

A V -COHOMOLOGY WITH RESPECT TO COMPLEX LIOUVILLE DISTRIBUTION

ADELINA MANEA AND CRISTIAN IDA

(Communicated by Murat TOSUN)

ABSTRACT. We define a complex Liouville distribution on the holomorphic tangent bundle of a complex Finsler manifold. Some new operators on vertical forms with respect to Liouville distribution are defined, a Dolbeault type lemma is proved and new cohomology groups are studied.

1. INTRODUCTION AND PRELIMINARIES

The idea of decomposing the exterior derivative for real smooth or complex analytic foliated manifolds and the study of their cohomology is due to I. Vaisman (see [13, 14]). There, are proved some Poincaré type Lemmas with respect to some differential operators corresponding to $(0, 1)$ foliated type or to $(0, 1)$ mixed type for the analytic case, respectively. Latter on, in [12] is studied a decomposition of the exterior differential for the complex type forms on complex Finsler manifolds and it is proved a Grothendieck-Dolbeault type Lemma for the conjugated vertical differential operator d''^v which appears in the decomposition of the operator d'' corresponding to $(0, 1)$ complex type. Also, there the v - cohomology groups of a complex Finsler manifold are defined. Recently, in [9] the first author has studied a new type cohomology with respect to Liouville foliation on the tangent bundle of a real Finsler manifold and a de Rham type theorem is obtained.

The main purpose of the present paper is to extend this cohomology theory on complex Finsler manifolds and to find new cohomology groups related to this spaces. Firstly, following [3, 4] and [8], we define the complex Liouville distribution on the holomorphic tangent bundle of a complex Finsler space and we get an adapted basis on the holomorphic vertical bundle with respect to the orthogonal splitting $V'(\widetilde{M}) = L'(\widetilde{M}) \oplus \{\xi\}$, where ξ is the complex Liouville vector field. Next, by analogy with [9], we consider new type of vertical forms with respect to conjugated Liouville distribution of $\bar{v}(s, 0)$ type and $\bar{v}(s-1, 1)$ type, respectively, and we obtain a decomposition of the conjugated vertical differential operator $d''^v = d''^v_{1,0} + d''^v_{0,1}$ for vertical forms of $\bar{v}(s, 0)$ type. Finally, by applying the results from [12] concerning

Date: Received: June 28, 2010 and Accepted: April 10, 2012.

2000 Mathematics Subject Classification. 53B40, 53C12, 53A12.

Key words and phrases. Complex Finsler manifold, Liouville distribution, v -cohomology.

to the operator $d''v$ we prove a Grothendieck-Dolbeault type Lemma with respect to the operator $d''_{1,0}v$ and new cohomology groups are obtained and studied.

Let M be a n dimensional complex manifold and $(z^k), k = 1, \dots, n$ the complex coordinates in a local chart U . The complexified of the real tangent bundle $T_{\mathbb{R}}M$ denoted by $T_{\mathbb{C}}M$ splits into the direct sum of holomorphic and antiholomorphic tangent subbundles $T'M$ and $T''M$, respectively, namely $T_{\mathbb{C}}M = T'M \oplus T''M$. The total space of holomorphic tangent bundle $\pi : T'M \rightarrow M$ is in turn a $2n$ dimensional complex manifold with $u = (z^k, \eta^k), k = 1, \dots, n$ the induced complex coordinates in the local chart $\pi^{-1}(U)$, where $\eta = \eta^k \frac{\partial}{\partial z^k} \in T'_z M$.

A complex Finsler space is a pair (M, F) , where $F : T'M \rightarrow \mathbb{R}_+ \cup \{0\}$ is a continuous function satisfying the conditions

(i) $G := F^2$ is smooth on $\widetilde{M} := T'M - \{o\}$, where o denotes the zero section of $T'M$;

(ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

(iii) $F(z, \lambda\eta) = |\lambda|F(z, \eta)$ for any $\lambda \in \mathbb{C}$, the homogeneity condition;

(iv) the complex hessian $(G_{i\bar{j}}) = (\frac{\partial^2 G}{\partial \eta^i \partial \bar{\eta}^j})$ is positively definite, or equivalently, it mean that the indicatrix $I_z = \{\eta/G_{i\bar{j}}(z, \eta)\eta^i \bar{\eta}^j = 1\}$ is strongly pseudoconvex for any $z \in M$.

In the following we consider the notations:

$$G_i = \frac{\partial G}{\partial \eta^i}, G_{\bar{j}} = \frac{\partial G}{\partial \bar{\eta}^j}, G_{ij} = \frac{\partial^2 G}{\partial \eta^i \partial \eta^j}, G_{i\bar{j}} = \frac{\partial^2 G}{\partial \eta^i \partial \bar{\eta}^j} \text{ etc.}$$

According to [5], the strongly pseudoconvex complex Finsler structure has the following properties:

$$(1.1) \quad G_{ij}\eta^i = 0; G_{i\bar{j}}\bar{\eta}^i = 0; G_i\eta^i = G; G_{\bar{j}}\bar{\eta}^j = G$$

$$(1.2) \quad G_{i\bar{j}k}\eta^i = 0; G_{i\bar{j}\bar{k}}\bar{\eta}^j = 0; G_{i\bar{j}k}\bar{\eta}^j = G_{ik}$$

$$(1.3) \quad G_{i\bar{j}}\eta^i = G_{\bar{j}}; G_{i\bar{j}}\bar{\eta}^j = G_i; G_{i\bar{j}}\eta^i \bar{\eta}^j = G$$

Let $V'(\widetilde{M}) \subset T'(\widetilde{M})$ be the holomorphic vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$ and $V''(\widetilde{M})$ be its conjugate, locally spanned by $\{\frac{\partial}{\partial \bar{\eta}^k}\}$. A *complex nonlinear connection*, briefly c.n.c., on \widetilde{M} is given by a supplementary complex subbundle to $V'(\widetilde{M})$ in $T'(\widetilde{M})$, namely $T'(\widetilde{M}) = H'(\widetilde{M}) \oplus V'(\widetilde{M})$. The horizontal subbundle $H'(\widetilde{M})$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z, \eta)$ are the coefficients of the c.n.c., which obey a certain rule of change at the local charts change such that $\frac{\delta}{\delta z^k} = \frac{\partial z'^j}{\partial z^k} \frac{\delta}{\delta z'^j}$ performs. Obviously, we also have that $\frac{\partial}{\partial \eta^k} = \frac{\partial z'^j}{\partial \eta^k} \frac{\partial}{\partial z'^j}$. The pair $\{\delta_k := \frac{\delta}{\delta z^k}; \dot{\delta}_k := \frac{\partial}{\partial \eta^k}\}$, $k = 1, \dots, n$ will be called the adapted frames of the c.n.c. By conjugation an adapted frame $\{\delta_{\bar{k}}; \dot{\delta}_{\bar{k}}\}$ is obtained on $T''(\widetilde{M})$. The dual adapted bases are given by $\{dz^k\}$, $\{\delta\eta^k = d\eta^k + N_j^k dz^j\}$, $\{d\bar{z}^k\}$ and $\{\delta\bar{\eta}^k = d\bar{\eta}^k + N_{\bar{j}}^{\bar{k}} d\bar{z}^j\}$ which span the dual bundles $H'^*(\widetilde{M})$, $V'^*(\widetilde{M})$, $H''^*(\widetilde{M})$ and $V''^*(\widetilde{M})$, respectively.

According to [1, 2, 11], a c.n.c. related only to the fundamental function of the complex Finsler space (M, F) is almost classical now, the Chern-Finsler c.n.c., locally given by $N_k^j = G^{m\bar{j}} \frac{\partial G_{l\bar{m}}}{\partial z^k} \eta^l$, where $(G^{m\bar{j}})$ denotes the inverse of $(G_{j\bar{m}})$.

Throughout this paper we consider the adapted frames and coframes with respect to this c.n.c.

2. A COMPLEX LIOUVILLE DISTRIBUTION

It is well known that a strongly pseudoconvex complex Finsler structure F defines a hermitian structure on the holomorphic vertical bundle $V'(\widetilde{M})$ by

$$(2.1) \quad G^v(X, \bar{Y})(z, \eta) = G_{i\bar{j}}(z, \eta) \dot{X}^i(z, \eta) \overline{\dot{Y}^j(z, \eta)}$$

for all $X = \dot{X}^i(z, \eta) \dot{\partial}_i$, $Y = \dot{Y}^j(z, \eta) \dot{\partial}_j \in \Gamma(V'(\widetilde{M}))$.

An important global vertical vector field is defined by $\xi = \eta^i \dot{\partial}_i$ and it is called *the complex Liouville vector field* (or radial vertical vector field). We notice that the third equation of (1.3) says that

$$(2.2) \quad G = G^v(\xi, \bar{\xi}) > 0$$

so ξ is an embedding of \widetilde{M} into $V'(\widetilde{M})$.

Let $\{\xi\}$ be the complex line bundle over \widetilde{M} spanned by ξ and we define the complex Liouville distribution as the complementary orthogonal distribution $L'(\widetilde{M})$ to $\{\xi\}$ in $V'(\widetilde{M})$ with respect to G^v , namely $V'(\widetilde{M}) = L'(\widetilde{M}) \oplus \{\xi\}$. Hence, $L'(\widetilde{M})$ is defined by

$$(2.3) \quad \Gamma(L'(\widetilde{M})) = \{X \in \Gamma(V'(\widetilde{M})); G^v(X, \bar{\xi}) = 0\}$$

Consequently, let us consider the vertical vector fields

$$(2.4) \quad X_k = \dot{\partial}_k - t_k \xi, \quad k = 1, \dots, n$$

where the functions $t_k(z, \eta)$ are defined by the conditions

$$(2.5) \quad G^v(X_k, \bar{\xi}) = 0, \quad k = 1, \dots, n$$

Thus, the above conditions become

$$G^v(\dot{\partial}_k, \bar{\eta}^j \dot{\partial}_{\bar{j}}) - t_k G^v(\xi, \bar{\xi}) = 0, \quad k = 1, \dots, n$$

so, taking into account (1.3) and (2.1), we obtain the local expression of the functions t_k in a local chart $(U, (z^i, \eta^i))$

$$(2.6) \quad t_k = \frac{G_k}{G}, \quad k = 1, \dots, n$$

If $(U', (z'^i, \eta'^i))$ is another local chart on \widetilde{M} , then on $U \cap U' \neq \emptyset$, we have

$$t'_j = \frac{G'_{j\bar{l}} \bar{\eta}'^l}{G'} = \frac{1}{G'} \frac{\partial \bar{z}'^l}{\partial \bar{z}^k} \bar{\eta}^k \frac{\partial z^i}{\partial z'^j} \frac{\partial \bar{z}^k}{\partial \bar{z}'^l} G_{i\bar{k}} = \frac{\partial z^i}{\partial z'^j} t_i$$

so we obtain the following changing rule for the vector fields from (2.4)

$$(2.7) \quad X'_j = \frac{\partial z^k}{\partial z'^j} X_k, \quad j = 1, \dots, n$$

By conjugation we obtain the decomposition $V_{\mathbb{C}}(\widetilde{M}) = L'(\widetilde{M}) \oplus \{\xi\} \oplus L''(\widetilde{M}) \oplus \{\bar{\xi}\}$.

Proposition 2.1. *The functions $\{t_k\}$, $k = 1, \dots, n$, locally given by (2.6) satisfies*

$$(2.8) \quad t_k \eta^k = t_{\bar{k}} \bar{\eta}^k = 1 ; X_k \eta^k = X_{\bar{k}} \bar{\eta}^k = 0$$

$$(2.9) \quad \dot{\partial}_k (t_l) = \frac{G_{lk}}{G} - t_l t_k ; \dot{\partial}_{\bar{k}} (t_l) = \frac{G_{l\bar{k}}}{G} - t_l t_{\bar{k}}$$

$$(2.10) \quad \xi t_k = -t_k ; \bar{\xi} t_k = 0 ; \eta^k \dot{\partial}_j (t_k) = -t_j ; \eta^k (\xi t_k) = -1$$

Proof. We have that $t_k \eta^k = \frac{G_k}{G} \eta^k = 1$ and similarly $t_{\bar{k}} \bar{\eta}^k = \frac{G_{\bar{k}}}{G} \bar{\eta}^k = 1$, where we used (2.6) and the last two equalities from (1.1). Now, $X_k \eta^k = (\dot{\partial}_k - t_k \xi) \eta^k = 1 - t_k \eta^k = 0$ and similarly for conjugated. Thus, the relations (2.8) are proved. Similarly, by direct calculations using (1.1), (1.3) and (2.6), one gets (2.9) and (2.10). \square

Proposition 2.2. *There are the relations*

$$(2.11) \quad [X_i, X_j] = t_i X_j - t_j X_i ; [X_i, \xi] = X_i$$

$$(2.12) \quad [X_i, X_{\bar{j}}] = 0 ; [X_{\bar{i}}, \xi] = 0$$

and its conjugates.

Proof. Taking into account (2.10), $[\dot{\partial}_i, \dot{\partial}_j] = 0$, $G_{ij} = G_{ji}$ and the classical properties of Lie brackets we obtain $[X_i, X_j] = -t_j \dot{\partial}_i + t_i \dot{\partial}_j - t_i t_j \xi + t_j t_i \xi = t_i X_j - t_j X_i$ and $[X_i, \xi] = [\dot{\partial}_i, \xi] + \xi(t_i) \xi = \delta_i^k \dot{\partial}_k - t_i \xi = X_i$. Now, by direct calculations we have

$$\begin{aligned} [X_i, X_{\bar{j}}] &= X_{\bar{j}}(t_i) \xi - X_i(t_{\bar{j}}) \bar{\xi} = \dot{\partial}_{\bar{j}} (t_i) \xi - \dot{\partial}_i (t_{\bar{j}}) \bar{\xi} \\ &= \left(\frac{G_{\bar{j}}}{G} - t_{\bar{j}} \right) \dot{\partial}_i - \left(\frac{G_i}{G} - t_i \right) \dot{\partial}_{\bar{j}} = 0 \end{aligned}$$

and $[X_{\bar{i}}, \xi] = [\dot{\partial}_{\bar{i}}, \xi] - t_{\bar{i}} [\bar{\xi}, \xi] + \xi(t_{\bar{i}}) \bar{\xi} = 0$. \square

The above proposition says that the distribution $L'(\widetilde{M})$ is one integrable.

By the conditions (2.5), $\{X_1, \dots, X_n\}$ are n vectors fields orthogonal to ξ , so they belong to the $(n-1)$ dimensional distribution $L'(\widetilde{M})$. It results that they are linear dependent and, from (2.8)

$$(2.13) \quad X_n = -\frac{1}{\eta^n} \sum_{i=1}^{n-1} \eta^i X_i$$

since the local coordinate η^n is nonzero everywhere.

We have

Proposition 2.3. *The system of complex vertical vector fields $\{X_1, \dots, X_{n-1}, \xi\}$ is a locally adapted basis on $V'(\widetilde{M})$ with respect to the complex Liouville distribution.*

Proof. The proof is similar with the analogue result from real case (see [8]), and it consist to check that the rank of the matrix of change from the natural basis $\{\dot{\partial}_k\}$, $k = 1, \dots, n$ of $V'(\widetilde{M})$ to $\{X_1, \dots, X_{n-1}, \xi\}$ is equal to n . \square

As in the real case (see [9]), we have the following concludent remark: Let $(U', (z'^k, \eta'^k))$ and $(U, (z^k, \eta^k))$ be two local charts which domains overlap, where η'^k and η^n are nonzero functions (in every local chart on \widetilde{M} there is at least one nonzero coordinate function η^i).

The adapted basis in U' is $\{X'_1, \dots, X'_{k-1}, X'_{k+1}, \dots, X'_n, \xi\}$. In $U \cap U' \neq \phi$ we have (2.7) and (2.13), hence

$$X'_j = \sum_{i=1}^{n-1} \left(\frac{\partial z^i}{\partial z'^j} - \frac{\eta^i}{\eta^n} \frac{\partial z^n}{\partial z'^j} \right) X_i; \quad X_i = \sum_{j=1, j \neq k}^n \left(\frac{\partial z'^j}{\partial z^i} - \frac{\eta'^j}{\eta'^k} \frac{\partial z'^k}{\partial z^i} \right) X'_j$$

for all $j = 1, \dots, n$, $i = 1, \dots, n-1$. One can see that the above relation also imply

$$\frac{\partial z^i}{\partial z'^k} - \frac{\eta^i}{\eta^n} \frac{\partial z^n}{\partial z'^k} = - \sum_{j=1, j \neq k}^n \frac{\eta'^j}{\eta'^k} \left(\frac{\partial z'^j}{\partial z^i} - \frac{\eta^i}{\eta^n} \frac{\partial z^n}{\partial z'^j} \right)$$

By a straightforward calculation we have that the determinant of the change matrix

$$\{X_1, \dots, X_{n-1}, \xi\} \rightarrow \{X'_1, \dots, X'_{k-1}, X'_{k+1}, \dots, X'_n, \xi\}$$

on $V'(\widetilde{M})$ is equal to $(-1)^{n+k} \frac{\eta'^k}{\eta^n} \det\left(\frac{\partial z^i}{\partial z'^j}\right)$

3. NEW OPERATORS ON VERTICAL FORMS WITH RESPECT TO COMPLEX LIOUVILLE DISTRIBUTION

According to [12], let us consider $\mathcal{A}^{p,q,r,s}(\widetilde{M})$ the set of all (p, q, r, s) -forms with complex values on \widetilde{M} locally defined by

$$(3.1) \quad \varphi = \frac{1}{p!q!r!s!} \varphi_{I_p J_q \overline{H}_r \overline{K}_s}(z, \eta) dz^{I_p} \wedge \delta \eta^{J_q} \wedge d\overline{z}^{H_r} \wedge \delta \overline{\eta}^{K_s}$$

where $I_p = (i_1 \dots i_p)$, $J_q = (j_1 \dots j_q)$, $H_r = (h_1 \dots h_r)$, $K_s = (k_1 \dots k_s)$ and these forms can be nonzero only when they act on p vectors from $\Gamma(H'(\widetilde{M}))$, on q vectors from $\Gamma(V'(\widetilde{M}))$, on r vectors from $\Gamma(H''(\widetilde{M}))$ and s vectors from $\Gamma(V''(\widetilde{M}))$, respectively.

We also consider the conjugated vertical differential operator $d''^v : \mathcal{A}^{p,q,r,s}(\widetilde{M}) \rightarrow \mathcal{A}^{p,q,r,s+1}(\widetilde{M})$, locally given by

$$(3.2) \quad d''^v \varphi = \sum_k \dot{\partial}_k (\varphi_{I_p J_q \overline{H}_r \overline{K}_s}) \delta \overline{\eta}^k \wedge dz^{I_p} \wedge \delta \eta^{J_q} \wedge d\overline{z}^{H_r} \wedge \delta \overline{\eta}^{K_s}$$

where the sum is after $i_1 < \dots < i_p$, $j_1 < \dots < j_q$, $h_1 < \dots < h_r$ and $k_1 < \dots < k_s$.

This operator has the property $(d''^v)^2 = 0$ and it locally satisfies a Grothendieck-Dolbeault type lemma (for details see [12] and [11] p. 88).

Proposition 3.1. *The $(0, 0, 0, 1)$ vertical form $\overline{\omega}_0 = t_{\bar{i}} \delta \overline{\eta}^i$ is globally defined and satisfies*

$$(3.3) \quad \overline{\omega}_0(\overline{\xi}) = 1, \quad \overline{\omega}_0(X_{\overline{a}}) = 0, \quad \overline{\omega}_0 = d''^v(\ln G)$$

for all $a = 1, \dots, n-1$, X_a given by (2.4) and $G = F^2$ is the fundamental function.

Proof. In $U \cap U' \neq \phi$ we have

$$\overline{\omega}'_0 = t'_{\bar{j}} \delta \overline{\eta}'^j = \frac{\partial \overline{z}^i}{\partial \overline{z}'^j} t_{\bar{i}} \frac{\partial \overline{z}'^j}{\partial \overline{z}^k} \delta \overline{\eta}^k = t_{\bar{i}} \delta \overline{\eta}^i = \overline{\omega}_0$$

We also have $\delta\bar{\eta}^i(\bar{\xi}) = \bar{\eta}^i$, for all $i = 1, \dots, n$, and taking into account the first relation of (2.8) it results

$$\bar{\omega}_0(\bar{\xi}) = 1, \bar{\omega}_0(X_{\bar{a}}) = t_{\bar{i}}\delta\bar{\eta}^i(\partial_{\bar{a}} - t_{\bar{a}}\bar{\xi}) = t_{\bar{i}}\delta_{\bar{a}}^{\bar{i}} - t_{\bar{a}}t_{\bar{i}}\bar{\eta}^i = 0$$

where $\delta_{\bar{a}}^{\bar{i}}$ denotes the Kronecker symbols. By conjugation in the relation (2.13) it results also $\bar{\omega}_0(X_{\bar{n}}) = 0$.

Now, we have

$$d''^v(\ln G) = \dot{\partial}_{\bar{k}}(\ln G)\delta\bar{\eta}^k = \frac{G_{\bar{k}}}{G}\delta\bar{\eta}^k = t_{\bar{k}}\delta\bar{\eta}^k = \bar{\omega}_0$$

which ends the proof. \square

We notice that the equality $\bar{\omega}_0 = d''^v(\ln G)$ shows that $\bar{\omega}_0$ is an d''^v -exact vertical $(0, 0, 0, 1)$ -form and the conjugated complex Liouville distribution is defined by the equation $\bar{\omega}_0 = 0$.

In the following, we will consider $\Omega_{\bar{v}}^s(\widetilde{M}) := \mathcal{A}^{0,0,0,s}(\widetilde{M}) \subset \mathcal{A}^{p,q,r,s}(\widetilde{M})$ the subspace of all vertical forms of $(0, 0, 0, s)$ type on \widetilde{M} .

Definition 3.1. A vertical $(0, 0, 0, s)$ -form $\varphi \in \Omega_{\bar{v}}^s(\widetilde{M})$ is called a $\bar{v}(s_1, s_2)$ -form iff for any vertical vector fields $X_1, \dots, X_s \in \Gamma(V''(\widetilde{M}))$ we have $\varphi(X_1, \dots, X_s) \neq 0$ only if s_1 arguments are in $\Gamma(L''(\widetilde{M}))$ and s_2 arguments are in $\Gamma(\{\bar{\xi}\})$.

Since $\{\bar{\xi}\}$ is a line distribution, we can talk only about $\bar{v}(s_1, s_2)$ -forms with $s_2 \in \{0, 1\}$. We will denote the space of $\bar{v}(s_1, s_2)$ -forms by $\Omega_{\bar{v}}^{s_1, s_2}(\widetilde{M})$. By the above definition, we have the equivalence

$$(3.4) \quad \varphi \in \Omega_{\bar{v}}^{s-1, 1}(\widetilde{M}) \Leftrightarrow \varphi(X_1, \dots, X_s) = 0, \forall X_1, \dots, X_s \in \Gamma(L''(\widetilde{M}))$$

Proposition 3.2. Let φ be a nonzero vertical $(0, 0, 0, s)$ -form on \widetilde{M} . The following assertions are true

- (i) $\varphi \in \Omega_{\bar{v}}^{s, 0}(\widetilde{M})$ iff $i_{\bar{\xi}}\varphi = 0$, where i_X denotes the interior product.
- (ii) The vertical $(0, 0, 0, s-1)$ -form $i_{\bar{\xi}}\varphi$ is a $\bar{v}(s-1, 0)$ -form.
- (iii) $\varphi \in \Omega_{\bar{v}}^{s-1, 1}(\widetilde{M})$ implies $i_{\bar{\xi}}\varphi \neq 0$.
- (iv) If there is a $\bar{v}(s-1, 0)$ -form α such that $\varphi = \bar{\omega}_0 \wedge \alpha$ then $\varphi \in \Omega_{\bar{v}}^{s-1, 1}(\widetilde{M})$.

Proof. (i) Let $\varphi \in \Omega_{\bar{v}}^{s, 0}(\widetilde{M})$, hence $\varphi(X_1, \dots, X_s) \neq 0$ only if all the arguments are in $\Gamma(L''(\widetilde{M}))$. So $i_{\bar{\xi}}\varphi$ is a $(0, 0, 0, s-1)$ -form and $(i_{\bar{\xi}}\varphi)(X_1, \dots, X_{s-1}) = \varphi(\bar{\xi}, X_1, \dots, X_{s-1}) = 0$ for every vertical vector fields X_1, \dots, X_{s-1} . So, $i_{\bar{\xi}}\varphi = 0$. Conversely, if φ is a $(0, 0, 0, s)$ -form such that $i_{\bar{\xi}}\varphi = 0$, then $\varphi(X_1, \dots, X_s) = 0$ since there is an index $i \in \{1, \dots, s\}$ such that $X_i = \bar{\xi}$. Hence φ does not vanish only on $L''(\widetilde{M})$, and by definition it is a $\bar{v}(s, 0)$ -form.

(ii) We have $i_{\bar{\xi}}i_{\bar{\xi}}\varphi = 0$ and taking into account (i), it results that $i_{\bar{\xi}}\varphi \in \Omega_{\bar{v}}^{s-1, 0}(\widetilde{M})$.

(iii) If φ is a nonzero $\bar{v}(s-1, 1)$ -form, then $\varphi(X_1, \dots, X_s) \neq 0$ only if exactly one of the arguments is from the line distribution $\bar{\xi}$. Then $(i_{\bar{\xi}}\varphi)(X_1, \dots, X_{s-1}) \neq 0$ for some vertical vector fields $X_1, \dots, X_{s-1} \in \Gamma(L''(\widetilde{M}))$.

(iv) Let α be a form like in hypothesis and X_1, \dots, X_s , s arbitrary vertical vector fields from $\Gamma(L''(\widetilde{M}))$. Then

$$\varphi(X_1, \dots, X_s) = (\bar{\omega}_0 \wedge \alpha)(X_1, \dots, X_s) = \sum_{\sigma \in S_s} \varepsilon(\sigma) \bar{\omega}_0(X_{\sigma(1)}) \alpha(X_{\sigma(2)}, \dots, X_{\sigma(s)})$$

But, $\bar{\omega}_0$ vanishes on $L''(\widetilde{M})$ and for the all arguments $X_1, \dots, X_s \in \Gamma(L''(\widetilde{M}))$ the all terms of the above sum vanish. Then by (3.4), we have $\varphi \in \Omega_{\bar{v}}^{s-1,1}(\widetilde{M})$. \square

Proposition 3.3. *For any vertical $(0, 0, 0, s)$ -form φ there are $\varphi_1 \in \Omega_{\bar{v}}^{s,0}(\widetilde{M})$ and $\varphi_2 \in \Omega_{\bar{v}}^{s-1,1}(\widetilde{M})$ such that $\varphi = \varphi_1 + \varphi_2$, uniquely.*

Proof. Let φ be a nonzero vertical $(0, 0, 0, s)$ -form. If $i_{\bar{\xi}}\varphi = 0$, then by Proposition 3.2., we have $\varphi \in \Omega_{\bar{v}}^{s,0}(\widetilde{M})$. So $\varphi = \varphi + 0$.

If $i_{\bar{\xi}}\varphi \neq 0$, then let φ_2 be the vertical $(0, 0, 0, s)$ -form given by $\bar{\omega}_0 \wedge i_{\bar{\xi}}\varphi$. By Proposition 3.2. (iv), it results φ_2 is a $\bar{v}(s-1, 1)$ -form. Moreover, putting $\varphi_1 = \varphi - \varphi_2$, we have

$$i_{\bar{\xi}}\varphi_1 = i_{\bar{\xi}}\varphi - i_{\bar{\xi}}(\bar{\omega}_0 \wedge i_{\bar{\xi}}\varphi) = i_{\bar{\xi}}\varphi - \bar{\omega}_0(\bar{\xi})i_{\bar{\xi}}\varphi = 0$$

since $\bar{\omega}_0(\bar{\xi}) = 1$. So, φ_1 is a $\bar{v}(s, 0)$ -form and φ_1, φ_2 are unique defined by φ . Obviously $\varphi = \varphi_1 + \varphi_2$. \square

We have to remark that only the zero form can be simultaneous a $\bar{v}(s, 0)$ - and a $\bar{v}(s-1, 1)$ -form, respectively. The above Proposition leads to the decomposition

$$(3.5) \quad \Omega_{\bar{v}}^s(\widetilde{M}) = \Omega_{\bar{v}}^{s,0}(\widetilde{M}) \oplus \Omega_{\bar{v}}^{s-1,1}(\widetilde{M})$$

A consequence of the Propositions 3.2. and 3.3. is

Proposition 3.4. *Let φ be a $(0, 0, 0, s)$ -form. We have the equivalence*

$$(3.6) \quad \varphi \in \Omega_{\bar{v}}^{s-1,1}(\widetilde{M}) \Leftrightarrow \exists \alpha \in \Omega_{\bar{v}}^{s-1,0}(\widetilde{M}), \varphi = \bar{\omega}_0 \wedge \alpha$$

Moreover, the form α is uniquely determined.

Taking into account the characterization given in Proposition 3.2. (i) and the relation (3.6), it follows

Proposition 3.5. *We have the following facts*

- (i) *If $\varphi \in \Omega_{\bar{v}}^{s,0}(\widetilde{M})$ and $\psi \in \Omega_{\bar{v}}^{t,0}(\widetilde{M})$, then $\varphi \wedge \psi \in \Omega_{\bar{v}}^{s+t,0}(\widetilde{M})$.*
- (ii) *If $\varphi \in \Omega_{\bar{v}}^{s,1}(\widetilde{M})$ and $\psi \in \Omega_{\bar{v}}^{t,0}(\widetilde{M})$, then $\varphi \wedge \psi \in \Omega_{\bar{v}}^{s+t,1}(\widetilde{M})$.*
- (iii) *If $\varphi \in \Omega_{\bar{v}}^{s,1}(\widetilde{M})$ and $\psi \in \Omega_{\bar{v}}^{t,1}(\widetilde{M})$, then $\varphi \wedge \psi = 0$.*

Example 3.1. (i) $\bar{\omega}_0 \in \Omega_{\bar{v}}^{0,1}(\widetilde{M})$ since there is the $\bar{v}(0, 0)$ -form, the constant 1 function on \widetilde{M} , such that $\bar{\omega}_0 = \bar{\omega}_0 \cdot 1$.

(ii) $\theta^{\bar{i}} = \delta\bar{\eta}^i - \bar{\eta}^i\bar{\omega}_0 \in \Omega_{\bar{v}}^{1,0}(\widetilde{M})$, for each $i = 1, \dots, n$. Indeed

$$\theta^{\bar{i}}(\bar{\xi}) = \delta\bar{\eta}^i(\bar{\xi}) - \bar{\omega}_0(\bar{\xi})\bar{\eta}^i = 0$$

so $i_{\bar{\xi}}\theta^{\bar{i}} = 0$. We have to remark that the vertical $(0, 0, 0, 1)$ -forms $\{\theta^{\bar{i}}\}$, $i = 1, \dots, n$ are linear dependent, since $\sum t_{\bar{i}}\theta^{\bar{i}} = 0$.

(iii) $i_{\bar{\xi}}(\theta^{\bar{i}} \wedge \theta^{\bar{j}})(X) = \theta^{\bar{i}}(\bar{\xi})\theta^{\bar{j}}(X) - \theta^{\bar{j}}(\bar{\xi})\theta^{\bar{i}}(X) = 0$, for any vertical vector field $X \in \Gamma(V''(\widetilde{M}))$, hence $\theta^{\bar{i}} \wedge \theta^{\bar{j}} \in \Omega_{\bar{v}}^{2,0}(\widetilde{M})$.

Proposition 3.6. *The conjugated vertical differential operator d''^v has the following property: for any $\bar{v}(s-1, 1)$ -form φ , $d''^v\varphi$ is a $\bar{v}(s, 1)$ -form.*

Proof. Let φ be a $\bar{v}(s-1, 1)$ -form. By (3.6), there is a $\bar{v}(s-1, 0)$ -form α such that $\varphi = \bar{\omega}_0 \wedge \alpha$. By Proposition 3.3. we have also that $\alpha = i_{\bar{\xi}}\varphi$. Taking into account that $\bar{\omega}_0$ is an d''^v -exact form, it follows

$$d''^v\varphi = d''^v(\bar{\omega}_0 \wedge \alpha) = -\bar{\omega}_0 \wedge d''^v\alpha = -\bar{\omega}_0 \wedge \beta_1 - \bar{\omega}_0 \wedge \beta_2$$

where β_1 and β_2 are the $\bar{v}(s, 0)$ - and $\bar{v}(s-1, 1)$ -forms, respectively, components of the $(0, 0, 0, s)$ -form $d''^v\alpha$. By (3.6) we have $\beta_2 = \bar{\omega}_0 \wedge \gamma$ with $\gamma \in \Omega_{\bar{v}}^{s-1, 0}(\widetilde{M})$, so $d''^v\varphi = -\bar{\omega}_0 \wedge \beta_1$. Then $d''^v\varphi \in \Omega_{\bar{v}}^{s, 1}(\widetilde{M})$. \square

We can write

$$(3.7) \quad d''^v(\Omega_{\bar{v}}^{s-1, 1}(\widetilde{M})) \subset \Omega_{\bar{v}}^{s, 1}(\widetilde{M})$$

Now, we can consider p_1 and p_2 the projections of the module $\Omega_{\bar{v}}^s(\widetilde{M})$ on its direct summands from the relation (3.5), namely

$$(3.8) \quad p_1 : \Omega_{\bar{v}}^s(\widetilde{M}) \rightarrow \Omega_{\bar{v}}^{s, 0}(\widetilde{M}), \quad p_1\varphi = \varphi - \bar{\omega}_0 \wedge i_{\bar{\xi}}\varphi$$

$$(3.9) \quad p_2 : \Omega_{\bar{v}}^s(\widetilde{M}) \rightarrow \Omega_{\bar{v}}^{s-1, 1}(\widetilde{M}), \quad p_2\varphi = \bar{\omega}_0 \wedge i_{\bar{\xi}}\varphi$$

for any $\varphi \in \Omega_{\bar{v}}^s(\widetilde{M})$.

For an arbitrary $(0, 0, 0, s)$ -form φ , we have $d''^v\varphi = d''^v(p_1\varphi) + d''^v(p_2\varphi)$. The relation (3.5) shows that $d''^v(p_2\varphi)$ is a $\bar{v}(s, 1)$ -form, hence $p_1 d''^v(p_2\varphi) = 0$. It results

$$(3.10) \quad p_1 d''^v\varphi = p_1 d''^v(p_1\varphi), \quad p_2 d''^v\varphi = p_2 d''^v(p_1\varphi) + p_2 d''^v(p_2\varphi)$$

The above relations prove that

$$(3.11) \quad d''^v(\Omega_{\bar{v}}^{s, 0}(\widetilde{M})) \subset \Omega_{\bar{v}}^{s+1, 0}(\widetilde{M}) \oplus \Omega_{\bar{v}}^{s, 1}(\widetilde{M})$$

which allows to define the following operators:

$$(3.12) \quad d''_{1,0} : \Omega_{\bar{v}}^{s, 0}(\widetilde{M}) \rightarrow \Omega_{\bar{v}}^{s+1, 0}(\widetilde{M}), \quad d''_{1,0}\varphi = p_1 d''^v\varphi$$

$$(3.13) \quad d''_{0,1} : \Omega_{\bar{v}}^{s, 0}(\widetilde{M}) \rightarrow \Omega_{\bar{v}}^{s, 1}(\widetilde{M}), \quad d''_{0,1}\varphi = p_2 d''^v\varphi$$

so that

$$(3.14) \quad d''^v|_{\Omega_{\bar{v}}^{s, 0}(\widetilde{M})} = d''_{1,0} + d''_{0,1}$$

Proposition 3.7. *The operator $d''_{1,0}$ satisfies*

$$(i) \quad d''_{1,0}(\varphi \wedge \psi) = d''_{1,0}\varphi \wedge \psi + (-1)^s \varphi \wedge d''_{1,0}\psi, \quad \forall \varphi \in \Omega_{\bar{v}}^{s, 0}(\widetilde{M}), \quad \psi \in \Omega_{\bar{v}}^{t, 0}(\widetilde{M}).$$

$$(ii) \quad (d''_{1,0})^2 = 0.$$

Proof. (i) Let $\varphi \in \Omega_{\bar{v}}^{s, 0}(\widetilde{M})$ and $\psi \in \Omega_{\bar{v}}^{t, 0}(\widetilde{M})$. According to [12], we have

$$d''^v(\varphi \wedge \psi) = d''^v\varphi \wedge \psi + (-1)^s \varphi \wedge d''^v\psi$$

and by (3.14) it follows

$$d''_{1,0}(\varphi \wedge \psi) + d''_{0,1}(\varphi \wedge \psi) = d''_{1,0}\varphi \wedge \psi + d''_{0,1}\varphi \wedge \psi + (-1)^s \varphi \wedge d''_{1,0}\psi + (-1)^s \varphi \wedge d''_{0,1}\psi$$

By equating the $\bar{v}(s+t+1, 0)$ components in the both members of above relation, we get the desired result.

(ii) Let φ be a $\bar{v}(s, 0)$ -form. By (3.8) and (3.12) we have that $d''_{1,0} \varphi = d''^v \varphi - \bar{\omega}_0 \wedge i_{\bar{\xi}}(d''^v \varphi)$. Thus, using $(d''^v)^2 = 0$, $d''^v \bar{\omega}_0 = 0$ and $i_{\bar{\xi}} \bar{\omega}_0 = 1$, by direct calculations, one gets

$$\begin{aligned} (d''_{1,0})^2 \varphi &= d''_{1,0}(d''^v \varphi) - d''_{1,0}(\bar{\omega}_0 \wedge i_{\bar{\xi}}(d''^v \varphi)) \\ &= -d''^v(\bar{\omega}_0 \wedge i_{\bar{\xi}}(d''^v \varphi)) + \bar{\omega}_0 \wedge i_{\bar{\xi}}(d''^v(\bar{\omega}_0 \wedge i_{\bar{\xi}}(d''^v \varphi))) \\ &= \bar{\omega}_0 \wedge d''^v(i_{\bar{\xi}}(d''^v \varphi)) + \bar{\omega}_0 \wedge i_{\bar{\xi}}(-\bar{\omega}_0 \wedge d''^v(i_{\bar{\xi}}(d''^v \varphi))) \\ &= \bar{\omega}_0 \wedge d''^v(i_{\bar{\xi}}(d''^v \varphi)) - \bar{\omega}_0 \wedge d''^v(i_{\bar{\xi}}(d''^v \varphi)) = 0 \end{aligned}$$

□

Example 3.2. (i) For a $(0, 0, 0, 1)$ -form φ , we have $p_1 \varphi = \varphi - \varphi(\bar{\xi})\bar{\omega}_0$ and $p_2 \varphi = \varphi(\bar{\xi})\bar{\omega}_0$.

(ii) Let $f \in \mathcal{F}(\widetilde{M})$ and $d''^v f = \dot{\partial}_{\bar{k}}(f)\delta\bar{\eta}^k$ its conjugated vertical derivative. Locally, we have

$$d''_{0,1} f = p_2 d''^v f = (d''^v f)(\bar{\xi})\bar{\omega}_0 = \bar{\xi}(f)\bar{\omega}_0$$

and

$$d''_{1,0} f = p_1 d''^v f = d''^v f - (d''^v f)(\bar{\xi})\bar{\omega}_0 = \dot{\partial}_{\bar{k}}(f)\delta\bar{\eta}^k - \bar{\eta}^k \dot{\partial}_{\bar{k}}(f)\bar{\omega}_0 = \dot{\partial}_{\bar{k}}(f)\theta^{\bar{k}}$$

where $\theta^{\bar{k}}$ are the $\bar{v}(1, 0)$ -forms given in Example 3.1. Moreover, taking into account the relation (2.4) and the fact $\sum t_{\bar{i}}\theta^{\bar{i}} = 0$, it results that locally

$$(3.15) \quad d''_{1,0} f = (X_{\bar{i}} f)\theta^{\bar{i}}$$

We have

$$d''_{1,0} \bar{\eta}^k = (X_{\bar{i}} \bar{\eta}^k)\theta^{\bar{i}} = \delta_{\bar{i}}^{\bar{k}} \theta^{\bar{i}} - t_{\bar{i}} \bar{\xi}(\bar{\eta}^k)\theta^{\bar{i}} = \theta^{\bar{k}} - (t_{\bar{i}} \theta^{\bar{i}})\bar{\eta}^k = \theta^{\bar{k}}$$

so the $\bar{v}(1, 0)$ -forms $\theta^{\bar{k}}$ are exactly the $d''_{1,0}$ -derivatives of the local coordinates $\bar{\eta}^k$, for all $k = 1, \dots, n$.

(iii) The $\bar{v}(2, 0)$ -forms $d''_{1,0} \bar{\eta}^k \wedge d''_{1,0} \bar{\eta}^j$ are $d''_{1,0}$ -closed, for all $j, k = 1, \dots, n$.

Let us consider an arbitrary $(0, 0, 0, 1)$ -form on \widetilde{M} . It is locally given in U by $\varphi = \varphi_{\bar{i}} \delta \bar{\eta}^i$, with $\varphi_{\bar{i}} \in \mathcal{F}(U)$ such that in $U \cap U' \neq \emptyset$ we have $\varphi'_{\bar{j}} = \frac{\partial \bar{z}^i}{\partial \bar{z}'^j} \varphi_{\bar{i}}$. By the Proposition 3.2., φ is a $\bar{v}(1, 0)$ -form on \widetilde{M} iff $i_{\bar{\xi}} \varphi = 0$ which is equivalent locally with $\varphi_{\bar{i}} \bar{\eta}^i = 0$. Then, locally we have

$$\varphi = \varphi_{\bar{i}} \delta \bar{\eta}^i = \varphi_{\bar{i}} (d''_{1,0} \bar{\eta}^i + \bar{\eta}^i \bar{\omega}_0) = \varphi_{\bar{i}} d''_{1,0} \bar{\eta}^i + (\varphi_{\bar{i}} \bar{\eta}^i) \bar{\omega}_0 = \varphi_{\bar{i}} d''_{1,0} \bar{\eta}^i$$

Conversely, the expression locally given by $\varphi_{\bar{i}} d''_{1,0} \bar{\eta}^i$, with the functions $\varphi_{\bar{i}}$ satisfying $\varphi'_{\bar{j}} = \frac{\partial \bar{z}^i}{\partial \bar{z}'^j} \varphi_{\bar{i}}$ is a $\bar{v}(1, 0)$ -form because $d''_{1,0} \bar{\eta}^i(\bar{\xi}) = 0$, for all $i = 1, \dots, n$.

4. A $d''_{1,0}$ -COHOMOLOGY

In [12] a classical theory of de Rham cohomology is developed for the conjugated vertical differential d''^v . The sequence

$$O \rightarrow \Phi^0 \xrightarrow{i} \mathcal{F}_v^0 \xrightarrow{d''^v} \mathcal{F}_v^1 \xrightarrow{d''^v} \mathcal{F}_v^2 \xrightarrow{d''^v} \dots \xrightarrow{d''^v} \dots,$$

is a fine resolution for the sheaf Φ^0 of germs of d''^v -closed functions on \widetilde{M} , where $\mathcal{F}_{\bar{v}}^s$ are the sheaves of germs of vertical $(0,0,0,s)$ -forms. It is proved a de Rham type theorem for the v -cohomology groups of the complex Finsler manifold:

$$H^s(\widetilde{M}, \Phi^0) \approx Z_{\bar{v}}^s(\widetilde{M})/B_{\bar{v}}^s(\widetilde{M})$$

where $Z_{\bar{v}}^s(\widetilde{M})$ is the space of d''^v -closed $(0,0,0,s)$ -forms and $B_{\bar{v}}^s(\widetilde{M})$ is the space of d''^v -exact $(0,0,0,s)$ -forms globally defined on \widetilde{M} .

In this section we define new cohomology groups on \widetilde{M} and we study the relations between these groups and $H^s(\widetilde{M}, \Phi^0)$.

Definition 4.1. We say that a $\bar{v}(s,0)$ -form φ is $d''_{1,0}{}^v$ -closed if $d''_{1,0}{}^v\varphi = 0$ and it is called $d''_{1,0}{}^v$ -exact if $\varphi = d''_{1,0}{}^v\psi$ for some $\psi \in \Omega_{\bar{v}}^{s-1,0}(\widetilde{M})$.

An important property of the operator $d''_{1,0}{}^v$ is a Grothendieck-Dolbeault type lemma, namely

Theorem 4.1. *Let $\varphi \in \Omega_{\bar{v}}^{s,0}(U)$ be a $d''_{1,0}{}^v$ -closed form and $s \geq 1$. Then there exists $\psi \in \Omega_{\bar{v}}^{s-1,0}(U')$, $U' \subset U$ and such that $\varphi = d''_{1,0}{}^v\psi$ on U' .*

Proof. Let $\varphi \in \Omega_{\bar{v}}^{s,0}(U)$ such that $d''_{1,0}{}^v\varphi = 0$. Then

$$d''^v\varphi = d''_{1,0}{}^v\varphi + d''_{0,1}{}^v\varphi = d''_{0,1}{}^v\varphi = \bar{\omega}_0 \wedge i_{\bar{\xi}}(d''^v\varphi)$$

so $d''^v\varphi = 0$ (modulo terms containing $\bar{\omega}_0$).

Hence on the space $\bar{\omega}_0 = 0$ we have that φ is d''^v -closed. But the operator d''^v satisfies a Grothendieck-Dolbeault type lemma (see [12]), so there exists a $(0,0,0,s-1)$ -form τ defined on $U' \subset U$ such that

$$(4.1) \quad \varphi = d''^v\tau + \lambda \wedge \bar{\omega}_0, \quad \lambda \in \Omega_{\bar{v}}^{s-1}(U')$$

Following the Proposition 3.3. we have that $\tau = \tau_1 + \bar{\omega}_0 \wedge i_{\bar{\xi}}\tau$, with $\tau_1 = p_1\tau \in \Omega_{\bar{v}}^{s-1,0}(U')$. Now, the relation (4.1) become

$$\varphi = d''^v\tau_1 - \bar{\omega}_0 \wedge d''^v i_{\bar{\xi}}\tau + \lambda \wedge \bar{\omega}_0$$

Here $\varphi \in \Omega_{\bar{v}}^{s,0}(U')$, $\bar{\omega}_0 \wedge (\lambda + d''^v i_{\bar{\xi}}\tau) \in \Omega_{\bar{v}}^{s-1,1}(U')$ and $d''^v\tau_1 = d''_{1,0}{}^v\tau_1 + d''_{0,1}{}^v\tau_1 \in \Omega_{\bar{v}}^{s,0}(U') \oplus \Omega_{\bar{v}}^{s-1,1}(U')$. It results $\varphi = d''_{1,0}{}^v\tau_1$ on U' . \square

Let $\Phi_{\bar{v}}$ be the sheaf of germs of functions on \widetilde{M} which satisfies $d''_{1,0}{}^vf = 0$ and $\mathcal{F}_{\bar{v}}^{s,0}$ be the sheaf of germs of $\bar{v}(s,0)$ -forms on \widetilde{M} . We denote by $i : \Phi_{\bar{v}} \rightarrow \mathcal{F}_{\bar{v}}^{0,0}$ be the natural inclusion. The sheaves $\mathcal{F}_{\bar{v}}^{s,0}$ are fine and taking into account the Theorem 4.1., it follows that the sequence of sheaves

$$0 \longrightarrow \Phi_{\bar{v}} \xrightarrow{i} \mathcal{F}_{\bar{v}}^{0,0} \xrightarrow{d''_{1,0}{}^v} \mathcal{F}_{\bar{v}}^{1,0} \xrightarrow{d''_{1,0}{}^v} \dots \xrightarrow{d''_{1,0}{}^v} \mathcal{F}_{\bar{v}}^{s,0} \xrightarrow{d''_{1,0}{}^v} \dots$$

is a fine resolution of $\Phi_{\bar{v}}$ and we denote by $H^s(\widetilde{M}, \Phi_{\bar{v}})$ the cohomology groups of \widetilde{M} with the coefficients in the sheaf $\Phi_{\bar{v}}$. Then, we obtain a de Rham type theorem, namely

Theorem 4.2. *The \bar{v} -cohomology groups with respect to the operator $d''_{1,0}{}^v$ of $\bar{v}(s,0)$ -forms on a complex Finsler manifold are given by*

$$(4.2) \quad H^s(\widetilde{M}, \Phi_{\bar{v}}) \approx Z_{\bar{v}}^{s,0}(\widetilde{M})/B_{\bar{v}}^{s,0}(\widetilde{M})$$

where $Z_{\bar{v}}^{s,0}(\widetilde{M})$ is the space of d''^v -closed $\bar{v}(s,0)$ -forms and $B_{\bar{v}}^{s,0}(\widetilde{M})$ is the space of d''^v -exact $\bar{v}(s,0)$ -forms globally defined on \widetilde{M} .

By (3.7), the de Rham complex

$$O \rightarrow \mathcal{F}_{\bar{v}}^{0,0}(\widetilde{M}) \xrightarrow{d''^v} \Omega_{\bar{v}}^1(\widetilde{M}) \xrightarrow{d''^v} \Omega_{\bar{v}}^2(\widetilde{M}) \xrightarrow{d''^v} \dots \xrightarrow{d''^v} \Omega_{\bar{v}}^s(\widetilde{M}) \xrightarrow{d''^v} \dots,$$

admits the subcomplex

$$O \rightarrow \Phi_{\bar{v}}(\widetilde{M}) \xrightarrow{d''^v} \Omega_{\bar{v}}^{0,1}(\widetilde{M}) \xrightarrow{d''^v} \Omega_{\bar{v}}^{1,1}(\widetilde{M}) \xrightarrow{d''^v} \dots \xrightarrow{d''^v} \Omega_{\bar{v}}^{s,1}(\widetilde{M}) \xrightarrow{d''^v} \dots,$$

We denote by $Z_{\bar{v}}^{s,1}(\widetilde{M})$ and $B_{\bar{v}}^{s,1}(\widetilde{M})$ the spaces of the d''^v -closed and d''^v -exact $\bar{v}(s,1)$ -forms, respectively, and let

$$(4.3) \quad H_{\bar{v}}^{s,1}(\widetilde{M}) = Z_{\bar{v}}^{s,1}(\widetilde{M})/B_{\bar{v}}^{s,1}(\widetilde{M}).$$

be the s -cohomology group of the last complex.

Theorem 4.3. *The cohomology groups $H_{\bar{v}}^{s,1}(\widetilde{M})$ and $H^s(\widetilde{M}, \Phi_{\bar{v}})$ are isomorphic.*

Proof. By Proposition 3.4 we can define the map

$$\zeta : Z_{\bar{v}}^{s,1}(\widetilde{M}) \rightarrow Z_{\bar{v}}^{s,0}(\widetilde{M}), \quad \zeta(\varphi) = \alpha$$

for $\alpha \in \Omega_{\bar{v}}^{s,0}(\widetilde{M})$ such that $\varphi = \alpha \wedge \bar{\omega}_0$. It is a well-defined map since the equality

$$0 = d''^v \varphi = d''^v \alpha \wedge \bar{\omega}_0 = d''^v \alpha \wedge \bar{\omega}_0 + d''^v \alpha \wedge \bar{\omega}_0 = d''^v \alpha \wedge \bar{\omega}_0$$

implies $d''^v \alpha = 0$. Moreover, ζ is a bijective morphism of groups and $\zeta(B_{\bar{v}}^{s,1}(\widetilde{M})) = B_{\bar{v}}^{s,0}(\widetilde{M})$. Indeed, for $\varphi \in B_{\bar{v}}^{s,1}(\widetilde{M})$, there is $\theta \in \Omega_{\bar{v}}^{s-1,1}(\widetilde{M})$ such that $\varphi = d''^v \theta$. By (3.6), there are $\alpha \in \Omega_{\bar{v}}^{s,0}(\widetilde{M})$, $\beta \in \Omega_{\bar{v}}^{s-1,0}(\widetilde{M})$ such that $\varphi = \alpha \wedge \bar{\omega}_0$ and $\theta = \beta \wedge \bar{\omega}_0$. Then, we have

$$\alpha \wedge \bar{\omega}_0 = d''^v(\beta \wedge \bar{\omega}_0) = d''^v \beta \wedge \bar{\omega}_0$$

It follows $\alpha \in B_{\bar{v}}^{s,0}(\widetilde{M})$. Conversely, $\alpha = d''^v \beta$ implies $\alpha \wedge \bar{\omega}_0 = d''^v(\beta \wedge \bar{\omega}_0)$. We obtain that $\zeta^* : H_{\bar{v}}^{s,1}(\widetilde{M}) \rightarrow Z_{\bar{v}}^{s,0}(\widetilde{M})/B_{\bar{v}}^{s,0}(\widetilde{M})$, $\zeta^*([\varphi]) = [\zeta(\varphi)]$, for $\varphi \in Z_{\bar{v}}^{s,1}(\widetilde{M})$, is bijective. \square

Finally, we remark that the projection p_1 from (3.8) induces the morphism

$$p'_1 : Z_{\bar{v}}^s(\widetilde{M}) \rightarrow Z_{\bar{v}}^{s,0}(\widetilde{M})$$

Indeed, taking into account the Propositions 3.3. and 3.6. and the relations (3.7) and (3.14), we obtain for $\varphi \in Z_{\bar{v}}^s(\widetilde{M})$:

$$0 = d''^v \varphi = d''^v(\varphi_1 + \varphi_2) = d''^v \varphi_1 + d''^v \varphi_1 + d''^v \varphi_2$$

which implies $d''^v \varphi_1 = 0$ since $d''^v \varphi_1 \in \Omega_{\bar{v}}^{s+1,0}(\widetilde{M})$ and $d''^v \varphi_1 + d''^v \varphi_2 \in \Omega_{\bar{v}}^{s,1}(\widetilde{M})$.

Moreover, for every d''^v -exact form $\varphi = d''^v \theta$, with $\theta \in \Omega_{\bar{v}}^{s-1}(\widetilde{M})$, we have by Proposition 3.3. that

$$\varphi = \varphi_1 + \varphi_2 = d''^v(\theta_1 + \theta_2) = d''^v \theta_1 + d''^v \theta_1 + d''^v \theta_2$$

Equating the $\bar{v}(s,0)$ -components we obtain $\varphi_1 = d''^v \theta_1$.

So, the morphism p'_1 satisfies $p'_1(B_{\bar{v}}^s(\widetilde{M})) = B_{\bar{v}}^{s,0}(\widetilde{M})$. Hence p'_1 induces a morphism of cohomology groups

$$p_1^* : Z_{\bar{v}}^s(\widetilde{M})/B_{\bar{v}}^s(\widetilde{M}) \rightarrow Z_{\bar{v}}^{s,0}(\widetilde{M})/B_{\bar{v}}^{s,0}(\widetilde{M})$$

which is not always injective.

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DEPARTMENT OF MATHEMATICS AND INFORMATICS, TRANSILVANIA UNIVERSITY OF BRAŞOV,
ROMÂNIA

E-mail address: amanea28@yahoo.com; cristian.ida@unitbv.ro