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SOME RESULTS ON THE COMPARATIVE GROWTH ANALYSIS OF ENTIRE FUNCTIONS UNDER THE TREATMENT OF THEIR MAXIMUM TERMS AND GENERALIZED RELATIVE L^*-ORDERS

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Abstract. In this paper we estimate some comparative growth properties of composition of entire functions in terms of their maximum terms on the basis of their generalized relative L^* order (respectively generalized relative L^* lower order) with respect to another entire function.

1. Introduction, Definitions and Notations

The value distribution theory deals with various aspects of the behavior of entire functions one of which is the study of comparative growth properties. For any entire function f defined in the open complex plane $\mathbb{C}$, $M_f(r)$, a function of r is defined as follows:

$$M_f(r) = \max_{|z|=r} |f(z)| \ .$$

If f is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f^{-1}(r) : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_f^{-1}(s) = \infty$.

An entire function f has an everywhere convergent power series expansion as

$$f = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

The maximum term $\mu_f(r)$ of f can be defined in the following way:

$$\mu_f(r) = \max_{n\geq 0} (|a_n| r^n) \ .$$

In fact $\mu_f(r)$ is much weaker than $M_f(r)$ in some sense. For another entire function g, $\mu_g(r)$ is also defined and the ratio $\frac{\mu_f(r)}{\mu_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g interms of their maximum term.

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Bernal [1] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^{\mu}) \text{ for all } r > r_0(\mu) > 0\}\\ &= \limsup_{r\to\infty}\frac{\log M_g^{-1}M_f(r)}{\log r}\ .\end{aligned}$$

Similarly, one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \liminf_{r\to\infty}\frac{\log M_g^{-1}M_f(r)}{\log r}\ .$$

If we consider $g(z) = \exp z$, the above definition coincides with the classical definition { cf. [12] } of order (lower order) of an entire function f which is as follows:

Definition 1. *The order ρ_f and the lower order λ_f of an entire function f are defined as*

$$\rho_f = \limsup_{r\to\infty}\frac{\log^{[2]} M_f(r)}{\log r} \textit{ and } \lambda_f = \liminf_{r\to\infty}\frac{\log^{[2]} M_f(r)}{\log r},$$

where

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right), k = 1, 2, 3, ... \textit{and } \log^{[0]} x = x\ .$$

Using the inequalities $\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R)$ $\{cf.\ [11]\}$, for $0 \leq r < R$ one may give an alternative definition of the order ρ_f and the lower order λ_f of an entire function f in the following manner:

$$\rho_f = \limsup_{r\to\infty}\frac{\log^{[2]}\mu_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty}\frac{\log^{[2]}\mu_f(r)}{\log r}\ .$$

Lahiri and Banerjee [7] gave a more generalized concept of relative order in the following way:

Definition 2. [7] *If $k \geq 1$ is a positive integer, then the k- th generalized relative order of f with respect to g, denoted by $\rho_g^{[k]}(f)$ is defined by*

$$\begin{aligned}\rho_g^{[k]}(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g\left(\exp^{[k-1]} r^{\mu}\right) \textit{ for all } r > r_0(\mu) > 0\right\}\\ &= \limsup_{r\to\infty}\frac{\log^{[k]} M_g^{-1}M_f(r)}{\log r}\ .\end{aligned}$$

Clearly $\rho_g^1(f) = \rho_g(f)$ and $\rho_{\exp z}^1(f) = \rho_f$.

Likewise one can define the generalized relative lower order of f with respect to g denoted by $\lambda_g^{[k]}(f)$ as

$$\lambda_g^{[k]}(f) = \liminf_{r\to\infty} \frac{\log^{[k]} M_g^{-1} M_f(r)}{\log r} .$$

Now let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. Singh and Barker [8] defined it in the following way:

Definition 3. [8] *A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon\,(>0)$,*

$$\frac{1}{k^{\varepsilon}} \le \frac{L(kr)}{L(r)} \le k^{\varepsilon} \text{ for } r \ge r(\varepsilon) \text{ and}$$

uniformly for $k\,(\ge 1)$.

Somasundaram and Thamizharasi [9] introduced the notions of L-order for entire function where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalised concept for L-order for entire function is L^*-order and its definition is as follows:

Definition 4. [9] *The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} .$$

In view of the inequalities $\mu_f(r) \le M_f(r) \le \frac{R}{R-r}\mu_f(R)$ $\{cf.$ [11]$\}$, for $0 \le r < R$ one may verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} \mu_f(r)}{\log\left[re^{L(r)}\right]} .$$

In the line of Somasundaram and Thamizharasi [9] and Bernal [1], Datta and Biswas [2] gave the definition of relative L^*-order of an entire function in the following way:

Definition 5. [2] *The relative L^*-order of an entire function f with respect to another entire function g , denoted by $\rho_g^{L^*}(f)$ in the following way*

$$\begin{aligned} \rho_g^{L^*}(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g\left\{re^{L(r)}\right\}^{\mu} \text{ for all } r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]} . \end{aligned}$$

Similarly, one can define the relative L^*-lower order of f with respect to g denoted by $\lambda_g^{L^*}(f)$ as follows :

$$\lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log\left[re^{L(r)}\right]} .$$

In the case of relative L^*-order (relative L^*-lower order) , it therefore seems reasonable to define suitably an alternative definition of relative L^*-order (relative L^*-lower order) of entire function in terms of its maximum terms. Datta , Biswas and Ali [4] also introduced such definition in the following way:

Definition 6. [4] *The relative order $\rho_g^{L^*}(f)$ and the relative lower order $\lambda_g(f)$ of an entire function f with respect to another entire function g are defined as*

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1} \mu_f(r)}{\log\left[re^{L(r)}\right]} .$$

Similarly in the line of Lahiri and Banerjee [7], Biswas and Ali [4] one can define the generalized relative L^*-order and generalized relative L^*-lower order of an entire function in the following way :

Definition 7. *Let k be an integer ≥ 1. The generalized relative L^*-order and generalized relative L^*- lower order of an entire function f with respect to another entire function g , denoted respectively by $\rho_g^{[k]L^*}(f)$ and $\lambda_g^{[k]L^*}(f)$ are defined in the following way*

$$\rho_g^{[k]L^*}(f) = \limsup_{r\to\infty} \frac{\log^{[k]} \mu_g^{-1} \mu_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_g^{[k]L^*}(f) = \liminf_{r\to\infty} \frac{\log^{[k]} \mu_g^{-1} \mu_f(r)}{\log\left[re^{L(r)}\right]} .$$

In this paper we will establish some results related to the growth rates of composite entire functions in terms of their maximum terms on the basis of generalized relative L^*-order (generalized relative L^*-lower order). Also we extend some results of Datta et al. {[5], [6]}. We do not explain the standard definitions and notations in the theory of entire functions since those are available in [13].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [10] *Let f and g be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \leq \frac{\alpha}{\alpha - 1} \mu_f\left(\frac{\alpha R}{R - r} \mu_g(R)\right) .$$

Lemma 2. [10] *If f and g are any two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \geq \frac{1}{2} \mu_f\left(\frac{1}{8} \mu_g\left(\frac{r}{4}\right) - |g(0)|\right) .$$

Lemma 3. [3] *If f be an entire function and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,*

$$\mu_f(\alpha r) \geq \beta \mu_f(r) \ .$$

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f, g and h be any three entire functions such that*

$$\begin{aligned} (i)\ \limsup_{r\to\infty} \frac{\log^{[k]} \mu_h^{-1}\left(\mu_g(r)\right)}{\left(\log re^{L(r)}\right)^{\alpha}} &= A,\ \textit{a real number } > 0, \\ (ii)\ \liminf_{r\to\infty} \frac{\log^{[k]} \mu_h^{-1}\left(\mu_f(r)\right)}{\left(\log^{[k]} \mu_h^{-1}(r)\right)^{\beta+1}} &= B,\ \textit{a real number } > 0 \end{aligned}$$

and $g(0) = 0$ for any pair of α, β satisfying $0 < \alpha < 1$, $\beta > 0$ and $\alpha(\beta + 1) > 1$. Then

$$\rho_h^{[k]L^*}(f \circ g) = \infty \ ,$$

where $k = 2, 3, 4 \cdots$.

Proof. From (i), we get for a sequence of values of r tending to infinity that

$$\log^{[k]} \mu_h^{-1}\left(\mu_g(r)\right) \geq (A - \varepsilon)\left(\log re^{L(r)}\right)^{\alpha} \tag{1}$$

and from (ii), it follows for all sufficiently large values of r that

$$\log^{[k]} \mu_h^{-1}\left(\mu_f(r)\right) \geq (B - \varepsilon)\left(\log^{[k]} \mu_h^{-1}(r)\right)^{\beta+1} \ .$$

As $\mu_g(r)$ is continuous, increasing and unbounded function of r, we obtain from above for all sufficiently large values of r that

$$\log^{[k]} \mu_h^{-1}\left(\mu_f(\mu_g(r))\right) \geq (B - \varepsilon)\left(\log^{[k]} \mu_h^{-1}\left(\mu_g(r)\right)\right)^{\beta+1} \ . \tag{2}$$

Since $\mu_h^{-1}(r)$ is an increasing function of r, we have from Lemma 2, Lemma 3, equations (1) and (2) for a sequence of values of r tending to infinity that

$$\begin{aligned}
\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r) &\geq \log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{2}\right)\right)\right\}\\
i.e.,\ \log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r) &\geq \log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}\\
i.e.,\ \log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r) &\geq (B-\varepsilon)\left(\log^{[k]}\mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)\right)^{\beta+1}\\
i.e.,\ \log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r) &\geq (B-\varepsilon)\left[(A-\varepsilon)\left(\log\left(\frac{r}{100}\right)e^{L\left(\frac{r}{100}\right)}\right)^{\alpha}\right]^{\beta+1}\\
i.e.,\ \log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r) &\geq (B-\varepsilon)(A-\varepsilon)^{\beta+1}\left(\log\left(\frac{r}{100}\right)e^{L\left(\frac{r}{100}\right)}\right)^{\alpha(\beta+1)}\\
i.e.,\ \frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]} &\geq \frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}\left[\log\left(\frac{r}{100}\right)e^{L\left(\frac{r}{100}\right)}\right]^{\alpha(\beta+1)}}{\log\left[re^{L(r)}\right]}
\end{aligned}$$

$$\begin{aligned}
& i.e.,\ \limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\\
\geq\ & \liminf_{r\to\infty}\frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}\left[\log re^{L(r)}+O(1)\right]^{\alpha(\beta+1)}}{\log\left[re^{L(r)}\right]}\ .
\end{aligned}$$

As $\varepsilon\,(>0)$ is arbitrary and $\alpha(\beta+1)>1$, it follows from above that

$$\rho_h^{[k]L^*}(f\circ g)=\infty\ .$$

Thus the theorem follows. □

In the line of Theorem 1, one may state the following two theorems without their proofs :

Theorem 2. *Let f, g and h be any three entire functions such that*

$$\begin{aligned}
\liminf_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left(\mu_g(r)\right)}{\left(\log re^{L(r)}\right)^{\alpha}} &= A,\ \text{a real number } >0,\\
\limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left(\mu_f(r)\right)}{\left(\log^{[k]}\mu_h^{-1}(r)\right)^{\beta+1}} &= B,\ \text{a real number } >0,
\end{aligned}$$

and $g(0)=0$ for any pair of α,β satisfying $0<\alpha<1,\ \beta>0$ and $\alpha(\beta+1)>1$. Then

$$\rho_h^{[k]L^*}(f\circ g)=\infty,$$

where $k=2,3,4\cdots$.

Theorem 3. *Let f, g and h be any three entire functions such that*

$$\liminf_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left(\mu_g(r)\right)}{\left(\log re^{L(r)}\right)^{\alpha}} = A, \text{ a real number } >0,$$

$$\liminf_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left(\mu_f(r)\right)}{\left(\log^{[k]}\mu_h^{-1}(r)\right)^{\beta+1}} = B, \text{ a real number } >0,$$

and $g(0)=0$ for any pair of α,β satisfying $0<\alpha<1,\ \beta>0$ and $\alpha(\beta+1)>1$. Then

$$\lambda_h^{[k]L^*}(f\circ g)=\infty,$$

where $k=2,3,4\cdots$.

Theorem 4. *Let f, g and h be any three entire functions such that*

$$(i)\ \limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left(\mu_g(r)\right)}{\left(\log^{[2]}r\right)^{\alpha}} = A, \text{ a real number } >0,$$

$$(ii)\ \liminf_{r\to\infty}\frac{\log\left[\frac{\log^{[k]}\mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[k]}\mu_h^{-1}(r)}\right]}{\left[\log^{[k]}\mu_h^{-1}(r)\right]^{\beta}} = B, \text{ a real number } >0$$

and $g(0)=0$ for any pair of α,β satisfying $\alpha>1,\ 0<\beta<1$ and $\alpha\beta>1$. Then

$$\rho_h^{[k]L^*}(f\circ g)=\infty,$$

where $k=2,3,4\cdots$.

Proof. From (i), we get for a sequence of values of r tending to infinity that

$$\log^{[k]}\mu_h^{-1}\left(\mu_g(r)\right)\geq(A-\varepsilon)\left(\log^{[2]}r\right)^{\alpha} \tag{3}$$

and from (ii), we obtain for all sufficiently large values of r that

$$\begin{aligned}\log\left[\frac{\log^{[k]}\mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[k]}\mu_h^{-1}(r)}\right] &\geq (B-\varepsilon)\left[\log^{[k]}\mu_h^{-1}(r)\right]^{\beta}\\ i.e.,\ \frac{\log^{[k]}\mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[k]}\mu_h^{-1}(r)} &\geq \exp\left[(B-\varepsilon)\left[\log^{[k]}\mu_h^{-1}(r)\right]^{\beta}\right].\end{aligned}$$

As $\mu_g(r)$ is continuous, increasing and unbounded function of r, we have from above for all sufficiently large values of r that

$$\frac{\log^{[k]}\mu_h^{-1}\left(\mu_f(\mu_g(r))\right)}{\log^{[k]}\mu_h^{-1}\left(\mu_g(r)\right)}\geq\exp\left[(B-\varepsilon)\left[\log^{[k]}\mu_h^{-1}\left(\mu_g(r)\right)\right]^{\beta}\right]. \tag{4}$$

Further $\mu_h^{-1}(r)$ is increasing function of r, it follows from Lemma 2, Lemma 3, equations (3) and (4) for a sequence of values of r tending to infinity that

$$\frac{\log^{[k]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log\left[re^{L(r)}\right]} \geq \frac{\log^{[k]} \mu_h^{-1}\left\{\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r}{4}\right)\right)\right\}}{\log\left[re^{L(r)}\right]}$$

$$i.e., \ \frac{\log^{[k]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log\left[re^{L(r)}\right]} \geq \frac{\log^{[k]} \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log\left[re^{L(r)}\right]}$$

$$\begin{aligned} & i.e., \ \frac{\log^{[k]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log\left[re^{L(r)}\right]} \\ \geq \quad & \frac{\log^{[k]} \mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log^{[k]} \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)} \cdot \frac{\log^{[k]} \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)}{\log\left[re^{L(r)}\right]} \end{aligned}$$

$$\begin{aligned} & i.e., \ \frac{\log^{[k]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log\left[re^{L(r)}\right]} \\ \geq \quad & \exp\left[(B-\varepsilon)\left[\log^{[k]} \mu_h^{-1}\left(\mu_g\left(\frac{r}{100}\right)\right)\right]^{\beta}\right] \cdot \frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log\left[re^{L(r)}\right]} \\ & i.e., \ \frac{\log^{[k]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log\left[re^{L(r)}\right]} \\ \geq \quad & \exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta}\right] \cdot \frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log\left[re^{L(r)}\right]} \end{aligned}$$

$$\begin{aligned} & i.e., \ \frac{\log^{[k]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log\left[re^{L(r)}\right]} \\ \geq \quad & \exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha\beta-1}\log^{[2]}\left(\frac{r}{100}\right)\right] \cdot \frac{(A-\varepsilon)\left(\log^{[2]}\left(\frac{r}{100}\right)\right)^{\alpha}}{\log\left[re^{L(r)}\right]} \end{aligned}$$

$$i.e., \ \frac{\log^{[k]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log \left[r e^{L(r)} \right]}$$

$$\geq \left(\log \left(\frac{r}{100} \right) \right)^{(B-\varepsilon)(A-\varepsilon)^{\beta} \left(\log^{[2]} \left(\frac{r}{100} \right) \right)^{\alpha\beta - 1}} \cdot \frac{(A-\varepsilon) \left(\log^{[2]} \left(\frac{r}{100} \right) \right)^{\alpha}}{\log \left[r e^{L(r)} \right]}$$

$$i.e., \ \limsup_{r \to \infty} \frac{\log^{[k]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log \left[r e^{L(r)} \right]}$$

$$\geq \liminf_{r \to \infty} \left(\log \left(\frac{r}{100} \right) \right)^{(B-\varepsilon)(A-\varepsilon)^{\beta} \left(\log^{[2]} \left(\frac{r}{100} \right) \right)^{\alpha\beta - 1}} \cdot \frac{(A-\varepsilon) \left(\log^{[2]} \left(\frac{r}{100} \right) \right)^{\alpha}}{\log \left[r e^{L(r)} \right]} .$$

Since $\varepsilon \, (> 0)$ is arbitrary and $\alpha > 1$, $\alpha\beta > 1$, the theorem follows from above. □

In the line of Theorem 4, one may also state the following two theorems without their proofs :

Theorem 5. *Let f, g and h be any three entire functions such that*

$$\liminf_{r \to \infty} \frac{\log^{[k]} \mu_h^{-1} \left(\mu_g(r) \right)}{\left(\log^{[2]} r \right)^{\alpha}} = A, \ \text{a real number } > 0,$$

$$\limsup_{r \to \infty} \frac{\log \left[\frac{\log^{[k]} \mu_h^{-1} \left(\mu_f(r) \right)}{\log^{[k]} \mu_h^{-1}(r)} \right]}{\left[\log^{[k]} \mu_h^{-1}(r) \right]^{\beta}} = B, \ \text{a real number } > 0$$

and $g(0) = 0$ for any pair of α, β with $\alpha > 1$, $0 < \beta < 1$ and $\alpha\beta > 1$. Then

$$\rho_h^{[k]L^*}(f \circ g) = \infty,$$

where $k = 2, 3, 4 \cdots$.

Theorem 6. *Let f, g and h be any three entire functions such that*

$$\liminf_{r \to \infty} \frac{\log^{[k]} \mu_h^{-1} \left(\mu_g(r) \right)}{\left(\log^{[2]} r \right)^{\alpha}} = A, \ \text{a real number } > 0,$$

$$\liminf_{r \to \infty} \frac{\log \left[\frac{\log^{[k]} \mu_h^{-1} \left(\mu_f(r) \right)}{\log^{[k]} \mu_h^{-1}(r)} \right]}{\left[\log^{[k]} \mu_h^{-1}(r) \right]^{\beta}} = B, \ \text{a real number } > 0$$

and $g(0) = 0$ for any pair of α, β satisfying $\alpha > 1$, $0 < \beta < 1$ and $\alpha\beta > 1$. Then

$$\lambda_h^{[k]L^*}(f \circ g) = \infty,$$

where $k = 2, 3, 4 \cdots$.

Theorem 7. *Let f, g and h be any three entire functions such that* $0<\lambda_h^{[k]L^*}(g)\leq\rho_h^{[k]L^*}(g)<\infty$ *where* $k=2,3,4\cdots$, $g(0)=0$ *and*

$$\limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[k]}\mu_h^{-1}(r)}=A,\ \textit{a real number }<\infty.$$

Then

$$\lambda_h^{[k]L^*}(f\circ g)\leq A\cdot\lambda_h^{[k]L^*}(g)\ \textit{ and }\rho_h^{[k]L^*}(f\circ g)\leq A\cdot\rho_h^{[k]L^*}(g)\ .$$

Proof. Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 1 for all sufficiently large values of r that

$$\frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\leq\frac{\log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$
$$\leq\ \frac{\log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}}{\log^{[k]}\mu_h^{-1}\left(\mu_g(26r)\right)}\cdot\frac{\log^{[k]}\mu_h^{-1}\left(\mu_g(26r)\right)}{\log\left[re^{L(r)}\right]}\tag{5}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$
$$\leq\ \liminf_{r\to\infty}\left[\frac{\log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}}{\log^{[k]}\mu_h^{-1}\left(\mu_g(26r)\right)}\cdot\frac{\log^{[k]}\mu_h^{-1}\left(\mu_g(26r)\right)}{\log\left[re^{L(r)}\right]}\right]$$
$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$
$$\leq\ \limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}}{\log^{[k]}\mu_h^{-1}\left(\mu_g(26r)\right)}\cdot\liminf_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left(\mu_g(26r)\right)}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \lambda_h^{[k]L^*}(f\circ g)\leq A\cdot\lambda_h^{[k]L^*}(g)\ .\tag{6}$$

Also from (5), we obtain for all sufficiently large values of r that

$$\limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$
$$\leq\ \limsup_{r\to\infty}\left[\frac{\log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\mu_g(26r)\right)\right\}}{\log^{[k]}\mu_h^{-1}\left(\mu_g(26r)\right)}\cdot\frac{\log^{[k]}\mu_h^{-1}\left(\mu_g(26r)\right)}{\log\left[re^{L(r)}\right]}\right]$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$

$$\leq \limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(26r\right)\right)\right\}}{\log^{[k]}\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}\cdot\limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left(\mu_g\left(26r\right)\right)}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \rho_h^{[k]L^*}(fog)\leq A\cdot\rho_h^{[k]L^*}(g)\ . \tag{7}$$

Therefore the theorem follows from (6) and (7) . □

Theorem 8. *Let f, g and h be any three entire functions such that $0<\lambda_h^{[k]L^*}(g)<\infty$ where $k=2,3,4\cdots,$ $g(0)=0$ and*

$$\limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[k]}\mu_h^{-1}(r)}=A,\ \text{a real number }<\infty.$$

Then

$$\rho_h^{[k]L^*}(f\circ g)\geq B\cdot\lambda_h^{[k]L^*}(g)\ .$$

Proof. Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2 for all sufficiently large values of r that

$$\frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}\geq\frac{\log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$

$$\geq \frac{\log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log^{[k]}\mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)}\cdot\frac{\log^{[k]}\mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$

$$\geq \limsup_{r\to\infty}\left[\frac{\log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log^{[k]}\mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)}\cdot\frac{\log^{[k]}\mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)}{\log\left[re^{L(r)}\right]}\right]$$

$$i.e.,\ \limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\mu_{f\circ g}(r)}{\log\left[re^{L(r)}\right]}$$

$$\geq \limsup_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left\{\mu_f\left(\mu_g\left(\frac{r}{100}\right)\right)\right\}}{\log^{[k]}\mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)}\cdot\liminf_{r\to\infty}\frac{\log^{[k]}\mu_h^{-1}\left(\mu M_g\left(\frac{r}{100}\right)\right)}{\log\left[re^{L(r)}\right]}$$

$$i.e.,\ \rho_h^{[k]L^*}(f\circ g)\geq B\cdot\lambda_h^{[k]L^*}(g)\ .$$

Thus the theorem follows. □

Theorem 9. *Let f, g and h be any three entire functions such that* $0 < \lambda_h^{[k]L^*}(g) \leq \rho_h^{[k]L^*}(g) < \infty$ *where* $k = 2, 3, 4 \cdots$, $g(0) = 0$ *and*

$$\liminf_{r\to\infty} \frac{\log^{[k]} \mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[k]} \mu_h^{-1}(r)} = B, \text{ a real number } < \infty.$$

Then

$$\lambda_h^{[k]L^*}(f \circ g) \leq B \cdot \rho_h^{[k]L^*}(g) \ .$$

Theorem 10. *Let f, g and h be any three entire functions such that* $0 < \rho_h^{[k]L^*}(g) < \infty$ *where* $k = 2, 3, 4 \cdots$, $g(0) = 0$ *and*

$$\limsup_{r\to\infty} \frac{\log^{[k]} \mu_h^{-1}\left(\mu_f(r)\right)}{\log^{[k]} \mu_h^{-1}(r)} = A, \text{ a real number } < \infty.$$

Then

$$\rho_h^{[k]L^*}(f \circ g) \geq A \cdot \rho_h^{[k]L^*}(g) \ .$$

The proof of Theorem 9 and Theorem 10 are omitted because those can be carried out in the line of Theorem 7 and Theorem 8, respectively.

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