Article electronically published on March 08, 2017

Commun.Fac.Sci.Univ.Ank.Series A1
Volume 66, Number 2, Pages 253–262 (2017)
DOI: 10.1501/Commua1_0000000816
ISSN 1303–5991
http://communications.science.ankara.edu.tr/index.php?series=A1

ON A LIPSCHITZ STABILITY PROBLEM FOR $p-$LAPLACIAN BESSEL EQUATION

TUBA GULSEN, EMRAH YILMAZ, AND E. S. PANAKHOV

Abstract. In this study, we are enunciative of some asymptotic expansions and reconstruction formulas for inverse nodal problem of p-Laplacian Bessel equation. Furthermore, Lipschitz stability problem for this equation with Dirichlet boundary conditions is solved. And, it is also proved that the space of all potential functions w is homeomorphic to the partition set of all asymptotically equivalent nodal sequences induced by an equivalence relation.

1. Introduction

Let us consider the below $p-$Laplacian Bessel eigenvalue problem;

$$-\left(u^{'(p-1)}\right)'=(p-1)\left(\lambda-w_0(x)-\frac{l(l+1)}{x^2}\right)u^{(p-1)},\ 1\leq x\leq a, \tag{1.1}$$

with Dirichlet conditions

$$u(1)=u(a)=0, \tag{1.2}$$

where $l=0,1,2,...,$ $a,p>1$ are constants, λ is a spectral parameter; $w_0(x)\in L^2[1,a]$ is a real valued function and $u^{(p-1)}=|u|^{(p-1)}\,sgnu$ (see [1], [2]). Normally, the equation (1.1) is considered by a condition at the origin. In this instance, the problem becomes singular and it is not easy to solve inverse nodal problem for these type equations in $p-$Laplacian case. Therefore, we will study the Lipschitz stability of inverse nodal problem for $p-$Laplacian Bessel equation on a smooth interval.

Uniqueness and reconstruction problems of $p-$Laplacian Bessel equation have been considered in some works (see [2], [3]) just left a stability issue is worth considering and undone for eigenvalue problem (1.1)-(1.2). In an exact solution of inverse problems, the questions of existence, uniqueness, stability and construction are to be taken into account. The query of existence and uniqueness is of major

Received by the editors: August 02, 2016; Accepted: February 15, 2017.

2010 *Mathematics Subject Classification.* 34A55, 34L05, 34L20.

Key words and phrases. $p-$Laplacian Bessel equation, Inverse nodal problem, Lipschitz stability.

importance in testing the assumption behind any mathematical model. If the response in the uniqueness question is no, then we know that even perfect data do not contain enough data to recover the physical quantity to be estimated. In the query of stability, we should decide whether the solution depends continuously on the data. Stability is essential if we want to be sure that a variation of the given data in a sufficiently small range leads to an arbitrarily small modification in the solution. This notion was firstly defined by Hadamard in 1902 (see [4]). Due to this significant reason, we want to deal with a stability issue for the problem (1.1)-(1.2).

As clearly seen, equation (1.1) reduces to the following equation

$$-u'' + \left(w_0 + \frac{l(l+1)}{x^2}\right)u = \lambda u, \tag{1.3}$$

for $p = 2$ and it is known as Bessel equation which is obtained by separation of variables in the 3D radial Schrödinger equation. In case of a wave function with spherical symmetry, the wave equation can be seperated using spherical coordinates, and the equation for the radial component becomes (1.3) where λ is a constant refers to eigenvalue of the problem, physically proportional to the energy of the particle under consideration, w_0 is proportional to the potential energy and l is a positive integer or zero. Throughout this study, we will use this equality $w_0 + \frac{l(l+1)}{x^2} = w$, briefly. Inverse problems in spectral theory are divided into two parts; inverse eigenvalue problem and inverse nodal problem. Inverse eigenvalue problem is constructing operator by using some spectral datas as spectrum, norming constants. The first study is given by Ambarzumyan in 1929 about inverse eigenvalue problems [5]. Later, many researchers obtained some important results for different type operators. Inverse eigenvalue problem for classical Bessel equation was studied by various authors (see [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]).

In 1988, McLaughlin [16] proposed a new way to construct the Sturm-Liouville operator. This effective technique is called inverse nodal problem. These problems consist in recovering operators from given zeros of their eigenfunctions (nodal points). Physically, this is related to find, e.g., the density of a string or a beam from the zero-amplitude positions of their eigenvibrations. She is admitted to be the first mathematician to consider this kind of inverse problem. Independently, Shen, studied the relation between density function of string equation and nodal points [17]. Afterwards, inverse nodal problem has been kept in view by several authors (see [18], [19], [20]).

Assume that $X_n = \left\{x_j^n\right\}_{j=1}^{n-1}$ are the zeros of the eigenfunction $u_n(x)$ related to the eigenvalues $\{\lambda_n\}$ of the problem (1.1)-(1.2). Nodal length of this problem is denoted by $l_j^n = x_{j+1}^n - x_j^n$ for $j = 1, 2, ..., n-1$. Using these type nodal datas, some uniqueness and reconstruction results for different type of operators have been expressed by several authors (see [21], [22], [23], [24]).

Equation (1.1) gives us the following $p-$Laplacian Sturm-Liouville eigenvalue problem in case of $w(x)=0$,

$$-\left(u^{'(p-1)}\right)' = (p-1)\,\lambda u^{(p-1)}, u(1)=u(a)=0. \tag{1.4}$$

where the eigenvalues of the problem (1.4) associated eigenfunctions $u_n(x)$ are countably infinite real and simple [25]. Inverse and stability problems for (1.4) were studied many times in literature (see [26], [27], [28], [29], [30], [31]). Besides, inverse problems for different types of $p-$Laplacian operators have been proved by various mathematicians (see [32], [33], [34], [35], [36]).

To say something about the stability of the inverse nodal problem for the problem (1.1)-(1.2), we should define the function S_p that is the solution of the initial value problem (see [25], [32], [33])

$$-\left(S_p^{'(p-1)}\right)' = (p-1)S_p^{(p-1)}, S_p(0)=0,\ S_p'(0)=1. \tag{1.5}$$

S_p and S_p' are periodic functions which provide the identity

$$|S_p(x)|^p + \left|S_p'(x)\right|^p = 1,$$

for any $x\in\mathbb{R}$ which is similar with the known trigonometric identity $\sin^2 x + \cos^2 x = 1$ where $S_p(x)$ is called generalized sine function. These generalized functions are $p-$analogues of classical sine and cosine functions. It is well known that

$$\pi_p = 2\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}} = \frac{2\pi}{p\sin\left(\frac{\pi}{p}\right)},$$

is the first zero of S_p (see [33]). Presently, we will give some important properties of S_p.

Lemma 1.1. [25], [33]
a) *For* $S_p' \neq 0$,

$$\left(S_p'\right)' = -\left|\frac{S_p}{S_p'}\right|^{p-2} S_p.$$

b)

$$\left(S_p S_p^{'(p-1)}\right)' = \left|S_p'\right|^p - (p-1)S_p^p = 1-p\left|S_p\right|^p = (1-p)+p\left|S_p'\right|^p.$$

This study is organized as follows. In section 2, we express some asymptotic formulas for eigenvalues, nodal parameters and potential function of the problem (1.1)-(1.2) by using modified Prüfer substitution [2]. In section 3, we define a metric to clarify Lipschitz stability of inverse nodal problem for $p-$Laplacian Bessel equation. Finally, we give some conclusions in section 4.

2. Asymptotic formulas for p-Laplacian Bessel equation

In this section, we recur some features of $p-$Laplacian Bessel equation (1.1) with the conditions (1.2) which were solved in [2]. In accordance with this purpose, we need to define the following modified Prüfer substitution as

$$\begin{aligned} u(x) &= c(x)S_p\left(\lambda^{1/p}\ \theta(x,\lambda)\right), \\ u'(x) &= (l+1)\lambda^{1/p}c(x)S_p'\left(\lambda^{1/p}\ \theta(x,\lambda)\right), \end{aligned} \tag{2.1}$$

or

$$\frac{u'(x)}{u(x)} = (l+1)\lambda^{1/p}\frac{S_p'\left(\lambda^{1/p}\ \theta(x,\lambda)\right)}{S_p\left(\lambda^{1/p}\ \theta(x,\lambda)\right)}, \tag{2.2}$$

where $c(x)$ is amplitude and $\theta(x)$ is Prüfer variable [37]. Differentiating each side of the above equation (2.2) with respect to x and considering Lemma 1.1, it can be obtained as [2]

$$\theta'(x) = l+1+\left[-(l+1)+(l+1)^{1-p} - \frac{(l+1)^{1-p}}{\lambda}\left(w_0(x)+\frac{l(l+1)}{x^2}\right)\right]S_p^p\left(\lambda^{1/p}\ \theta(x,\lambda)\right). \tag{2.3}$$

This equality plays an important role throughout this study. Now, we are ready to express asymptotic estimations of nodal parameters and potential function for the problem (1.1), (1.2).

Theorem 2.1. [2] *The eigenvalues of the problem (1.1)-(1.2) have the form*

$$\lambda_n^{1/p} = \frac{n\pi_p}{\widetilde{l}\,(a-1)} + \frac{(l+1)^{1-p}\widetilde{l}^{p-2}(a-1)^{p-2}}{p(n\pi_p)^{p-1}}\int_1^a \left\{w_0(s)+\frac{l(l+1)}{s^2}\right\}ds + O\left(\frac{1}{n^{p-1}}\right),$$

as $n\to\infty$ where $\tilde{l} = (l+1)\left(1-\dfrac{1}{p}+\dfrac{1}{p(l+1)^p}\right)$.

Theorem 2.2. [2] *The nodal parameters of the problem (1.1)-(1.2) satisfy following equalities*

$$\begin{aligned} x_j^n &= 1+\frac{j(a-1)\tilde{l}}{(l+1)n} - \frac{j\widetilde{l}^p(a-1)^p}{p(l+1)^p n^{p+1}\pi_p^p}\int_1^a w(s)ds \\ &\quad + \int_1^{x_j^n} S_p^p ds - \frac{1}{(l+1)^p}\int_1^{x_j^n}\left\{1-\frac{\widetilde{l}^p(a-1)^p w(s)}{(n\pi_p)^p}\right\}S_p^p ds + O\left(\frac{j}{n^{p+1}}\right), \end{aligned}$$

and

$$l_j^n = \frac{a-1}{n} + \frac{(a-1)^p \widetilde{l}^{p-1}}{p\tilde{l}\ (l+1)^{p-1}\,(n\pi_p)^p} \int\limits_{x_j^n}^{x_{j+1}^n} w(t)dt + O\left(\frac{1}{n^p}\right),$$

respectively, as $n \to \infty$.

Theorem 2.3. [2] *Let* $w \in L^2\,[1,a]$. *Then*

$$w(x) = \lim_{n\to\infty} p(l+1)^{p-1}\lambda_n \left(\frac{\tilde{l}\ \lambda_n^{\frac{1}{p}}}{\pi_p} l_j^n - 1\right),$$

where $x \in (1,a)$ *and* $j = j_n(x) = \max\left\{j : x_j^n < x\right\}$.

3. Main results about Lipschitz stability

Here, we want to clarify Lipschitz stability problem for p-Laplacian Bessel equation. Lipschitz stability is about a continuity between two metric spaces. To denote this continuity, we will handle a homeomorphism between these metric spaces. Stability problems for different operators were studied by many authors (see [38], [39], [40], [41], [42]). The procedure that we have applied in the proof of Lipschitz stability is similar to the study [38].

Let define Ω_{dif} and Σ_{dif} by

$$\begin{aligned} \Omega_{dif} &= \left\{w \in L^2\,[1,a]\right\}, \\ \Sigma_{dif} &= \left\{X = \{x_k^n\} : X \text{ is the nodal set associated with some } w \in \Omega_{dif}\right\}. \end{aligned}$$

We will denote that Ω_{dif} and Σ_{dif} are homeomorphic to each other. Therefore, when $\overline{X}$ is the nodal set associated with $\overline{w}$ and $\overline{X}$ is close to X in Σ_{dif}, then $\overline{w}$ is close to w in Ω_{dif}. Hence, inverse nodal problem for $p-$Laplacian Bessel equation is Lipschitz stable. Herein, we use $L^m(1,a)$ $(m \geq 1)$ for Ω_{dif}. Let

$$S_n^m(X,\overline{X}) = \frac{(l+1)^{p-1}\pi_p^p}{\widetilde{l}^p (a-1)^{p+1-\frac{1}{m}}}\ n^{p+1-\frac{1}{m}} \left[\sum_{k=0}^{n-1} \left|l_k^n - \overline{l}_k^n\right|^m\right]^{\frac{1}{m}}, \tag{3.1}$$

where $l_k^n = x_{k+1}^n - x_k^n$, $\overline{l}_k^n = \overline{x}_{k+1}^n - \overline{x}_k^n$ and $m \geq 1$. Define the metric and a pseudometric on Σ_{dif}

$$d_0^m(X,\overline{X}) = \overline{\lim_{n\to\infty}} S_n^m(X,\overline{X}),$$

and

$$d_{\Sigma_{dif}}^m(X,\overline{X}) = \overline{\lim_{n\to\infty}} \frac{S_n^m(X,\overline{X})}{1+S_n^m(X,\overline{X})},$$

respectively. If we set a relation as $X \sim_m \overline{X}$ iff $d^m_{\Sigma_{dif}}(X,\overline{X}) = 0$, then $\sim_m$ is an equivalence relation on Σ_{dif} and $d^m_{\Sigma_{dif}}$ would be a metric for the partition set $\Sigma^*_{dif} = \Sigma_{dif}/\sim_m$.

Lemma 3.1. *$d^m_{\Sigma_{dif}}(.,.)$ is a pseudometric on Σ_{dif}.*

Proof: The proof can be expressed easily by considering the similar way with in [25].

Lemma 3.2. *Let $X, \overline{X} \in \Sigma_{dif}$. Then,*

***(a)*:** *The interval $I_{n,k}$ between the points x^n_k and $\overline{x}^n_k$ has length $O(n^{-p})$.*

***(b)*:** *The inequality $\left|j_n(x) - \overline{j}_n(x)\right| \leq 1$ holds when n is sufficiently large for all $x \in (1,a)$.*

Proof:

(a) Using the similar way with in [38] and the asymptotic expansions of nodal points yield

$$\begin{aligned} |I_{n,k}| &= |x^n_k - \overline{x}^n_k| \\ &\leq |x^n_k - A| + |A - \overline{x}^n_k| \\ &= O(n^{-p}) + O(n^{-p}) \\ &= O(n^{-p}), \end{aligned}$$

where $A = \dfrac{l+1}{\tilde{l}} + \dfrac{k(a-1)}{n} - \dfrac{l+1}{p\tilde{l}} + \dfrac{1}{p\tilde{l}(l+1)^{p-1}}$.

***(b)*:** It can be proved by using analogous method with [38].

Theorem 3.1. *$d^m_{\Sigma_{dif}}$ is a metric on the space $\Sigma_{dif}/\sim_m$ for any of $m \geq 1$. Intercalarily, the metric spaces $(\Omega_{dif}, \|.\|_m)$ and $\left(\Sigma_{dif}/\sim_m, d^m_{\Sigma_{dif}}\right)$ are homeomorphic to each other where $\sim_m$ is an equivalence relation induced by $d^m_{\Sigma_{dif}}$.*

Proof: It is sufficient to show that

$$\|w - \overline{w}\|_m = p d^m_0(X, \overline{X}).$$

By Theorem 2.3, we get

$$w(x) - \overline{w}(x) = \lim_{n\to\infty} \frac{p(l+1)^{p-1}\,(n\pi_p)^p}{\widetilde{l}^p(a-1)^p} \left[\frac{n}{a-1}\left(l^n_{j_n(x)} - \overline{l}^n_{\overline{j}_n(x)}\right)\right],$$

for each $x \in (1,a)$. Therefore, we obtain

$$\begin{aligned} \|w - \overline{w}\|_m &\leq \frac{p(l+1)^{p-1} n (n\pi_p)^p}{\widetilde{l}^p(a-1)^{p+1}} \underline{\lim_{n\to\infty}} \left\|l^n_{j_n(x)} - \overline{l}^n_{\overline{j}_n(x)}\right\|_m \\ &\leq \frac{p(l+1)^{p-1}\pi^p_p}{\widetilde{l}^p(a-1)^{p+1}} \overline{\lim_{n\to\infty}} \left[n^{p+1}\left\|l^n_{j_n(x)} - \overline{l}^n_{j_n(x)}\right\|_m\right]. \qquad (3.2) \end{aligned}$$

by Fatou's Lemma and the definition of norm on L_m. Here, Lemma 3.2. and Theorem 2.2. yield

$$\begin{aligned}
n^{p+1}\left\|\bar{l}^n_{j_n(x)}-\bar{l}^n_{\bar{j}_n(x)}\right\|_m &= n^{p+1}\left[\int_1^a\left|\bar{l}^n_{j_n(x)}-\bar{l}^n_{\bar{j}_n(x)}\right|^m dx\right]^{\frac{1}{m}} \\
&= n^{p+1}\left[\sum_{k=0}^{n-1}\left|\bar{l}^n_{k+1}-\bar{l}^n_k\right|^m I_{n,k}\right]^{\frac{1}{m}} \\
&= O\left(n^{1+\frac{1-p}{m}}\right)=o(1), \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
n^{p+1}\left\|l^n_{j_n(x)}-\bar{l}^n_{j_n(x)}\right\|_m &= n^{p+1}\left[\int_1^a\left|l^n_{j_n(x)}-\bar{l}^n_{j_n(x)}\right|^m dx\right]^{\frac{1}{m}} \\
&= n^{p+1}\left[\sum_{k=0}^{n-1}\left|l^n_k-\bar{l}^n_k\right|^m l^n_k\right]^{\frac{1}{m}} \\
&= n^{p+1-\frac{1}{m}}\,(a-1)^{\frac{1}{m}}\left[\sum_{k=0}^{n-1}\left|l^n_k-\bar{l}^n_k\right|^m\right]^{\frac{1}{m}}. \qquad (3.4)
\end{aligned}$$

Considering (3.3) and (3.4) together in (3.2), we obtain

$$\begin{aligned}
\|w-\overline{w}\|_m &\leq \frac{p(l+1)^{p-1}\pi_p^p}{\widetilde{l}^p(a-1)^{p+1-\frac{1}{m}}}\overline{\lim_{n\to\infty}}\, n^{p+1-\frac{1}{m}}\left[\sum_{k=0}^{n-1}\left|l^n_k-\bar{l}^n_k\right|^m\right]^{\frac{1}{m}} \\
&= p d_0^m(X,\overline{X}).
\end{aligned}$$

Contrarily, using the above derivations

$$\begin{aligned}
\left\|w-\overline{w}\right\|_m+o(1) &= \frac{p(l+1)^{p-1}\pi_p^p}{\widetilde{l}^p(a-1)^{p+1}}n^{p+1}\left\|l^n_{j_n(x)}-\overline{l}^n_{\overline{j}_n(x)}\right\|_m \\
&\geq \frac{p(l+1)^{p-1}\pi_p^p}{\widetilde{l}^p(a-1)^{p+1}}n^{p+1}\left\|l^n_{j_n(x)}-\overline{l}^n_{\overline{j}_n(x)}\right\|_m - O\left(n^{1+\frac{1-p}{m}}\right) \\
&= \frac{p(l+1)^{p-1}\pi_p^p}{\widetilde{l}^p(a-1)^{p+1}}n^{p+1}\left[\sum_{k=0}^{n-1}\left|l^n_k-\overline{l}^n_k\right|^m l^n_k\right]^{\frac{1}{m}} - O\left(n^{1+\frac{1-p}{m}}\right) \\
&= \frac{p(l+1)^{p-1}\pi_p^p n^{p+1-\frac{1}{m}}}{\widetilde{l}^p(a-1)^{p+1-\frac{1}{m}}}\left[\sum_{k=0}^{n-1}\left|l^n_k-\overline{l}^n_k\right|^m\right]^{\frac{1}{m}} - O\left(n^{1+\frac{1-p}{m}}\right).
\end{aligned}$$

Hereby, as n approaches to infinity,

$$\left\|w-\overline{w}\right\|_m \geq p d_0^m(X,\overline{X}).$$

So, the proof is complete.

4. Conclusion

In this study, we have underlined the significance of the stability issue for inverse nodal problem of $p-$Laplacian Bessel equation. Then, some asymptotic expansions for eigenvalues, nodal parameters and reconstruction formula for potential function of the problem (1.1)-(1.2) have been expressed. Moreover, we have solved the Lipschitz stability problem for (1.1)-(1.2).

Competing Interests

The authors declare to have no competing interests.

References

[1] Koyunbakan, H., Inverse nodal problem for $p-$Laplacian energy-dependent Sturm-Liouville equation, Boundary Value Problems, 2013:272 (2013) (Erratum: Inverse nodal problem for p-Laplacian energy-dependent Sturm-Liouville equation, Boundary Value Problems, 2014:222 (2014).

[2] Coskun, M., Gulsen, T., Koyunbakan, H., Inverse nodal problem for $p-$Laplacian Bessel Equation (Submitted to the journal).

[3] Hamadamen, M. H., Inverse nodal problem for $p-$Laplacian differential operators, Master Thesis, The institute of Natural and Applied Sciences, Firat University, (2017).

[4] Baumeister, J., Stable solution of inverse problems, Advanced Lectures in Mathematics, (1987).

[5] Ambarzumyan, V.A., Uber eine Frage der Eigenwerttheorie, Zeitschrift für Physik, 53, 690–695 (1929).

[6] Bas, E., Panakhov, E. S., Yilmazer, R., The uniqueness theorem for hydrogen atom equation, TWMS Journal of Pure and Applied Mathematics, 4(1), 20-28 (2013).

[7] Bondarenko, N., An inverse spectral problem for the matrix Sturm-Liouville operator with a Bessel-type singularity, International Journal of Differential Equations, Art. ID 647396, 4 pp. (2015).
[8] Guldu, Y., Amirov, R. Kh., Topsakal, N., On impulsive Sturm-Liouville operators with singularity and spectral parameter in boundary conditions, Ukrainian Mathematical Journal, 64(12), 1816–1838 (2013).
[9] McLeod, J. B., The distribution of the eigenvalues for the Hydrogen atom and similar cases, Proceedings of the London Mathematical Society, 3(1), 139-158 (1961).
[10] Masood, K., Messaoudi, S., Zaman, F. D., Initial inverse problem in heat equation with Bessel operator, International Journal of Heat and Mass Transfer, 45, 2959–2965 (2002).
[11] Albeverio, S., Hryniv, R., Mykytyuk, Y., Inverse spectral problems for Bessel operators, Journal of Differential Equations, 241(1), 130–159 (2007).
[12] Hryniv, R., Sacks, P., Numerical solution of the inverse spectral problem for Bessel operators, Journal of Computational and Applied Mathematics, 235(1), 120–136 (2010).
[13] Alekseeva, V. S., Ananieva, A. Y., On extensions of the Bessel operator on a finite interval and a half-line, Journal of Mathematical Sciences, 187(1), 1-8 (2012).
[14] Sat, M., Panakhov, E. S., A uniqueness theorem for Bessel operator from interior spectral data, Abstract and Applied Analysis, Volume 2013, Article ID 713654, 6 pages.
[15] Yilmaz, E., Koyunbakan, H., Some Ambarzumyan type theorems for Bessel operator on a finite interval, Differential Equations and Dynamical Systems, (2016).
[16] McLaughlin, J. R., Inverse spectral theory using nodal points as data-a uniqueness result, Journal of Differential Equations, 73(2), 354-362 (1988).
[17] Shen, C. L., On the nodal sets of the eigenfunctions of the string equations, SIAM Journal on Mathematical Analysis, 19(6), 1419–1424 (1988).
[18] Yurko, V. A., Inverse nodal problems for Sturm-Liouville operators on star-type graphs, Journal of Inverse and Ill-Posed Problems, 16(**7**), 715–722 (2008).
[19] Hald, O. H., McLaughlin, J. R., Solutions of the inverse nodal problems, Inverse Problems, 5(3), 307-347 (1989).
[20] Yang, C. F., Inverse nodal problems for the Sturm-Liouville operator with a constant delay, Journal of Differential Equations, 257(4), 1288-1306 (2014).
[21] Koyunbakan, H., Yilmaz, E., Reconstruction of the potential function and its derivatives for the diffusion operator, Zeitschrift für Naturforschung A, 63(3-4), 127-130 (2008).
[22] Yang, C. F., An inverse problem for a differential pencil using nodal points as data, Israel Journal of Mathematics, 204(1), 431-446 (2014).
[23] Law, C. K., Yang, C. F., Reconstruction of the potential function and its derivatives using nodal data, Inverse Problems, 14, 299-312 (1999).
[24] Buterin, S. A., Shieh, C. T., Inverse nodal problem for differential pencils, Applied Mathematics Letters, 22, no. 8, 1240–1247 (2009).
[25] Law, C. K., Lian, W. C., Wang, W. C., The inverse nodal problem and the Ambarzumyan problem for the $p-$Laplacian, Proceedings of the Royal Society of Edinburgh Section A Mathematics, 139(6), 1261-1273 (2009).
[26] Chen, H. Y., On generalized trigonometric functions, Master of Science, National Sun Yat-sen University, Kaohsiung, Taiwan, (2009).
[27] Elbert, A., On the half-linear second order differential equations, Acta Mathematica Hungarica, 49(3-4), 487–508 (1987).
[28] Binding, P. A., Rynne, B. P., Variational and non-variational eigenvalues of the $p-$ Laplacian, Journal of Differential Equations, 244(1), 24-39 (2008).
[29] Brown, B. M., Reichel, W., Eigenvalues of the radially symmetric $p-$Laplacian in R^n, Journal of the London Mathematical Society, 69(3), 657-675 (2004).
[30] Walter, W., Sturm-Liouville theory for the radial Δ_p-operator, Mathematische Zeitschrift, 227(1), 175-185 (1998).

[31] Binding, P., Drábek, P., Sturm–Liouville theory for the $p-$Laplacian, Studia Scientiarum Mathematicarum Hungarica, 40(4), 373–396 (2003).
[32] Wang, W. C., Cheng, Y. H., Lian, W. C., Inverse nodal problems for the p-Laplacian with eigenparameter dependent boundary conditions, Mathematical and Computer Modelling, 54(11-12), 2718-2724 (2011).
[33] Wang, W. C., Direct and inverse problems for one dimensional $p-$Laplacian operators, National Sun Yat-Sen University, PhD Thesis, (2010).
[34] Pinasco, J. P., Scarola, C. A., A nodal inverse problem for a quasi-linear ordinary differential equation in the half-line, Journal of Differential Equations, 261(2), 1000–1016 (2016).
[35] Gulsen, T., Yilmaz, E., Inverse nodal problem for $p-$Laplacian diffusion equation with polynomially dependent spectral parameter, Communications, Series A1; Mathematics and Statistics, 65(2), 23-36, (2016).
[36] Gulsen, T., Yilmaz, E., Koyunbakan, H., Inverse nodal problem for p-Laplacian Dirac system, Mathematical methods in applied sciences, Doi: 10.1002/mma.4141, (2016).
[37] Yantir, A., Oscillation theory for second order differential equations and dynamic equations on time scales, Master of Science, Izmir institue of Technology, Izmir, (2004).
[38] Law, C. K., Tsay, J., On the well-posedness of the inverse nodal problem, Inverse Problems, 17(5), 1493-1512 (2001).
[39] Marchenko, V. A., Maslov, K. V., Stability of the problem of recovering the Sturm-Liouville operator from the spectral function, Matematicheskii Sbornik, 123(4), 475-502 (1970).
[40] McLaughlin, J. R., Stability theorems for two inverse spectral problems, Inverse Problems, 4(2), 529-540 (1988).
[41] Yilmaz, E., Koyunbakan, H., On the high order Lipschitz stability of inverse nodal problem for string equation, Dynamics of Continuous, Discrete and Impulsive Systems. Series A. Mathematical Analysis, 21, 79-88 (2014).
[42] Yilmaz, E., Lipschitz stability of inverse nodal problem for energy-dependent Sturm-Liouville equation, New Trends in Mathematical Sciences, 3(1), 46-61 (2015).

Current address: Tuba GULSEN: Firat University, Department of Mathematics, 23119, Elazıg TURKEY

E-mail address: `tubagulsen87@hotmail.com`

Current address: Emrah YILMAZ (Corresponding author): Firat University, Department of Mathematics, 23119, Elazıg TURKEY

E-mail address: `emrah231983@gmail.com`

Current address: E. S. PANAKHOV: Baku State University, Institute of Applied Mathematics, Baku AZARBAIJAN

E-mail address: `epenahov@hotmail.com`