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# CONVEXITY PROPERTIES AND INEQUALITIES CONCERNING THE $(p, k)$-GAMMA FUNCTION

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Abstract. In this paper, some convexity properties and some inequalities for the $(p, k)$-analogue of the Gamma function, $\Gamma_{p,k}(x)$ are given. In particular, a $(p, k)$-analogue of the celebrated Bohr-Mollerup theorem is given. Furthermore, a $(p, k)$-analogue of the Riemann zeta function, $\zeta_{p,k}(x)$ is introduced and some associated inequalities are derived. The established results provide the $(p, k)$-generalizations of some known results concerning the classical Gamma function.

## 1. Introduction

In a recent paper [10], the authors introduced a $(p, k)$-analogue of the Gamma function defined for $p \in \mathbb{N}$, $k > 0$ and $x \in \mathbb{R}^+$ as

$$\Gamma_{p,k}(x) = \int_0^p t^{x-1}\left(1 - \frac{t^k}{pk}\right)^p dt \tag{1.1}$$

$$= \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)} \tag{1.2}$$

satisfying the basic properties

$$\Gamma_{p,k}(x+k) = \frac{pkx}{x+pk+k}\Gamma_{p,k}(x), \tag{1.3}$$

$$\Gamma_{p,k}(ak) = \frac{p+1}{p}k^{a-1}\Gamma_p(a), \quad a \in \mathbb{R}^+$$

$$\Gamma_{p,k}(k) = 1.$$

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The $(p,k)$-analogue of the Digamma function is defined for $x>0$ as

$$\psi_{p,k}(x)=\frac{d}{dx}\ln\Gamma_{p,k}(x)=\frac{1}{k}\ln(pk)-\sum_{n=0}^{p}\frac{1}{nk+x} \tag{1.4}$$
$$=\frac{1}{k}\ln(pk)-\int_0^\infty\frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}\,dt.$$

Also, the $(p,k)$-analogue of the Polygamma functions are defined as

$$\psi_{p,k}^{(m)}(x)=\frac{d^m}{dx^m}\psi_{p,k}(x)=\sum_{n=0}^{p}\frac{(-1)^{m+1}m!}{(nk+x)^{m+1}} \tag{1.5}$$
$$=(-1)^{m+1}\int_0^\infty\left(\frac{1-e^{-k(p+1)t}}{1-e^{-kt}}\right)t^m e^{-xt}\,dt$$

where $m\in\mathbb{N}$, and $\psi_{p,k}^{(0)}(x)\equiv\psi_{p,k}(x)$.

The functions $\Gamma_{p,k}(x)$ and $\psi_{p,k}(x)$ satisfy the following commutative diagrams.

$$\begin{array}{ccc} \Gamma_{p,k}(x) & \xrightarrow{p\to\infty} & \Gamma_k(x) \\ {\scriptstyle k\to1}\downarrow & & \downarrow{\scriptstyle k\to1} \\ \Gamma_p(x) & \xrightarrow[p\to\infty]{} & \Gamma(x) \end{array} \qquad \begin{array}{ccc} \psi_{p,k}(x) & \xrightarrow{p\to\infty} & \psi_k(x) \\ {\scriptstyle k\to1}\downarrow & & \downarrow{\scriptstyle k\to1} \\ \psi_p(x) & \xrightarrow[p\to\infty]{} & \psi(x) \end{array}$$

The $(p,k)$-analogue of the classical Beta function is defined as

$$B_{p,k}(x,y)=\frac{\Gamma_{p,k}(x)\Gamma_{p,k}(y)}{\Gamma_{p,k}(x+y)},\quad x>0,y>0. \tag{1.6}$$

The purpose of this paper is to establish some convexity properties and some inequalities involving the function, $\Gamma_{p,k}(x)$. In doing so, a $(p,k)$-analogue of the Bohr-Mollerup theorem is proved. Also, a $(p,k)$-analogue of the Riemann zeta function, $\zeta_{p,k}(x)$ is introduced and some associated inequalities relating $\Gamma_{p,k}(x)$ and $\zeta_{p,k}(x)$ are derived. We present our findings in the following sections.

## 2. Convexity Properties Involving the $(p,k)$-Gamma function

Let us begin by recalling the following basic definitions and concepts.

**Definition 1.** *A function $f:(a,b)\to\mathbb{R}$ is said to be convex if*

$$f(\alpha x+\beta y)\le\alpha f(x)+\beta f(y) \tag{2.1}$$

*for all $x,y\in(a,b)$, where $\alpha,\beta>0$ such that $\alpha+\beta=1$.*

**Lemma 1.** *Let $f:(a,b)\to\mathbb{R}$ be a twice differentiable function. Then $f$ is said to be convex if and only if $f''(x)\ge0$ for every $x\in(a,b)$.*

**Remark 1.** *A function $f$ is said to be concave if $-f$ is convex, or equivalently, if the inequality (2.1) is reversed.*

**Definition 2.** *A function $f:(a,b)\rightarrow\mathbb{R}^+$ is said to be logarithmically convex if the inequality*

$$\log f(\alpha x+\beta y)\leq \alpha\log f(x)+\beta\log f(y)$$

*or equivalently*

$$f(\alpha x+\beta y)\leq (f(x))^{\alpha}(f(y))^{\beta}$$

*holds for all $x,y\in(a,b)$ and $\alpha,\beta>0$ such that $\alpha+\beta=1$.*

**Theorem 1.** *The function, $\Gamma_{p,k}(x)$ is logarithmically convex.*

*Proof.* Let $x,y>0$ and $\alpha,\beta>0$ such that $\alpha+\beta=1$. Then, by the integral representation (1.1) and by the Hölder's inequality for integrals, we obtain

$$\begin{aligned}
\Gamma_{p,k}(\alpha x+\beta y) &= \int_0^p t^{\alpha x+\beta y-1}\left(1-\frac{t^k}{pk}\right)^p dt \\
&= \int_0^p t^{\alpha(x-1)}t^{\beta(y-1)}\left(1-\frac{t^k}{pk}\right)^{p(\alpha+\beta)} dt \\
&= \int_0^p t^{\alpha(x-1)}\left(1-\frac{t^k}{pk}\right)^{\alpha p} t^{\beta(y-1)}\left(1-\frac{t^k}{pk}\right)^{\beta p} dt \\
&\leq \left(\int_0^p t^{x-1}\left(1-\frac{t^k}{pk}\right)^p dt\right)^{\alpha}\left(\int_0^p t^{y-1}\left(1-\frac{t^k}{pk}\right)^p dt\right)^{\beta} \\
&= (\Gamma_{p,k}(x))^{\alpha}(\Gamma_{p,k}(y))^{\beta}
\end{aligned}$$

as required. □

**Remark 2.** *Since every logarithmically convex function is also convex* [13, p. 66], *it follows that the function $\Gamma_{p,k}(x)$ is convex.*

**Remark 3.** *Theorem 1 was proved in* [10] *by using a different procedure. In the present work, we provide a much simpler alternative proof by using the Hölder's inequality for integrals.*

The next theorem is the $(p,k)$-analogue of the celebrated Bohr-Mollerup theorem.

**Theorem 2.** *Let $f(x)$ be a positive function on $(0,\infty)$. Suppose that*

(a) $f(k)=1$,
(b) $f(x+k)=\frac{pkx}{x+pk+k}f(x)$,
(c) $\ln f(x)$ *is convex.*

*Then, $f(x)=\Gamma_{p,k}(x)$.*

*Proof.* Define $\phi$ by $e^{\phi(x)} = \frac{f(x)}{\Gamma_{p,k}(x)}$ for $x > 0$, $p \in \mathbb{N}$ and $k > 0$. Then by (a) we obtain

$$e^{\phi(k)} = \frac{f(k)}{\Gamma_{p,k}(k)} = 1$$

implying that $\phi(k) = 0$. Also by (b), we obtain

$$e^{\phi(x+k)} = \frac{f(x+k)}{\Gamma_{p,k}(x+k)} = \frac{f(x)}{\Gamma_{p,k}(x)} = e^{\phi(x)}$$

which implies $\phi(x+k) = \phi(x)$. Thus $\phi(x)$ is periodic with period $k$.

Next we want to show that $\phi(x) = \ln f(x) - \ln \Gamma_{p,k}(x)$ is a constant. That is

$$\phi'(x) = 0 \quad \Leftrightarrow \quad \lim_{h\to 0} \frac{\phi(x+h) - \phi(x)}{h} = 0.$$

By (c) and Theorem 1, the functions $\ln f(x)$ and $\ln \Gamma_{p,k}(x)$ are convex. This implies $\ln f(x)$ and $\ln \Gamma_{p,k}(x)$ are continuous. Then for $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$|\ln f(x+h) - \ln f(x)| < \frac{|h|\varepsilon}{2} \quad \text{whenever} \quad |h| < \delta_1$$

and

$$|\ln \Gamma_{p,k}(x+h) - \ln \Gamma_{p,k}(x)| < \frac{|h|\varepsilon}{2} \quad \text{whenever} \quad |h| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for $|h| < \delta$, we have

$$\begin{aligned}
\left|\frac{\phi(x+h) - \phi(x)}{h}\right| &= \left|\frac{\ln f(x+h) - \ln \Gamma_{p,k}(x+h) - \ln f(x) + \ln \Gamma_{p,k}(x)}{h}\right| \\
&\leq \left|\frac{\ln f(x+h) - \ln f(x)}{h}\right| + \left|\frac{\ln \Gamma_{p,k}(x+h) - \ln \Gamma_{p,k}(x)}{h}\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

proving that $\phi'(x) = 0$. Since $\phi(x)$ is a constant and $\phi(k) = 0$, then $\phi(x) = 0$ for every $x$. Hence $e^0 = \frac{f(x)}{\Gamma_{p,k}(x)}$. Therefore $f(x) = \Gamma_{p,k}(x)$. □

**Theorem 3.** *The function, $B_{p,k}(x,y)$ as defined by (1.6) is logarithmically convex on $(0,\infty) \times (0,\infty)$.*

*Proof.* For $x, y > 0$, let $B_{p,k}(x,y)$ be defined as in (1.6). Then

$$\ln B_{p,k}(x,y) = \ln \Gamma_{p,k}(x) + \ln \Gamma_{p,k}(y) - \ln \Gamma_{p,k}(x+y).$$

Without loss of generality, let $y$ be fixed. Then,

$$(\ln B_{p,k}(x,y))'' = \psi'_{p,k}(x) - \psi'_{p,k}(x+y) > 0$$

since $\psi'_{p,k}(x)$ is decreasing for $x > 0$. This completes the proof. □

**Remark 4.** *Theorem 3 is a $(p,k)$-analogue of Theorem 6 of* [1].

**Corollary 1.** *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$\psi'_{p,k}(x)\psi'_{p,k}(y) \geq \left[\psi'_{p,k}(x) + \psi'_{p,k}(y)\right] \psi'_{p,k}(x+y) \tag{2.2}$$

*is valid for $x, y > 0$.*

*Proof.* This follows from the logarithmic convexity of $B_{p,k}(x, y)$. Let

$$\phi(x, y) = \ln B_{p,k}(x, y) = \ln \Gamma_{p,k}(x) + \ln \Gamma_{p,k}(y) - \ln \Gamma_{p,k}(x + y).$$

Since $\phi(x, y)$ is convex on $(0, \infty) \times (0, \infty)$, then its discriminant, $\Delta$ is positive semidefinite. That is,

$$\frac{\partial^2 \phi}{\partial x^2} > 0, \quad \Delta = \frac{\partial^2 \phi}{\partial x^2} \cdot \frac{\partial^2 \phi}{\partial y^2} - \left(\frac{\partial^2 \phi}{\partial x \partial y}\right)\left(\frac{\partial^2 \phi}{\partial y \partial x}\right) \geq 0$$

implying that

$$\left[\psi'_{p,k}(x) - \psi'_{p,k}(x+y)\right]\left[\psi'_{p,k}(y) - \psi'_{p,k}(x+y)\right] - \left[\psi'_{p,k}(x+y)\right]^2 \geq 0.$$

Thus,

$$\psi'_{p,k}(x)\psi'_{p,k}(y) - \left[\psi'_{p,k}(x) + \psi'_{p,k}(y)\right] \psi'_{p,k}(x+y) \geq 0$$

which completes the proof. □

**Theorem 4.** *Let $x, y > 0$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. Then*

$$\psi_{p,k}(\alpha x + \beta y) \geq \alpha \psi_{p,k}(x) + \beta \psi_{p,k}(y). \tag{2.3}$$

*Proof.* It suffices to show that $-\psi_{p,k}(x)$ is convex on $(0, \infty)$. By (1.5) we obtain

$$-\psi''_{p,k}(x) = \sum_{n=0}^{p} \frac{2}{(nk + x)^3} > 0.$$

Then (2.3) follows from Definition 1. □

**Theorem 5.** *Let $p \in \mathbb{N}$, $k > 0$ and $a > 0$. Then the function $Q(x) = a^x \Gamma_{p,k}(x)$ is convex on $(0, \infty)$.*

*Proof.* Recall that $\Gamma_{p,k}(x)$ is logarithmically convex. Thus,

$$\Gamma_{p,k}(\alpha x + \beta y) \leq (\Gamma_{p,k}(x))^{\alpha}(\Gamma_{p,k}(y))^{\beta}$$

for $x, y > 0$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. Then,

$$Q(\alpha x + \beta y) = a^{\alpha x + \beta y}\Gamma_{p,k}(\alpha x + \beta y) \leq a^{\alpha x + \beta y}(\Gamma_{p,k}(x))^{\alpha}(\Gamma_{p,k}(y))^{\beta}. \tag{2.4}$$

Also recall from the Young's inequality that

$$u^{\alpha} v^{\beta} \leq \alpha u + \beta v \tag{2.5}$$

for $u, v > 0$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. Let $u = a^x \Gamma_{p,k}(x)$ and $v = a^y \Gamma_{p,k}(y)$. Then (2.5) becomes

$$a^{\alpha x + \beta y}(\Gamma_{p,k}(x))^{\alpha}(\Gamma_{p,k}(y))^{\beta} \leq \alpha a^x \Gamma_{p,k}(x) + \beta a^y \Gamma_{p,k}(y) = \alpha Q(x) + \beta Q(y). \tag{2.6}$$

Combining (2.4) and (2.6) yields $Q(\alpha x + \beta y) \leq \alpha Q(x) + \beta Q(y)$ which concludes the proof. □

**Theorem 6.** *Let $p \in \mathbb{N}$ and $k > 0$. Then the functions $A(x) = x\psi_{p,k}(x)$ is strictly convex on $(0, \infty)$.*

*Proof.* Direct computations yield

$$A''(x) = 2\psi'_{p,k}(x) - x\psi''_{p,k}(x)$$

which by (1.5) implies

$$A''(x) = 2\sum_{n=0}^{p} \frac{1}{(nk+x)^2} - 2\sum_{n=0}^{p} \frac{x}{(nk+x)^3} = 2\sum_{n=0}^{p} \frac{nk}{(nk+x)^3} > 0.$$

Thus, $A(x)$ is convex. □

**Remark 5.** *Corollary 1 and Theorems 4, 5 and 6 provide generalizations for some results proved in* [14] *and* [6].

**Definition 3** ([12],[15])**.** *Let $f : I \subseteq (0, \infty) \to (0, \infty)$ be a continuous function. Then $f$ is said to be geometrically (or multiplicatively) convex on $I$ if any of the following conditions is satisfied.*

$$f(\sqrt{x_1 x_2}) \leq \sqrt{f(x_1)f(x_2)}, \tag{2.7}$$

*or more generally*

$$f\left(\prod_{i=1}^{n} x_i^{\lambda_i}\right) \leq \prod_{i=1}^{n} [f(x_i)]^{\lambda_i}, \quad n \geq 2 \tag{2.8}$$

*where $x_1, x_2, \ldots, x_n \in I$ and $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. If inequalities (2.7) and (2.8) are reversed, then $f$ is said to be geometrically (or multiplicatively) concave on $I$.*

**Lemma 2** ([12])**.** *Let $f : I \subseteq (0, \infty) \to (0, \infty)$ be a differentiable function. Then $f$ is a geometrically convex function if and only if the function $\frac{xf'(x)}{f(x)}$ is nondecreasing.*

**Lemma 3** ([12])**.** *Let $f : I \subseteq (0, \infty) \to (0, \infty)$ be a differentiable function. Then $f$ is a geometrically convex function if and only if the function $\frac{f(x)}{f(y)} \geq \left(\frac{x}{y}\right)^{\frac{yf'(y)}{f(y)}}$ holds for any $x, y \in I$.*

**Theorem 7.** *Let $f(x) = e^x \Gamma_{p,k}(x)$ for $p \in \mathbb{N}$ and $k \geq 1$. Then $f$ is geometrically convex and the inequality*

$$\frac{e^y}{e^x}\left(\frac{x}{y}\right)^{y\left[1+\psi_{p,k}(y)\right]} \leq \frac{\Gamma_{p,k}(x)}{\Gamma_{p,k}(y)} \leq \frac{e^y}{e^x}\left(\frac{x}{y}\right)^{x\left[1+\psi_{p,k}(x)\right]} \tag{2.9}$$

*is valid for $x > 0$ and $y > 0$..*

*Proof.* We proceed as follows.

$$\ln f(x) = x + \ln \Gamma_{p,k}(x) \quad \text{implying} \quad \frac{f'(x)}{f(x)} = 1 + \psi_{p,k}(x).$$

Then,

$$\begin{aligned}\left(\frac{xf'(x)}{f(x)}\right)' &= 1+\psi_{p,k}(x)+x\psi'_{p,k}(x)\\ &= 1+\frac{1}{k}\ln(pk)-\sum_{n=0}^{p}\frac{1}{nk+x}+\sum_{n=0}^{p}\frac{x}{(nk+x)^2}\\ &= 1+\frac{1}{k}\ln(pk)+\sum_{n=1}^{p}\left[\frac{x}{(nk+x)^2}-\frac{1}{nk+x}\right]\\ &= 1+\frac{1}{k}\ln(pk)-\sum_{n=1}^{p}\frac{nk}{(nk+x)^2}\\ &\triangleq h(x).\end{aligned}$$

Then $h'(x)=2\sum_{n=0}^{p}\frac{nk}{(nk+x)^3}>0$ implying that $h$ is increasing. Moreover,

$$\begin{aligned}h(0) &= 1+\frac{1}{k}\ln(pk)-\sum_{n=1}^{p}\frac{1}{nk}\\ &= 1+\frac{1}{k}\ln k+\frac{1}{k}\left(\ln p-\sum_{n=1}^{p}\frac{1}{n}\right)\\ &> 1+\frac{1}{k}\ln k-\frac{1}{k}>0\end{aligned}$$

since $\ln p-\sum_{n=1}^{p}\frac{1}{n}>-1$ (See eqn. (6) of [2]). Then for $x>0$, we have $h(x)>h(0)>0$. Thus $\frac{xf'(x)}{f(x)}$ is nondecreasing. Therefore, by Lemmas 2 and 3, $f$ is geometrically convex and as a result, $\frac{f(x)}{f(y)}\geq\left(\frac{x}{y}\right)^{\frac{yf'(y)}{f(y)}}$. Consequently, we obtain

$$\frac{e^x\Gamma_{p,k}(x)}{e^y\Gamma_{p,k}(y)}\geq\left(\frac{x}{y}\right)^{y\left[1+\psi_{p,k}(y)\right]} \tag{2.10}$$

and

$$\frac{e^y\Gamma_{p,k}(y)}{e^x\Gamma_{p,k}(x)}\geq\left(\frac{y}{x}\right)^{x\left[1+\psi_{p,k}(x)\right]}. \tag{2.11}$$

Now combining (2.10) and (2.11) yields the result (2.9) as required. □

**Remark 6.** *In particular, by replacing $x$ and $y$ respectively by $x+k$ and $x+\frac{k}{2}$, inequality (2.9) takes the form:*

$$\frac{1}{\sqrt{e^k}}\left(\frac{x+k}{x+\frac{k}{2}}\right)^{(x+\frac{k}{2})\left[1+\psi_{p,k}(x+\frac{k}{2})\right]}\leq\frac{\Gamma_{p,k}(x+k)}{\Gamma_{p,k}(x+\frac{k}{2})}\leq\frac{1}{\sqrt{e^k}}\left(\frac{x+k}{x+\frac{k}{2}}\right)^{(x+k)\left[1+\psi_{p,k}(x+k)\right]}. \tag{2.12}$$

**Remark 7.** *Theorem 7 gives a $(p,k)$-analogue of the previous results:* [3, Theorem 1], [15, Theorem 1.2, Corollary 1.5] *and* [6, Theorem 3.5]. *In particular, by letting $k=1$, we recover the result of* [6].

**Remark 8.** *Results of type (2.9) and (2.12) can also be found in* [9].

## 3. Inequalities involving the $(p,k)$-Riemann zeta function

**Definition 4.** *For $p \in \mathbb{N}$, $k>0$ and $x>0$, let $\zeta_{p,k}(x)$ be the $(p,k)$-analogue of the Riemann zeta function, $\zeta(x)$. Then $\zeta_{p,k}(x)$ is defined as*

$$\zeta_{p,k}(x)=\frac{1}{\Gamma_{p,k}(x)}\int_0^p \frac{t^{x-k}}{\left(1+\frac{t^k}{pk}\right)^p-1}\,dt, \quad x>k. \tag{3.1}$$

The functions $\zeta_{p,k}(x)$ satisfies the commutative diagram:

$$\begin{array}{ccc} \zeta_{p,k}(x) & \xrightarrow{p\to\infty} & \zeta_k(x) \\ \Big\downarrow{\scriptstyle k\to 1} & & \Big\downarrow{\scriptstyle k\to 1} \\ \zeta_p(x) & \xrightarrow[p\to\infty]{} & \zeta(x) \end{array}$$

where $\zeta_p(x)$ and $\zeta_k(x)$ respectively denote the $p$ and $k$ analogues of the Riemann zeta function. See [5] and [4] for instance.

**Lemma 4** ([7])**.** *Let $f$ and $g$ be two nonnegative functions of a real variable, and $m$, $n$ be real numbers such that the integrals in (3.2) exist. Then*

$$\int_a^b g(t)\left(f(t)\right)^m\,dt\cdot\int_a^b g(t)\left(f(t)\right)^n\,dt \geq \left(\int_a^b g(t)\left(f(t)\right)^{\frac{m+n}{2}}\,dt\right)^2. \tag{3.2}$$

**Theorem 8.** *Let $p\in\mathbb{N}$, $k>0$ and $x>0$. Then the inequality*

$$\frac{x+pk+k}{x+pk+2k}\cdot\frac{\zeta_{p,k}(x)}{\zeta_{p,k}(x+k)} \geq \frac{x}{x+k}\cdot\frac{\zeta_{p,k}(x+k)}{\zeta_{p,k}(x+2k)}, \quad x>k \tag{3.3}$$

*holds.*

*Proof.* Let $g(t)=\frac{1}{\left(1+\frac{t^k}{pk}\right)^p-1}$, $f(t)=t$, $m=x-k$, $n=x+k$, $a=0$ and $b=p$. Then (3.2) implies

$$\int_0^p \frac{t^{x-k}}{\left(1+\frac{t^k}{pk}\right)^p-1}\,dt\cdot\int_0^p \frac{t^{x+k}}{\left(1+\frac{t^k}{pk}\right)^p-1}\,dt \geq \left(\int_0^p \frac{t^{x}}{\left(1+\frac{t^k}{pk}\right)^p-1}\,dt\right)^2$$

which by relation (3.1) gives

$$\zeta_{p,k}(x)\Gamma_{p,k}(x)\cdot\zeta_{p,k}(x+2k)\Gamma_{p,k}(x+2k) \geq \left(\zeta_{p,k}(x+k)\Gamma_{p,k}(x+k)\right)^2. \tag{3.4}$$

Then by the functional equation (1.3), inequality (3.4) can be rearranged to obtain the desired result (3.3). □

**Remark 9.** (1)

(i) *By letting $p \to \infty$ in (3.3), we obtain the result of Theorem 3.1 of* [4].
(ii) *By setting $k = 1$ in (3.3), we obtain the result of Theorem 6 of* [5].
(iii) *By letting $p \to \infty$ and $k = 1$ in (3.3), we obtain the result of Theorem 2.2 of* [7].

**Theorem 9.** *Let $p \in \mathbb{N}$ and $k > 0$. Then for $x > k$, $y > k$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ such that $\frac{x}{\alpha} + \frac{y}{\beta} > k$, the inequality*

$$\frac{\Gamma_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right)}{(\Gamma_{p,k}(x))^{\frac{1}{\alpha}} (\Gamma_{p,k}(y))^{\frac{1}{\beta}}} \leq \frac{\left(\zeta_{p,k}(x)\right)^{\frac{1}{\alpha}} \left(\zeta_{p,k}(y)\right)^{\frac{1}{\beta}}}{\zeta_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right)} \tag{3.5}$$

*holds.*

*Proof.* We employ the Hölder's inequality:

$$\int_a^b f(t)g(t)\, dt \leq \left(\int_a^b (f(t))^{\alpha}\, dt\right)^{\frac{1}{\alpha}} \left(\int_a^b (g(t))^{\beta}\, dt\right)^{\frac{1}{\beta}} \tag{3.6}$$

where $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Let $f(t) = \frac{t^{\frac{x-k}{\alpha}}}{\left(\left(1+\frac{t^k}{pk}\right)^p - 1\right)^{\frac{1}{\alpha}}}$, $g(t) = \frac{t^{\frac{y-k}{\beta}}}{\left(\left(1+\frac{t^k}{pk}\right)^p - 1\right)^{\frac{1}{\beta}}}$, $a = 0$ and $b = p$. Then (3.6) implies

$$\int_0^p \frac{t^{\frac{x}{\alpha}+\frac{y}{\beta}-k}}{\left(1+\frac{t^k}{pk}\right)^p - 1}\, dt \leq \left(\int_0^p \frac{t^{x-k}}{\left(1+\frac{t^k}{pk}\right)^p - 1}\, dt\right)^{\frac{1}{\alpha}} \left(\int_0^p \frac{t^{y-k}}{\left(1+\frac{t^k}{pk}\right)^p - 1}\, dt\right)^{\frac{1}{\beta}}.$$

By relation (3.1) we obtain

$$\Gamma_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right)\zeta_{p,k}\left(\frac{x}{\alpha} + \frac{y}{\beta}\right) \leq \left(\Gamma_{p,k}(x)\zeta_{p,k}(x)\right)^{\frac{1}{\alpha}} \left(\Gamma_{p,k}(y)\zeta_{p,k}(y)\right)^{\frac{1}{\beta}}$$

which when rearranged gives (3.5) as required. □

**Remark 10.** (1)

(i) *By letting $p \to \infty$ in (3.5), we obtain the result of Theorem 3.3 of* [4].
(ii) *By letting $p \to \infty$ and $k = 1$ in (3.5), we obtain the result of Theorem 7 of* [5].
(iii) *In particular, let $k = 1$ in (3.5). Then by replacing $x$ and $y$ respectively by $x - 1$ and $y + 1$, we obtain*

$$\frac{\Gamma_p\left(\frac{x-1}{\alpha} + \frac{y+1}{\beta}\right)}{(\Gamma_p(x-1))^{\frac{1}{\alpha}} (\Gamma_p(y+1))^{\frac{1}{\beta}}} \leq \frac{\left(\zeta_p(x-1)\right)^{\frac{1}{\alpha}} \left(\zeta_p(y+1)\right)^{\frac{1}{\beta}}}{\zeta_p\left(\frac{x-1}{\alpha} + \frac{y+1}{\beta}\right)}$$

*which corresponds to Theorem 2.7 of* [8].

**Lemma 5** ([11])**.** *Let $f:(0,\infty)\to(0,\infty)$ be a differentiable, logarithmically convex function. Then the function*

$$g(x)=\frac{(f(x))^\alpha}{f(\alpha x)},\quad \alpha\geq 1$$

*is decreasing on its domain.*

**Lemma 6.** *Let $p\in\mathbb{N}$, $k>0$ and $\alpha\geq 1$. Then the inequality*

$$\frac{[\Gamma_{p,k}(y+k)]^\alpha}{\Gamma_{p,k}(\alpha y+k)}\leq\frac{[\Gamma_{p,k}(x+k)]^\alpha}{\Gamma_{p,k}(\alpha x+k)}\leq 1 \tag{3.7}$$

*holds for $0\leq x\leq y$.*

*Proof.* Note that the function $f(x)=\Gamma_{p,k}(x+k)$ is differentiable and logarithmically convex. Then by Lemma 5, $G(x)=\frac{[\Gamma_{p,k}(x+k)]^\alpha}{\Gamma_{p,k}(\alpha x+k)}$ is decreasing and for $0\leq x\leq y$, we have $G(y)\leq G(x)\leq G(0)$ yielding the result. □

**Theorem 10.** *Let $p\in\mathbb{N}$, $k>0$ and $\alpha\geq 1$. Then the inequality*

$$\frac{\big[\Gamma_{p,k}(y+k)\zeta_{p,k}(y+k)\big]^\alpha}{\Gamma_{p,k}(\alpha y+k)\zeta_{p,k}(\alpha y+k)}\leq\frac{\big[\zeta_{p,k}(x+k)\big]^\alpha}{\zeta_{p,k}(\alpha x+k)} \tag{3.8}$$

*is satisfied for $0<x\leq y$.*

*Proof.* Let $H$ be defined $x>0$ by

$$H(x)=\Gamma_{p,k}(x+k)\zeta_{p,k}(x+k)=\int_0^p\frac{t^x}{\left(1+\frac{t^k}{pk}\right)^p-1}\,dt. \tag{3.9}$$

Then for $x,y>0$ and $a,b>0$ such that $a+b=1$, we have

$$\begin{aligned}
H(ax+by)&=\int_0^p\frac{t^{ax+by}}{\left(1+\frac{t^k}{pk}\right)^p-1}\,dt\\
&=\int_0^p\frac{t^{ax}}{\left(\left(1+\frac{t^k}{pk}\right)^p-1\right)^a}\cdot\frac{t^{by}}{\left(\left(1+\frac{t^k}{pk}\right)^p-1\right)^b}\,dt\\
&\leq\left(\int_0^p\frac{t^x}{\left(1+\frac{t^k}{pk}\right)^p-1}\,dt\right)^a\left(\int_0^p\frac{t^y}{\left(1+\frac{t^k}{pk}\right)^p-1}\,dt\right)^b\\
&=(H(x))^a(H(y))^b.
\end{aligned}$$

Therefore, $H(x)$ is logarithmically convex. Then by Lemma 5, the function

$$T(x)=\frac{\big[\Gamma_{p,k}(x+k)\zeta_{p,k}(x+k)\big]^\alpha}{\Gamma_{p,k}(\alpha x+k)\zeta_{p,k}(\alpha x+k)}$$

is decreasing. Hence for $0 < x \leq y$, we have

$$\frac{\left[\Gamma_{p,k}(y+k)\zeta_{p,k}(y+k)\right]^{\alpha}}{\Gamma_{p,k}(\alpha y+k)\zeta_{p,k}(\alpha y+k)} \leq \frac{\left[\Gamma_{p,k}(x+k)\zeta_{p,k}(x+k)\right]^{\alpha}}{\Gamma_{p,k}(\alpha x+k)\zeta_{p,k}(\alpha x+k)}.$$

Then by the right hand side of (3.7), we obtain

$$\frac{\left[\Gamma_{p,k}(y+k)\zeta_{p,k}(y+k)\right]^{\alpha}}{\Gamma_{p,k}(\alpha y+k)\zeta_{p,k}(\alpha y+k)} \leq \frac{\left[\zeta_{p,k}(x+k)\right]^{\alpha}}{\zeta_{p,k}(\alpha x+k)}$$

concluding the proof. □

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