SOME QUESTIONS ON PLANE CURVES

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Abstract. We consider some properties of simple plane curves, starting from a unusual metric formulation of the tangent line.

1. Introduction

In previous paper ([12]) a new notion of tangency is formulated in suitable metric spaces and an equivalent metric formulation of the notion of tangency at a point of the graphic of a real function ([12] (Example 2.6) and [11] (Teorema 1.1)) is considered.

Then it is possible to reconsider the notion of tangent line at a point of a plane curve (and also of normal line) in a very different way with respect the classical one. This paper is devoted to consider such notion and to investigate some new aspects and questions on plane curves.

It is well known that "... Not all curves are rectifiable; some do not have a tangent at any of their points..." (see[15] (pag.46)).

From your study at every point of a plane curve there is, in a some specified meaning, at least a "tangent direction" and it is possible, from a pure theoretical point of view, to have many "tangent directions" at every point.

In the last section we present some open problems which seem new and interesting.

Following [12], we consider two abstract operations. Let \((X, d)\) be a metric space and \(A, B\) be non-empty, compact (or locally compact) subsets of \(X\). Assume that \(x_0 \in A \cap B\) is an accumulation point of \(A\). We define the functions:

\[
\underline{D}_{x_0}(A, B) = \liminf_{A \setminus \{x_0\} \ni x \to x_0} \frac{d(x, B)}{d(x, x_0)};
\]

\[
\overline{D}_{x_0}(A, B) = \limsup_{A \setminus \{x_0\} \ni x \to x_0} \frac{d(x, B)}{d(x, x_0)};
\]

where \(d(x, B) = \inf\{d(x, y) : y \in B\}\).

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When \( \overline{D}_{x_0}(A, B) = \overline{D}_{x_0}(A, B) \), we write \( D_{x_0}(A, B) \). We remark that: \( 0 \leq D_{x_0}(A, B) \leq \overline{D}_{x_0}(A, B) \leq 1 \); hence we have: \( \overline{D}_{x_0}(A, B) = 0 \iff D_{x_0}(A, B) = D_{x_0}(A, B) = 0 \).

The previous operations are investigated, early and with other motivations, from different authors (see [6] [16], [17], [7], [8], [9], [13], [14]).

In [11] we prove the following result, establishing the connexion with the usual notion of tangent line.

Take as metric space \( R^2 \), endowed with the usual euclidean metric, let \( f : R \rightarrow R \) be a continuous function and \( G_f \subset R^2 \) its graph. Let \( p_0 = (x_0, f(x_0)) \in G_f \), and consider the functions \( \overline{D}_{p_0}, \overline{D}_{p_0}. \) For \( r \) a straight line through \( p_0 \), the following conditions are equivalent:

(i) \( r \) is the tangent line to \( G_f \) at \( p_0 \);
(ii) \( \overline{D}_{p_0}(r, G_f) = 0 \) and \( \overline{D}_{p_0}(s, G_f) > 0 \) for every line \( s \neq r \) through \( p_0 \);
(iii) \( \overline{D}_{p_0}(G_f, r) = 0 \) and \( \overline{D}_{p_0}(G_f, s) > 0 \) for every line \( s \neq r \) through \( p_0 \).

Then one can agree the following definition:

**Definition 1.1.** Let \( A, B \) be non-empty, compact (or locally compact) sets of the metric space \( X \) and let \( x_0 \) be an accumulation point of \( A \) and \( B \). We say that \( A \) is tangent to \( B \) in \( x_0 \) if and only if \( D_{x_0}(A, B) = 0 \).

We say that \( A, B \) are tangent in \( x_0 \) if and only if both \( D_{x_0}(A, B) \) and \( D_{x_0}(B, A) \) exist and \( D_{x_0}(A, B) = D_{x_0}(B, A) = 0 \).

We remark that if \( A \) is tangent to \( B \) or \( A, B \) are tangent in \( x_0 \) with respect to the metric \( d \), then the same occur with respect every metric \( d_1 \) equivalent to \( d \).

Furthermore for \( \overline{D} \) and \( \overline{D} \) we have the following result:

**Proposition 1.1.** Let \( A, B, C \) be non-empty, compact (or locally compact) subsets of the metric space \( X \); let \( x_0 \) be an accumulation point for \( A, B \) and \( C \). We have the following

\[
(1.3) \quad \overline{D}_{x_0}(A, C) - \overline{D}_{x_0}(A, B) \leq \overline{D}_{x_0}(B, C) \cdot [1 + \overline{D}_{x_0}(A, B)]
\]
\[
(1.4) \quad D_{x_0}(A, C) + \overline{D}_{x_0}(A, B) \geq D_{x_0}(B, C) \cdot [1 - \overline{D}_{x_0}(A, B)].
\]
\[
(1.5) \quad D_{x_0}(A, B) = 0 \iff D_{x_0}(B, A) = 0.
\]

and if \( D_{x_0}(A, B) \) and \( D_{x_0}(B, A) \) exist, then: \( D_{x_0}(A, B) = 0 \iff D_{x_0}(B, A) = 0 \).

For the proof, compare [12](cfr. Propositions 2.1 and 2.3 and remark 2.2).

Now we consider the metric space \( (X, d) \), where \( X = R^2 \) and \( d \) is the usual euclidean metric; we denote with \( \mathcal{R}(x) \) the set of half-straight lines through the point \( x \), that is: \( r \in \mathcal{R}(x) \iff \exists v \in R^2, \quad ||v|| = 1, \quad r = \{x + tv|t \geq 0\} \).

The following condition hold:

\[
(1.6) \quad \forall r \in \mathcal{R}(x) \implies x \in r;
\]
\[
(1.7) \quad \forall r \in \mathcal{R}(x), \text{ every bounded sequence in } r \text{ is compact};
\]
\[
(1.8) \quad \forall r, s \in \mathcal{R}(x) \implies \exists D_x(r, s), \exists D_x(s, r) \text{ and } D_x(r, s) = D_x(s, r);
\]
\[
(1.9) \quad \forall r, s \in \mathcal{R}(x) : \quad D_x(r, s) = 0 \implies r = s;
\]
\[
(1.10) \quad 0 < d(x, y) \implies \exists r \in \mathcal{R}(x), \quad \exists s \in \mathcal{R}(y) : y \in r \quad \text{and} \quad x \in s;
\]
Let $D$ be a non-empty subset, if there exists $r^* \in \mathbb{R}(x_0)$ such that $D_{x_0}(A, r^*) = 0$, then $r^*$ is unique.

**Proof.** We assume there exist $r^*, r^{**}$ such that $\overline{D}_{x_0}(A, r^*) = \overline{D}_{x_0}(A, r^{**}) = 0$. By (1.4), we have:

$$0 = \overline{D}_{x_0}(A, r^*) + \overline{D}_{x_0}(A, r^{**}) \geq \overline{D}_{x_0}(r^*, r^{**})(1 - \overline{D}_{x_0}(A, r^*)) = \overline{D}_{x_0}(r^*, r^{**}) = D_{x_0}(r^*, r^{**}).$$

Then, by (1.9), $r^* = r^{**}$.\hfill \Box

Furthermore, we have the proposition:

**Proposition 1.3.** Let $A \subset \mathbb{R}^2$ be a non-empty subset, if there exists $r^*, r^{**} \in \mathbb{R}(x_0)$ such that: $\overline{D}_{x_0}(A, r^{**}) = \overline{D}_{x_0}(r^*, A) = 0$, then $r^* = r^{**}$.

**Proof.** From (1.3), we have:

$$\overline{D}_{x_0}(r^*, r^{**}) - \overline{D}_{x_0}(r^*, A) \leq \overline{D}_{x_0}(r^*, r^{**})(1 + \overline{D}_{x_0}(r^*, A)) = 0.$$

Hence $\overline{D}_{x_0}(r^*, r^{**}) = 0$.

**Proposition 1.4.** Let $C \subset \mathbb{R}^2$ be a non-empty subset, if we assume that:

\begin{equation}
D_{x_0}(r_1, C) > D_{x_0}(r_2, C)
\end{equation}

then

\begin{equation}
r_1 \neq r_2 \quad \text{and} \quad 0 < \frac{D_{x_0}(r_1, C) - D_{x_0}(r_2, C)}{1 + D_{x_0}(r_1, C)} \leq D_{x_0}(r_1, r_2).
\end{equation}

**Proof.** Because, by (1.4) with $A = r_2, B = r_1$,

$$D_{x_0}(r_1, C)(1 - D_{x_0}(r_1, r_2)) \leq D_{x_0}(r_1, r_2) + D_{x_0}(r_2, C),$$

then we receive

$$0 \leq \frac{D_{x_0}(r_1, C) - D_{x_0}(r_2, C)}{1 + D_{x_0}(r_1, C)} \leq D_{x_0}(r_1, r_2).$$

Hence, by (1.9), $r_1 \neq r_2$.\hfill \Box

## 2. Some Results on Simple Plane Curves

In this section we will restrict the considerations on a given simple plane curve. Now, let $\gamma : [0, 1] \to \mathbb{R}^2$ be continuous and injective, denote $\Gamma = \{\gamma(t)|t \in [0, 1]\}$ and $x_0 = \gamma(t_0), t_0 \in [0, 1]$.

We prove the following proposition:

**Proposition 2.1.**

\begin{align*}
\exists r_1 \in \mathbb{R}(x_0) & \text{ such that } D_{x_0}(r_1, \Gamma) = \min\{D_{x_0}(s, \Gamma)|s \in \mathbb{R}(x_0)\} \\
\exists r_2 \in \mathbb{R}(x_0) & \text{ such that } D_{x_0}(r_2, \Gamma) = \max\{D_{x_0}(s, \Gamma)|s \in \mathbb{R}(x_0)\} \\
\exists r_3 \in \mathbb{R}(x_0) & \text{ such that } D_{x_0}(r_3, \Gamma) = \min\{D_{x_0}(s, \Gamma)|s \in \mathbb{R}(x_0)\}
\end{align*}
(2.1) \( \exists r_1 \in \mathbb{R}(x_0) \) such that \( \overline{D}_{x_0}(r_1, \Gamma) = \max \{ \overline{D}_{x_0}(s, \Gamma) | s \in \mathbb{R}(x_0) \} \)

\text{Proof.} (2.1). Let \((r_n)\) be a sequence in \( \mathbb{R}(x_0) \) such that:

\[ \lim_{n} \underline{D}_{x_0}(r_n, \Gamma) = \inf \{ \underline{D}_{x_0}(s, \Gamma) | s \in \mathbb{R}(x_0) \}. \]

By (1.11) we can assume, without loss of generality:

\[ \exists \tau \in \mathbb{R}(x_0) \text{ such that } \lim_{n} D_{x_0}(r_n, \tau) = 0. \]

Moreover, by (1.4), with \( A = r_n, B = \tau, C = \Gamma \), and by (1.8), we argue:

\[ \overline{D}_{x_0}(\tau, \Gamma)(1 - D_{x_0}(r_n, \tau)) \leq \overline{D}_{x_0}(r_n, \Gamma) + D_{x_0}(r_n, \tau) \quad \forall n \in N, \]

then we have:

\[ D_{x_0}(\tau, \Gamma) \leq \lim_{n} \underline{D}_{x_0}(r_n, \Gamma) = \inf \{ \underline{D}_{x_0}(s, \Gamma) | s \in \mathbb{R}(x_0) \}, \]

and so the thesis follows.

(2.2) Arguing as in the previous proof, let \((r_n)\) be a sequence in \( \mathbb{R}(x_0) \) and \( \tau \in \mathbb{R}(x_0) \) such that:

\[ \lim_{n} \underline{D}_{x_0}(r_n, \Gamma) = \sup \{ \underline{D}_{x_0}(s, \Gamma) | s \in \mathbb{R}(x_0) \}. \]

By (1.4), with \( A = \tau, B = r_n, C = \Gamma \), and by (1.8), we obtain:

\[ \overline{D}_{x_0}(r_n, \Gamma)(1 - D_{x_0}(r_n, \tau)) \leq \overline{D}_{x_0}(\tau, \Gamma) + \overline{D}_{x_0}(r_n, r) \quad \forall n \in N, \]

Then

\[ D_{x_0}(\tau, \Gamma) \geq \lim_{n} \underline{D}_{x_0}(r_n, \Gamma) = \sup \{ \underline{D}_{x_0}(s, \Gamma) | s \in \mathbb{R}(x_0) \}. \]

The proof of (2.3) (and (2.4)) is similar.

The elements of \( \mathbb{R}(x_0) \) in (2.1), (2.2) can be considered as different types of "tangent direction" to the curve \( \gamma \) at the point \( x_0 \) and the elements in (2.4) as "normal direction".

In the classical theory of tangency, the tangent line is strictly related to the limit of secant lines to \( \Gamma \); this can be a justification for the following proposition:

**Proposition 2.2.**

(2.7) \( \exists s_1 \in \mathbb{R}(x_0) \) such that \( D_{x_0}(s_1, \Gamma) = \lim \inf \frac{D_{x_0}(r_{xx_0}, \Gamma)}{r_{xx_0} \not\in \mathbb{R}(x_0)} \)

(2.8) \( \exists s_2 \in \mathbb{R}(x_0) \) such that \( D_{x_0}(s_2, \Gamma) = \lim \sup \frac{D_{x_0}(r_{xx_0}, \Gamma)}{r_{xx_0} \not\in \mathbb{R}(x_0)} \)

(2.9) \( \exists s_3 \in \mathbb{R}(x_0) \) such that \( \overline{D}_{x_0}(s_3, \Gamma) = \lim \inf \frac{\overline{D}_{x_0}(r_{xx_0}, \Gamma)}{r_{xx_0} \not\in \mathbb{R}(x_0)} \)

(2.10) \( \exists s_4 \in \mathbb{R}(x_0) \) such that \( \overline{D}_{x_0}(s_4, \Gamma) = \lim \sup \frac{\overline{D}_{x_0}(r_{xx_0}, \Gamma)}{r_{xx_0} \not\in \mathbb{R}(x_0)} \)

where \( r_{xx_0} \) is the half-straight line with \( v = \frac{x - x_0}{||x - x_0||} \).

\text{Proof.} (2.7). We can assume that:

\[ \exists (x_n) \in \mathbb{R}(x_0) \text{ such that } : \lim_{n} \underline{D}_{x_0}(r_n, \Gamma) = \lim \inf \frac{D(r_{xx_0}, \Gamma)}{r_{xx_0} \not\in \mathbb{R}(x_0)}, \]

where \( r_n = r_{xx_0}. \) Without loss of generality, we assume:

\( \exists \tau \in \mathbb{R}(x_0) \) for which \( \lim_{n} \underline{D}_{x_0}(r_n, \tau) = 0. \)

Then we have:

\[ \lim_{n} \underline{D}_{x_0}(r_n, \Gamma) = \underline{D}_{x_0}(\tau, \Gamma). \]
In fact, we consider (1.4) twice; first with $A = r_n, B = \mathcal{T}, C = \Gamma$ and next changing $A$ and $B$. Then
\[
D_{x_0}(\mathcal{T}, \Gamma)(1 - D_{x_0}(r_n, \mathcal{T})) \leq D_{x_0}(r_n, \Gamma) + D_{x_0}(r_n, \mathcal{T});
\]
\[
D_{x_0}(r_n, \Gamma)(1 - D_{x_0}(r_n, \mathcal{T})) \leq D_{x_0}(\mathcal{T}, \Gamma) + D_{x_0}(r_n, \mathcal{T});
\]
hence
\[
D_{x_0}(\mathcal{T}, \Gamma) \leq \lim_{n \to \infty} D_{x_0}(r_n, \Gamma) \leq D_{x_0}(\mathcal{T}, \Gamma).
\]
In a similar way we prove the other equalities. \hfill \Box

We remark that the following general inequalities are true:

\[
\min \{D_{x_0}(r, \Gamma)| r \in \mathbb{R}(x_0)\} \leq \inf \{D_{x_0}(r_{xx_0}, \Gamma)| x \in \Gamma \backslash x_0\} \leq \lim_{r \to x_0} \inf D_{x_0}(r_{xx_0}, \Gamma) \leq \lim_{r \to x_0} \sup D_{x_0}(r_{xx_0}, \Gamma) \leq \sup \{D_{x_0}(r, \Gamma)| r \in \mathbb{R}(x_0)\};
\]

\[
\min \{\overline{D}_{x_0}(r, \Gamma)| r \in \mathbb{R}(x_0)\} \leq \inf \{\overline{D}_{x_0}(r_{xx_0}, \Gamma)| x \in \Gamma \backslash x_0\} \leq \lim_{r \to x_0} \inf \overline{D}_{x_0}(r_{xx_0}, \Gamma) \leq \lim_{r \to x_0} \sup \overline{D}_{x_0}(r_{xx_0}, \Gamma) \leq \sup \{\overline{D}_{x_0}(r, \Gamma)| r \in \mathbb{R}(x_0)\}.
\]

We can prove the following proposition:

**Proposition 2.3.** If

\[
\lim_{r \to x} \inf \overline{D}_{x_0}(r_{xx_0}, \Gamma) = 0,
\]
then

\[
\exists r^* \in \mathbb{R}(x_0) \text{ for which } D_{x_0}(r^*, \Gamma) = 0.
\]

**Proof.** By the previous inequalities, we deduce:

\[
\min \{\overline{D}_{x_0}(s, \Gamma)| s \in \mathbb{R}(x_0)\} = 0.
\]

Then, from Proposition 2.1, $\exists r^* \in \mathbb{R}(x_0)$ such that $0 \leq \overline{D}_{x_0}(r^*, \Gamma) = 0$; hence the thesis follows. \hfill \Box

Furthermore, we have the following theorem:

**Theorem 2.1.**

\[
\min \{\overline{D}_{x_0}(s, \Gamma)| s \in \mathbb{R}(x_0)\} = 0.
\]

**Proof.** We can assume: $\min \{\overline{D}_{x_0}(s, \Gamma)| s \in \mathbb{R}(x_0)\} = \lambda > 0$; then, for every $s \in \mathbb{R}(x_0)$ we have: $\overline{D}_{x_0}(s, \Gamma) > \frac{\lambda}{2}$. Now, we consider two directions $s_1$ and $s_2$; then there exists a ball $B$ about the centre $x_0$ for which, in such ball, $\Gamma$ has the half-lines about directions $s_1$ and $s_2$. We can assume: $D(s_1, s_2) < \frac{\lambda}{2}$, we consider, in $B$, the convex (open) cone $C$ given by $s_1$ and $s_2$. We have $C \cap \Gamma = \emptyset$. In effect, if this fact is not true, eventually with a change of the radius of the ball $B$, we have, for $x \in s_1$:

\[
\frac{d(x, s_2)}{d(x, x_0)} = \frac{d(x, x_2)}{d(x, x_0)} = \frac{d(x, s_2)}{d(x, x_0)} \geq \frac{d(x, \Gamma)}{d(x, x_0)} = \frac{d(x, \Gamma)}{d(x, x_0)},
\]

where $x_2$ is a point of $s_2$ and $x_s$ is a point of $\Gamma$. 
After a finite steps, we can consider a ball $B^*$ of $x_0$ such that: $B^* \cap \Gamma = \{x_0\}$. Then a contradiction.

Now the following proposition holds:

**Proposition 2.4.**

(2.14) \[ \exists r^* \in \mathbb{R}(x_0) \text{ such that } D_{x_0}(r^*, \Gamma) = 0. \]

By proposition 1.4, we deduce easily the following result.

**Proposition 2.5. If**

(2.15) \[ \exists r^* \in \mathbb{R}(x_0) \text{ such that } D_{x_0}(\Gamma, r^*) = 0 \]

then \[ D_{x_0}(\Gamma, r^*) = D_{x_0}(r^*, \Gamma) = 0. \]

Hence the direction $r^*$ and $\Gamma$ are tangent in $x_0$.

The following theorem is true:

**Theorem 2.2.**

(2.16) \[ \liminf_{\Gamma \ni \{x_0\} \ni r \to x_0} \mathcal{D}_{x_0}(r_{x_0}, \Gamma) = 0. \]

**Proof.** Let $s^* \in \mathbb{R}(x_0)$ such that: $\mathcal{D}_{x_0}(s^*, \Gamma) = 0$; then there exists a sequence $(x_n)$ for which: $x_n \in s^*$; $x_n \neq x_0$ and $\lim_{n} d(x_n, x_0) = 0$. Moreover we can assume there exists a sequence $(y_n) \subset \Gamma$ (and $y_n \neq x_0$) such that we have: $\lim_{n} d(x_n, y_n) = d(x, y) = d(x_0, y_0)$. Hence: $\lim_{n} d(y_n, x_0) = 0$.

We denote $r_n = r_{x_0,y_n}$; we can assume there exists $r^* \in \mathbb{R}(x_0)$ such that $\lim_{n} D_{x_0}(r_n, r^*) = 0$.

Then: $\lim_{n} d_{x_0}(s^*, r_n) \leq \lim_{n} d_{x_0}(r, r_n) = 0$.

Moreover, from: $D_{x_0}(s^*, r_n) = D_{x_0}(r_n, s^*) \leq D_{x_0}(r_n, s^*)(1 + D_{x_0}(s^*, r_n))$, we deduce $D_{x_0}(s^*, r^*) = 0$. Hence: $s^* = r^*$.

Now, because $D_{x_0}(r_n, \Gamma) = D_{x_0}(r_n, s^*) \leq D_{x_0}(s^*, \Gamma)(1 + D_{x_0}(r_n, s^*)) = 0$; the following condition holds: $0 \leq D_{x_0}(r_n, \Gamma) \leq D_{x_0}(r_n, s^*)$. Hence: $\lim_{n} D_{x_0}(r_n, \Gamma) = 0$.

3. Open problems

From a theoretical point of view some questions arise in consideration of the obtained results.

The first is the question to reconsider this point of view with respect to other metric in $R^2$ not equivalent to euclidean metric, for example with a non homogenous metric. In such situation it is possible to need a change of the sets $\mathbb{R}(x)$ with some appropriate ”geodetic” directions and that the theorem 2.2 is not true.

In the framework of the euclidean metric we can consider one of the following non-empty sets (see proposition 2.1))

\[ T_1 = \{ r \in \mathbb{R}(x) | D_{x_0}(r, \Gamma) = \min \{ D_{x_0}(s, \Gamma) \} \} \]
\[ T_2 = \{ r \in \mathbb{R}(x) | D_{x_0}(r, \Gamma) = \max \{ D_{x_0}(s, \Gamma) \} \} \]
\[ N_1 = \{ r \in \mathbb{R}(x) | D_{x_0}(r, \Gamma) = \min \{ D_{x_0}(s, \Gamma) \} \} \]
\[ N_2 = \{ r \in \mathbb{R} | \overline{D}_{x_0}(r, \Gamma) = \max \{ \overline{D}_{x_0}(s, \Gamma) \} \}. \]

Then we can consider the more interesting question of the existence of plane simple curve for which one of the sets have, for every point \( x \) a number, fixed and greater than 2, of elements.

Of course, starting from proposition 2.2, analogous set can be defined and for it the same question considered.

References


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