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ON DOUGLAS SPACES WITH VANISHING $\bar{E}$-CURVATURE

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Abstract. In this paper, we prove that every compact Douglas space is a Berwald space, when the mean Berwald curvature is covariantly constant along all horizontal directions on the slit tangent bundle.

1. Introduction

There are two well-known projective invariants of Finsler metrics namely, Douglas curvature [8] and Weyl curvature [18]. The Douglas curvature is a non-Riemannian projective invariant constructed from the Berwald curvature. The notion of Douglas curvature was proposed by *Bácsó* and Matsumoto as a generalization of Berwald curvature [4][11].

On the other hand, there are several important non-Riemannian quantities such as the Cartan torsion **C**, the Berwald curvature **B**, the mean Berwald curvature **E** and the Landsberg curvature **L**, etc [16]. They all vanish for Riemannian metrics, hence they are said to be non-Riemannian.

The study shows that the above mentioned non-Riemannian quantities are closely related to the Douglas metrics, namely Bácsó-Matsumoto proved that every Douglas metric with vanishing Landsberg curvature is a Berwald metric [3]. Is there any other interesting non-Riemannian quantity with such property? In [12], Shen find a new non-Riemannian quantity for Finsler metrics that is closely related to the **E**-curvature and call it $\bar{\mathbf{E}}$-curvature. Recall that $\bar{\mathbf{E}}$ is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics.

In this paper, we prove that every compact Douglas space with vanishing $\bar{\mathbf{E}}$-curvature is a Berwald space. More precisely, we prove the following.

Theorem 1.1. *Let (M, F) be a complete Douglas space with bounded Cartan torsion. Suppose that $\bar{\mathbf{E}}$-curvature of F is vanishing. Then F reduces to a Berwald metric. In particular, every compact Douglas space with $\bar{\mathbf{E}} = 0$ is a Berwald space.*

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The completeness in Theorem 1.1, can not be dropped. Consider following Finsler metric on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$,

$$F(y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - <x, y>^2)} + <x, y>}{1 - |x|^2}, \quad y \in T_x\mathbb{B}^n = \mathbb{R}^n$$

where $|.|$ and $<,>$ denote the Euclidean norm and inner product in $\mathbb{R}^n$, respectively. F is called the Funk metric which is a Randers metric on $\mathbb{B}^n$. One can show that F is positively complete on $\mathbb{B}^n$ [7]. Funk metric is a Douglas metric and satisfies $\bar{\mathbf{E}} = 0$ while $\mathbf{B} \neq 0$.

There are many connections in Finsler geometry [6][14][15]. Throughout this paper, we use the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) F is C^∞ on TM_0,
(ii) F is positively 1-homogeneous on the fibers of TM,
(iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive definite,

$$g_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[g_{y+tw}(u, v)\right]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C}=\mathbf{0}$ if and only if F is Riemannian.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i(y)$ are local functions on TM given by

$$G^i := \frac{1}{4}g^{il}\Big\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l}y^k - \frac{\partial[F^2]}{\partial x^l}\Big\}, \quad y \in T_xM.$$

$\mathbf{G}$ is called the associated spray to (M, F). The projection of an integral curve of $\mathbf{G}$ is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x, \quad \mathbf{E}_y(u, v) := E_{jk}(y)u^jv^k,$$

where

$$B^i{}_{jkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y), \quad E_{jk}(y) := \frac{1}{2} B^m{}_{jkm}(y),$$

$u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. **B** and **E** are called the Berwald curvature and mean Berwald curvature, respectively. A Finsler metric is called a Berwald metric and mean Berwald metric if $\mathbf{B} = 0$ or $\mathbf{E} = 0$, respectively [12].

For a tangent vector $y \in T_xM_0$, define $\bar{\mathbf{E}}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\bar{\mathbf{E}}_y(u, v, w) := \bar{E}_{jkl}(y) u^i v^j w^k$, where

$$\bar{E}_{ijk} := E_{ij|k}.$$

From a Bianchi identity, we have

$$B^i{}_{jml|k} - B^i{}_{jkm|l} = R^i{}_{jkl.m},$$

where $R^i{}_{jkl}$ is the Riemannian curvature of Berwald connection [13][17]. By putting $i = m$ in the above relation, we get

$$\bar{E}_{jlk} - \bar{E}_{jkl} = 2R^m{}_{jkl,m}.$$

Then $\bar{E}_{ijk}$ is not totally symmetric in all three of its indices. It is easy to see that if $\bar{\mathbf{E}}$-curvature is vanishing, then **E**-curvature is covariantly constant along all horizontal directions on the slit tangent bundle TM_0.

The quantity $\mathbf{H}_y = H_{ij} dx^i \otimes dx^j$ is defined as the covariant derivative of **E** along geodesics [1][10]. More precisely

$$H_{ij} := E_{ij|m} y^m = \bar{E}_{ijm} y^m.$$

Define $\mathbf{D}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{D}_y(u, v, w) := D^i{}_{jkl}(y) u^i v^j w^k \frac{\partial}{\partial x^i}|_x$ where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l} y^i\}.$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y \in TM_0}$ the Douglas curvature [8]. A Finsler metric with $\mathbf{D} = 0$ is called a Douglas metric [9]. It is remarkable that, the notion of Douglas metrics was proposed by B*á*cs*ó*-Matsumoto as a generalization of Berwald metrics (see [4] and [5]).

Define $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{L}_y(u, v, w) := L_{ijk}(y) u^i v^j w^k$ where $L_{ijk} := C_{ijk|s} y^s$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature [17]. F is called a Landsberg metric if **L=0**. Every Berwald metric is a Landsberg metric.

Theorem 2.1. ([2][3]) For a Douglas metric F on a manifold M, if $\mathbf{L} = \mathbf{0}$, then $\mathbf{B} = \mathbf{0}$.

3. Proof of Theorem 1.1

To prove the Theorem 1.1, we need the following:

Lemma 3.1.

$$E_{jk,l|m} y^m = H_{jk,l} - \bar{E}_{jkl}. \tag{3.1}$$

Proof. The following Ricci identity for E_{ij} is hold:

$$E_{ij,l|k} - E_{ij|k,l} = E_{pj}B^p{}_{ikl} + E_{ip}B^p{}_{jkl}. \tag{3.2}$$

It follows from (3.2) that

$$E_{jk,l|m}y^m = E_{jk|m,l}y^m = [E_{jk|m}y^m]_{,l} - E_{jk|l}. \tag{3.3}$$

This yields the (3.1). □

Proposition 3.1. *Let (M, F) be a Douglas space. Suppose that F satisfies $\bar{\mathbf{E}} = 0$. Then for any geodesic $c(t)$ and any parallel vector field $U(t)$ along c, the following functions*

$$\mathbf{C}(t) = \mathbf{C}_{\dot{c}}(U(t), U(t), U(t)), \tag{3.4}$$

satisfying in the following equation

$$\mathbf{C}(t) = \mathbf{L}(0)t + \mathbf{C}(0). \tag{3.5}$$

Proof.

$$D^i{}_{jkl} = B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{lj}\delta^i{}_k + E_{jk,l}y^i\}. \tag{3.6}$$

Then

$$D^i{}_{jkl|m}y^m = B^i{}_{jkl|m}y^m - \frac{2}{n+1}\{H_{jk}\delta^i{}_l + H_{kl}\delta^i{}_j + H_{lj}\delta^i{}_k + E_{jk,l|m}y^m y^i\}. \tag{3.7}$$

By Lemma 3.1, we get

$$B^i{}_{jkl|m}y^m = \frac{2}{n+1}\{H_{jk}\delta^i{}_l + H_{kl}\delta^i{}_j + H_{lj}\delta^i{}_k + H_{jk,l}y^i - \bar{E}_{jkl}y^i\}. \tag{3.8}$$

From assumption, we have

$$B^i{}_{jkl|m}y^m = 0. \tag{3.9}$$

Contracting with y_i yields

$$L_{jkl|m}y^m = 0. \tag{3.10}$$

Let

$$\mathbf{L}(t) = \mathbf{L}_{\dot{c}}(U(t), U(t), U(t)). \tag{3.11}$$

From the definition of $\mathbf{L}_y$, we have

$$\mathbf{L}(t) = \mathbf{C}^{'}(t). \tag{3.12}$$

By (3.10) we get

$$\mathbf{L}^{'}(t) = 0. \tag{3.13}$$

The equation (3.13) implies that

$$\mathbf{L}(t) = \mathbf{L}(0). \tag{3.14}$$

Then we get the equation (3.5). □

Remark 3.1. Let (M, F) be a Finsler space and $c : [a, b] \to M$ be a geodesic. For a parallel vector field $V(t)$ along c,

$$g_{\dot{c}}(V(t), V(t)) = constant. \tag{3.15}$$

Proof of Theorem 1.1: Take an arbitrary unit vector $y \in T_xM$ and an arbitrary vector $v \in T_xM$. Let $c(t)$ be the geodesic with $\dot{c}(0) = y$ and $V(t)$ the parallel vector field along c with $V(0) = v$. Define $\mathbf{C}(t)$ and $\mathbf{L}(t)$ as in (3.4) and (3.11), respectively. Then

$$\mathbf{C}(t) = t\,\mathbf{L}(0) + \mathbf{C}(0). \tag{3.16}$$

Suppose that $\mathbf{C}_y$ is bounded, i.e., there is a constant $N < \infty$ such that

$$||\mathbf{C}||_x := \sup_{y \in T_xM_0} \sup_{v \in T_xM} \frac{\mathbf{C}_y(v,v,v)}{[g_y(v,v)]^{\frac{3}{2}}} \leq N. \tag{3.17}$$

By (3.15), we know that

$$T := g_{\dot{c}}(V(t), V(t)) = constant.$$

is positive constant. Thus

$$|\mathbf{C}(t)| \leq NT^{\frac{3}{2}} < \infty, \tag{3.18}$$

and $\mathbf{C}(t)$ is a bounded function on $[0, \infty)$. This implies

$$\mathbf{L}_y(v,v,v) = \mathbf{L}(0) = 0. \tag{3.19}$$

Therefore $\mathbf{L} = 0$ and by the Theorem 2.1, F is a Berwald metric. □

Corollary 3.1. *Let (M, F) be a compact Douglas manifold. Then $\mathbf{L} = 0$ if and only if $\bar{\mathbf{E}} = 0$.*

Proof. If $\mathbf{D} = \mathbf{L} = 0$, then by Theorem 2.1 $\mathbf{B} = 0$ which implies that $\bar{\mathbf{E}} = 0$. Conversely let F be a compact Douglas metric with $\bar{\mathbf{E}} = 0$. By Theorem 1.1, $\mathbf{B} = 0$ and then $\mathbf{L} = 0$. □

Corollary 3.2. *Let (M, F) be a complete Finsler space with Randers metric $F = \alpha + \beta$ such that α is a Riemannian metric and β is a close 1-form on M. Suppose that F satisfies $\bar{\mathbf{E}} = 0$. Then F is a Berwald metric.*

Proof. It is known that for a Randers metric $F = \alpha + \beta$ the Cartan tensor is bounded [12]. In fact

$$\|\mathbf{C}\| \leq \frac{3}{\sqrt{2}}.$$

Bácsó-Matsumoto showed that the Randers metric $F = \alpha + \beta$ is a Douglas metric if and only if β is a closed form [4]. Then by Theorem 1.1, we obtain the corollary. □

Corollary 3.3. *For any complete submanifold M in a Minkowski space (V, F), if the induced metric $\bar{F}$ satisfies $\bar{\mathbf{E}} = 0$, then $\bar{F}$ is a Berwald metric.*

Proof. For a submanifold M in a Minkowski space (V, F), the Cartan tensor is bounded [12]. Then by Theorem 1.1, $\bar{F}$ is a Berwald metric. □

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