\,X(n) - \left(\int_{n}^{n+1} Y(s)\,A(s)\,ds \right) X(n)\,, \tag{35}$$

$$C\left(n^{-}\right) = \lim_{t \to n^{-}} C(t) = Y\left(n^{-}\right) X\left(n^{-}\right). \tag{36}$$

From (30), we have

$$\Delta Y(n) = Y(n^{+}) - Y(n^{-}) = \int_{n}^{n+1} Y(s)\,A(s)\,ds.$$

Since Y is right continuous at the points $n \in \mathbb{Z}^{+}$, we get

$$Y\left(n^{-}\right) = Y(n) - \int_{n}^{n+1} Y(s)\,A(s)\,ds. \tag{37}$$

Similarly, from (24)

$$\begin{aligned} \Delta X(n) = X(n^{+}) - X(n^{-}) &= D(n) \\ X\left(n^{-}\right) &= X(n) - D(n) \end{aligned} \tag{38}$$

in view of the right continuity of X at the points $n \in \mathbb{Z}^+$.
Substituting (37) and (38) in (36), gives us

$$C\left(n^-\right) = \left(Y\left(n\right) - \int_n^{n+1} Y\left(s\right) A\left(s\right) ds\right)\left(X\left(n\right) - D\left(n\right)\right). \tag{39}$$

Considering (35) and (39) in $\Delta C\left(n\right) = C\left(n^+\right) - C\left(n^-\right)$ we get

$$\Delta C\left(n\right) = D\left(n\right),$$

and this is the second part of the Eq.(33). □

Theorem 5. *Suppose that assumptions* $(i), (ii), (iv)$ *in Theorem 2 and the condition* (26) *are satisfied. Then the limit value of the solution of* $X\left(t\right)$ *of IVP* (23)-(25), *when* $t \to \infty$*, is given by the formula*

$$\lim_{t\to\infty} X\left(t\right) = X^0 + \int_0^{\infty} Y\left(s\right) F\left(s\right) ds + \sum_{i=1}^{\infty} D\left(i\right) \tag{40}$$

where Y *is a solution of the Eq.*(27)*.*

Proof. Let $X\left(t\right)$ be the solution of IVP (23)-(25). For the proof it is sufficient to show that

$$\lim_{t\to\infty} X\left(t\right) = C\left(0\right) + \int_0^{\infty} Y\left(s\right) F\left(s\right) ds + \sum_{i=1}^{\infty} D\left(i\right), \tag{41}$$

where Y and C are given by (27) and (31), respectively.
From (32), we have for

$$\begin{aligned} & X\left(t\right) - C\left(0\right) - \int_0^{\infty} Y\left(s\right) F\left(s\right) ds - \sum_{i=1}^{\infty} D\left(i\right) \\ &= X\left(t\right) - \left(C\left(0\right) + \int_0^{t} Y\left(s\right) F\left(s\right) ds + \sum_{i=1}^{\lfloor t\rfloor} D\left(i\right)\right) - \int_t^{\infty} Y\left(s\right) F\left(s\right) ds - \sum_{i=\lfloor t\rfloor+1}^{\infty} D\left(i\right) \\ &= X\left(t\right) - C\left(t\right) - \int_t^{\infty} Y\left(s\right) F\left(s\right) ds - \sum_{i=\lfloor t\rfloor+1}^{\infty} D\left(i\right). \end{aligned} \tag{42}$$

Considering (31) in (42), we have

$$X(t) - C(0) - \int_0^{\infty} Y(s) F(s) \, ds - \sum_{i=1}^{\infty} D(i)$$
$$= X(t) - Y(t) X(t) + \int_t^{[t+1]} Y(s) A(s) X(\lfloor s \rfloor) \, ds - \int_t^{\infty} Y(s) F(s) \, ds - \sum_{i=\lfloor t \rfloor + 1}^{\infty} D(i). \tag{43}$$

On the other hand, multiplying (27) by $X(t)$ yields

$$X(t) = Y(t) X(t) - \int_t^{[t+1]} Y(s) A(s) X(t) \, ds. \tag{44}$$

Substituting (44) into (43), we obtain

$$X(t) - C(0) - \int_0^{\infty} Y(s) F(s) \, ds - \sum_{i=1}^{\infty} D(i)$$
$$= \int_t^{\lfloor t+1 \rfloor} Y(s) A(s) (X(\lfloor s \rfloor) - X(t)) \, ds - \int_t^{\infty} Y(s) F(s) \, ds - \sum_{i=\lfloor t \rfloor + 1}^{\infty} D(i). \tag{45}$$

From (45), it is found that

$$\|X(t) - C(0) - \int_0^{\infty} Y(s) F(s) \, ds - \sum_{i=1}^{\infty} D(i) \|$$
$$\leq \|Y\|_B (L + M) \int_t^{\lfloor t+1 \rfloor} \|A(s)\| ds + \|Y\|_B \int_t^{\infty} \|F(s)\| ds + \sum_{i=\lfloor t \rfloor + 1}^{\infty} \|D(i)\|.$$

Here (13), (19) and the boundedness of $Y(t)$ is used. Thus we conclude that (41) is true for $t \to \infty$. Taking into account (31), it can be easily verified that the limit relation (41) is reduced to (40). □

Now let us give an example to illustrate our results.

Example 1. *Consider the following IVP:*

$$X'(t) = \begin{pmatrix} \frac{1}{(1+2t)^2} & 0 \\ 0 & 0 \end{pmatrix} (X(t) - X(\lfloor t \rfloor)) + \begin{pmatrix} 0 \\ \frac{1}{(1+2t)^2} \end{pmatrix}, \ t \neq n, \tag{46}$$

$$\Delta X(n) = \begin{pmatrix} \frac{1}{2^n} \\ 0 \end{pmatrix}, \ n \in \mathbb{Z}^+, \tag{47}$$

$$X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{48}$$

First we show that the solutions of the corresponding difference equation of (46)-(48) *are bounded.* $\Phi(t) = \begin{pmatrix} e^{\frac{-1}{2(1+2t)}} & 0 \\ 0 & 1 \end{pmatrix}$ *is the fundamental matrix of the homogeneous system* $X'(t) = \begin{pmatrix} \frac{1}{(1+2t)^2} & 0 \\ 0 & 0 \end{pmatrix} X(t)$. *Considering this fundamental matrix in the formulas* (11) *and* (12), *we find*

$$M(n) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ N(n) = \begin{pmatrix} \frac{1}{2^{n+1}} \\ \frac{1}{(3+2n)(1+2n)}, \end{pmatrix}$$

respectively. Thus the corresponding difference system is found as

$$Z_{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Z_n + \begin{pmatrix} \frac{1}{2^{n+1}} \\ \frac{1}{(3+2n)(1+2n)} \end{pmatrix}, n \geq 0. \tag{49}$$

The solution of the system (49) *with the initial condition* (48) *is given as*

$$Z(n) = \begin{pmatrix} 1 + \sum_{r=0}^{n-1} \frac{1}{2^{r+1}} \\ 1 + \sum_{r=0}^{n-1} \frac{1}{(3+2r)(1+2r)} \end{pmatrix}$$

and it is clear that this solution is bounded.

Now let us verify the hypotheses of Theorem 5:

By considering the convenient matrix and vector norms, we have

$$(i) \int_0^\infty \|A(s)\| \, ds = \int_0^\infty \frac{1}{(1+2s)^2} ds = \frac{1}{2} < \infty,$$

$$(ii) \int_0^\infty \|F(s)\| \, ds = \int_0^\infty \frac{1}{(1+2s)^2} ds = \frac{1}{2} < \infty,$$

$$(iv) \sum_{i=1}^\infty \|D(i)\| = \sum_{i=1}^\infty \frac{1}{2^i} = 1 < \infty.$$

On the other hand, for $n \leq t < n+1$ *the condition* (26) *can be written as*

$$\int_{t}^{\lfloor t+1 \rfloor} \|A(s)\| \, ds \leq \int_{n}^{n+1} \|A(s)\| \, ds = \int_{n}^{n+1} \frac{1}{(1+2s)^2} ds = \frac{1}{(1+2n)(3+2n)} < 1.$$

So, all hypotheses of Theorem 5 are satisfied. Hence the limit of $X(t)$ *of* (46)-(48) *is computed as*

$$\lim_{t\to\infty} X(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \int_{0}^{\infty} Y(s) \begin{pmatrix} 0 \\ \frac{1}{(1+2s)^2} \end{pmatrix} ds + \sum_{i=1}^{\infty} \begin{pmatrix} \frac{1}{2^i} \\ 0 \end{pmatrix}$$

or

$$\lim_{t\to\infty} X(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \int_{0}^{\infty} Y(s) \begin{pmatrix} 0 \\ \frac{1}{(1+2s)^2} \end{pmatrix} ds$$

where $Y(t)$ *is the solution of*

$$Y(t) = I + \int_{t}^{\lfloor t+1 \rfloor} Y(s) \begin{pmatrix} \frac{1}{(1+2s)^2} & 0 \\ 0 & 0 \end{pmatrix} ds.$$

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