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REGULAR POLYGONS IN SOME MODELS OF PROTRACTOR GEOMETRY

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Abstract. Regular polygons in the taxicab and Chinese checker planes studied in [3] and [4], respectively. In this work we study regular polygons in the plane defined by the maximum metric. However, statements of all propositions are given not only for maximum plane but also for the taxicab and Chinese checker planes to show the common properties of them.

1. Introduction

A *metric geometry* consists of a set $\mathcal{P}$, whose elements are called *points*, together with a collection $\mathcal{L}$ of non-empty subsets of $\mathcal{P}$, called *lines*, and a distance function d, such that

1) Every two distinct points in $\mathcal{P}$ lie on a unique line,
2) There exist three points in $\mathcal{P}$, which do not lie all on one line,
3) There exists a bijective function $f : l \to \mathbb{R}$ for all lines in $\mathcal{L}$ such that $|f(P) - f(Q)| = d(P,Q)$ for each pair of points P and Q on l.

A metric geometry defined above is denoted by $\{\mathcal{P}, \mathcal{L}, d\}$. However, if a metric geometry satisfies the plane separation axiom below, and it has an angle measure function m, then it is called *protractor geometry* and denoted by $\{\mathcal{P}, \mathcal{L}, d, m\}$.

4) For every l in $\mathcal{L}$, there are two subsets H_1 and H_2 of $\mathcal{P}$ (called *half planes* determined by l) such that
(i) $H_1 \cup H_2 = \mathcal{P} - l$ ($\mathcal{P}$ with l removed),
(ii) H_1 and H_2 are disjoint and each is convex,
(iii) If $A \in H_1$ and $B \in H_2$, then $[AB] \cap l \neq \varnothing$.

Let $P=(x_1, y_1)$ and $Q=(x_2, y_2)$ be two points in the Cartesian coordinate plane, then $d_E(P,Q) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{1/2}$, $d_M(P,Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$, $d_T(P,Q) = |x_1 - x_2| + |y_1 - y_2|$ and $d_C(P,Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + (\sqrt{2} - 1)\min\{|x_1 - x_2|, |y_1 - y_2|\}$ are called *Euclidean*, *maximum*, *taxicab* and *Chinese*

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checker (CC) distances between P and Q, respectively. If L_E is the set of all lines in the Cartesian coordinate plane, and m_E is the standard angle measure function in the Euclidean plane, then $\{\mathbb{R}^2, L_E, d_M, m_E\}$, $\{\mathbb{R}^2, L_E, d_T, m_E\}$ and $\{\mathbb{R}^2, L_E, d_C, m_E\}$ are models of protractor geometry (see [7], [5], [8]), and they are called maximum, taxicab and CC plane, respectively. Clearly, these planes are almost the same as the Euclidean plane $\{\mathbb{R}^2, L_E, d_E, m_E\}$ since the points are the same, the lines are the same, and the angles are measured in the same way. However, the distance functions are different. Since maximum, taxicab and CC planes have distance functions different from that in the Euclidean plane, it is interesting to study the maximum, taxicab and CC analogues of topics that include the distance concept in the Euclidean plane. During the recent years, many such topics have been studied in these plane geometries (see references of [10] for some of studies). In this study, we define maximum regular polygons, and determine which Euclidean regular polygons are also maximum regular, and which are not. Finally, we investigate the existence or nonexistence of maximum regular polygons. However, statements of all propositions are given not only for maximum plane but also for the taxicab and Chinese checker planes to show the common properties of them.

2. **Maximum, Taxicab and CC Regular Polygons**

As in the Euclidean plane, a *polygon* in the maximum, taxicab and CC planes consists of three or more coplanar line segments; the line segments (*side*s) intersect only at endpoints; each endpoint (*vertex*) belongs to exactly two line segments; no two line segments with a common endpoint are collinear. If the number of sides of a polygon is n ($n \geqslant 3$, $n \in \mathbb{N}$), then the polygon is called an *n-gon*. The following definitions for the polygons in the Cartesian coordinate plane are given by means of the maximum, taxicab and CC lengths instead of the Euclidean lengths:

Definition 2.1. A polygon in the plane is said to be *maximum, taxicab* and *CC equilateral* if maximum, taxicab and CC lengths of its sides are equal, respectively.

Definition 2.2. A polygon in the plane is said to be *maximum, taxicab* and *CC equiangular* if measures of its interior angles are equal.

Definition 2.3. A polygon in the plane is said to be *maximum, taxicab* and *CC regular* if it is both maximum, taxicab and CC equilateral and equiangular, respectively.

Definition 2.2 does not give a new equiangular concept because angles are measured in the maximum, taxicab, CC and Euclidean plane in the same way (However, it is possible to define different angle measures in maximum, taxicab and CC planes; see [1] and [11] for example). That is, every Euclidean equiangular polygon is also the maximum, taxicab and CC equiangular, and vice versa. However, since the maximum, taxicab and CC planes has different distance functions, Definition 2.1, and therefore Definition 2.3 are new concepts. In the next section, we determine which Euclidean regular polygons in the plane are also maximum regular, and which are not.

The following equations, which relate the Euclidean distance to the maximum, taxicab and CC distances between two points in the Cartesian coordinate plane, plays an important role in our arguments.

Lemma 2.1. *For any two points P and Q in the Cartesian coordinate plane that do not lie on a vertical line, if m is the slope of the line through P and Q, then*

$$d_E(P,Q) = \rho_M(m)d_M(P,Q) = \rho_T(m)d_T(P,Q) = \rho_C(m)d_C(P,Q) \tag{2.1}$$

where $\rho_M(m) = (1+m^2)^{1/2} \big/ \max\{1,|m|\}$, $\rho_T(m) = (1+m^2)^{1/2} \big/ (1+|m|)$, and $\rho_C(m) = (1+m^2)^{1/2} \big/ (\max\{1,|m|\} + (\sqrt{2}-1)\min\{1,|m|\})$. If P and Q lie on a vertical line, then

$$d_E(P,Q) = d_M(P,Q) = d_T(P,Q) = d_C(P,Q). \tag{2.2}$$

Proof. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ with $x_1 \neq x_2$; then $m = (y_2 - y_1) \big/ (x_2 - x_1)$, $d_E(P,Q) = |x_1 - x_2|\,(1+m)^{1/2}$, $d_M(P,Q) = |x_1 - x_2|\,(\max\{1,|m|\})$, $d_T(P,Q) = |x_1 - x_2|\,(1+|m|)$ and $d_C(P,Q) = |x_1 - x_2|\,(\max\{1,|m|\} + (\sqrt{2} - 1)\min\{1,|m|\})$. Now, it is obvious that Equation (2.1) is true. Equation (2.2) is derived by straightforward calculations with the coordinate definitions of $d_E(P,Q)$, $d_M(P,Q)$, $d_T(P,Q)$ and $d_C(P,Q)$ given in Section 1. $\square$

Another useful fact is:

Lemma 2.2. *For $m_1, m_2 \in \mathbb{R}$-$\{0\}$,*

(i) $\rho_M(m_1) = \rho_M(m_2) \Leftrightarrow |m_1| = |m_2|$ *or* $|m_1 m_2| = 1$.

(ii) $\rho_T(m_1) = \rho_T(m_2) \Leftrightarrow |m_1| = |m_2|$ *or* $|m_1 m_2| = 1$.

(iii) $\rho_C(m_1) = \rho_C(m_2) \Leftrightarrow |m_1| = |m_2|$, $|m_1 m_2| = 1$, $|1 - |m_1 m_2|| = |m_1| + |m_2|$ *or* $||m_1| - |m_2|| = 1 + |m_1 m_2|$.

Proof. (*i*) Let $\rho(m_1) = \rho(m_2)$ for $m_1, m_2 \in \mathbb{R}$-$\{0\}$. If "$|m_1| \leq 1$ and $|m_2| \leq 1$" or "$|m_1| \geq 1$ and $|m_2| \geq 1$", then $m_1^2 = m_2^2$, which implies $|m_1| = |m_2|$. If "$|m_1| \leq 1$ and $|m_2| \geq 1$" or "$|m_1| \geq 1$ and $|m_2| \leq 1$", then $m_1^2 m_2^2 = 1$, which implies $|m_1 m_2| = 1$. The sufficient condition can be verified by direct calculations using $m_2 = m_1$, $m_2 = -m_1$, $m_2 = 1/m_1$ and $m_2 = -1/m_1$.

(*ii*) Let $\rho_T(m_1) = \rho_T(m_2)$ for $m_1, m_2 \in \mathbb{R}$-$\{0\}$. Then by simple calculations, one can easily get that $(|m_1| - |m_2|)(|m_1 m_2| - 1) = 0$, which implies $|m_1| = |m_2|$ or $|m_1 m_2| = 1$. The sufficient condition can be verified by direct calculations using $m_2 = m_1$, $m_2 = -m_1$, $m_2 = 1/m_1$ and $m_2 = -1/m_1$.

(*iii*) Let $\rho_C(m_1) = \rho_C(m_2)$ for $m_1, m_2 \in \mathbb{R}$-$\{0\}$. If "$|m_1| \leq 1$ and $|m_2| \leq 1$" or "$|m_1| \geq 1$ and $|m_2| \geq 1$", then one can easily obtain $(|m_1| - |m_2|)(|m_1 m_2| + |m_1| + |m_2| - 1) = 0$ and $(|m_1| - |m_2|)(|m_1 m_2| - |m_1| - |m_2| - 1) = 0$, respectively, which imply $|m_1| = |m_2|$ or $|1 - |m_1 m_2|| = |m_1| + |m_2|$. If "$|m_1| \leq 1$ and $|m_2|$" or "$|m_1| \geq 1$ and $|m_2| \leq 1$", then similarly one gets $(|m_1 m_2| - 1)(|m_1 m_2| + |m_1| - |m_2| + 1) = 0$ or $(|m_1 m_2| - 1)(|m_1 m_2| + |m_2| - |m_1| + 1) = 0$, respectively, which imply $|m_1 m_2| = 1$ or $||m_1| - |m_2|| = 1 + |m_1 m_2|$. The sufficient condition can be verified by direct calculations using $m_2 = m_1$, $m_2 = -m_1$, $m_2 = 1/m_1$, $m_2 = -1/m_1$, $m_2 = (1 - m_1)/(1 + m_1)$, $m_2 = (m_1 - 1)/(1 + m_1)$, $m_2 = (1 + m_1)/(1 - m_1)$ and $m_2 = (1 + m_1)/(m_1 - 1)$. $\square$

The following theorem gives the necessary and sufficient conditions for two line segments having the same Euclidean length to have the same maximum, taxicab or CC lengths, respectively, in terms of slopes of the line segments.

Theorem 2.1. *Let A, B, C, and D be four points in the Cartesian coordinate plane, such that $A \neq B$ and $d_E(A,B) = d_E(C,D)$; and let m_1 and m_2 denote the slopes of lines AB and CD, respectively.*

(i) *If $m_1, m_2 \in \mathbb{R}$-$\{0\}$, then*
$d_M(A,B) = d_M(C,D) \Leftrightarrow |m_1| = |m_2|$ *or* $|m_1 m_2| = 1$,
$d_T(A,B) = d_T(C,D) \Leftrightarrow |m_1| = |m_2|$ *or* $|m_1 m_2| = 1$,
$d_C(A,B) = d_C(C,D) \Leftrightarrow |m_1| = |m_2|$, $|m_1 m_2| = 1$, $|1 - |m_1 m_2|| = |m_1| + |m_2|$ *or* $||m_1| - |m_2|| = 1 + |m_1 m_2|$.

(ii) *For $i, j \in \{1,2\}$, $i \neq j$; if $m_i = 0$ or $m_i \to \infty$, then*
$d_M(A,B) = d_M(C,D) \Leftrightarrow m_j = 0$ *or* $m_j \to \infty$,
$d_T(A,B) = d_T(C,D) \Leftrightarrow m_j = 0$ *or* $m_j \to \infty$,
$d_C(A,B) = d_C(C,D) \Leftrightarrow m_j = 0$, $m_j = 1$, $m_j = -1$ *or* $m_j \to \infty$.

Proof. (i) This follows immediately from Equation (2.1) and Lemma 2.2.
(ii) Let $i = 1$ and $j = 2$. If $m_1 = 0$ or $m_1 \to \infty$, then $d_E(A,B) = d_M(A,B) = d_T(A,B) = d_C(A,B)$. Therefore $d_M(A,B) = d_M(C,D)$, $d_T(A,B) = d_T(C,D)$ and $d_C(A,B) = d_C(C,D)$ if and only if $\rho_M(m_2) = 1$, $\rho_T(m_2) = 1$ and $\rho_C(m_2) = 1$, respectively. It is clear that $\rho_M(m_2) = 1 \Leftrightarrow m_1 = 0$ or $m_1 \to \infty$; $\rho_T(m_2) = 1 \Leftrightarrow m_1 = 0$ or $m_1 \to \infty$; and $\rho_C(m_2) = 1 \Leftrightarrow m_1 = 0$, $m_1 = 1$, $m_1 = -1$ or $m_1 \to \infty$. The case of $i = 2$ and $j = 1$ is similar. $\square$

Let l_1 and l_2 be two non-vertical lines; let A be the measure of the non-obtuse angle between l_1 and l_2; and let m_1 and m_2 be slopes of l_1 and l_2, respectively. Then following properties can be verified by calculations:
(i) $|m_1| = |m_2|$ if and only if l_1 and l_2 are coincident, parallel or symmetric about a line parallel to x-axis or y-axis.
(ii) $|m_1 m_2| = 1$ if and only if $A = \pi/2$ or l_1 and l_2 symmetric about a line parallel to one of lines $y = x$ and $y = -x$.
(iii) $|1 - |m_1 m_2|| = |m_1| + |m_2|$ or $||m_1| - |m_2|| = 1 + |m_1 m_2|$ if and only if $A = \pi/4$ or l_1 and l_2 symmetric about a line parallel to one of lines $y = (\sqrt{2} - 1)x$, $y = -(\sqrt{2} - 1)x$, $y = (\sqrt{2} + 1)x$, $y = -(\sqrt{2} + 1)x$.

Now, let us denote by S_1 the set of lines $x = 0$, $y = 0$, $y = x$ and $y = -x$; and by S_2 the set of lines $y = (\sqrt{2} - 1)x$, $y = -(\sqrt{2} - 1)x$, $y = (\sqrt{2} + 1)x$, $y = -(\sqrt{2} + 1)x$, and the lines in S_1. Then one can immediately state following corollary:

Corollary 2.1. *Let A, B, and C be three non-collinear points in the Cartesian coordinate plane, such that $d_E(A,B) = d_E(B,C)$. Then,*
(i) *$d_M(A,B) = d_M(B,C)$ if and only if $\measuredangle(ABC) = \pi/2$ or A and C are symmetric about the line passing through B, and parallel to a line in S_1.*
(ii) *$d_T(A,B) = d_T(B,C)$ if and only if $\measuredangle(ABC) = \pi/2$ or A and C are symmetric about the line passing through B, and parallel to a line in S_1.*

(iii) $d_C(A,B) = d_C(B,C)$ *if and only if* $\measuredangle(ABC) \in \{\pi/4, \pi/2, 3\pi/4\}$ *or* A *and* C *are symmetric about the line passing through* B*, and parallel to a line in* S_2*.*

Note that Theorem 2.1 and Corollary 2.1 indicate also Euclidean isometries of the plane that do not change the maximum, taxicab and CC distances between any two points, respectively:

(*i*) Euclidean isometries of the plane that do not change the maximum distance between any two points are all translations, rotations of $\pi/2$, π and $3\pi/2$ radians around a point, reflections about lines parallel to a line in S_1, and their compositions; there is no other bijections of $\mathbb{R}^2$ onto $\mathbb{R}^2$ which preserve the maximum distance (see [8]).

(*ii*) Euclidean isometries of the plane that do not change the taxicab distance between any two points are all translations, rotations of $\pi/2$, π and $3\pi/2$ radians around a point, reflections about lines parallel to a line in S_1, and their compositions; there is no other bijections of $\mathbb{R}^2$ onto $\mathbb{R}^2$ which preserve the taxicab distance (see [9]).

(*iii*) Euclidean isometries of the plane that do not change the CC distance between any two points are all translations, rotations of $\pi/4$, $\pi/2$, $3\pi/4$, π, $5\pi/4$, $3\pi/2$ and $7\pi/4$ radians around a point, reflections about lines parallel to a line in S_2, and their compositions; there is no other bijections of $\mathbb{R}^2$ onto $\mathbb{R}^2$ which preserve the CC distance (see [6]).

3. **Euclidean Regular Polygons in the Maximum, Taxicab and CC Planes**

Since every Euclidean regular polygon is also maximum, taxicab and CC equiangular, it is obvious that a Euclidean regular polygon is maximum, taxicab and CC regular if and only if it is maximum, taxicab and CC equilateral, respectively. Therefore, to investigate the maximum, taxicab and CC regularity of a Euclidean regular polygon, it is sufficient to determine whether it is maximum, taxicab and CC equilateral or not, respectively. In doing so, we use following concepts:

Any Euclidean regular polygon can be inscribed in a circle, and a circle can be circumscribed about any Euclidean regular polygon. A point is called the *center* of a Euclidean regular polygon if it is the center of the circle circumscribed about the polygon. A line l is called *axis of symmetry* (AOS) of a polygon if the polygon is symmetric about l, and in addition, if l passes through a vertex of the polygon then l is called *radial axis of symmetry* (RAOS) of the polygon. Clearly, every AOS of a Euclidean regular polygon passes through the center of the polygon.

Now, we are ready to investigate the maximum, taxicab and CC regularity of the Euclidean regular polygons. In the following propositions proofs are given for only maximum regularity; see [3] and [4] for the proofs of taxicab and CC regularity.

Proposition 3.1. *No Euclidean regular triangle is maximum, taxicab or CC regular.*

Proof. Since the Euclidean lengths of two consecutive sides are the same, and the angle between two consecutive sides is not a right angle, by the Corollary 2.1, two consecutive sides must be symmetric about a line parallel to a line in S_1 in order to have the same maximum lengths. Suppose two consecutive sides are symmetric about a line that is parallel to a line in S_1. Figure 1 and Figure 2 show such Euclidean regular triangles. A simple calculation shows that none of the other

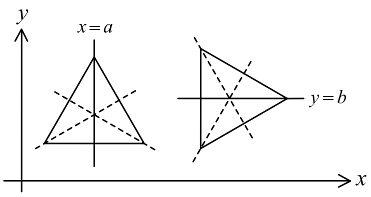

Figure 1

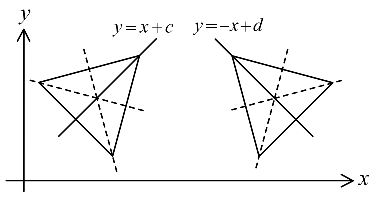

Figure 2

two AOS's is parallel to a line in S_1. Therefore, triangles in Figure 1 and Figure 2 are not maximum equilateral. Thus, no Euclidean regular triangle is maximum regular. □

Corollary 3.1. *No Euclidean regular hexagon is maximum, taxicab or CC regular.*

Proof. It is clear that every Euclidean regular hexagon is the union of six Euclidean regular triangles, and by Theorem 2.1, the maximum lengths of sides of one of the Euclidean regular triangles are the same as the maximum lengths of corresponding parallel sides of the Euclidean regular hexagon, respectively, as shown in Figure 3. Since no Euclidean regular triangle is maximum equilateral,

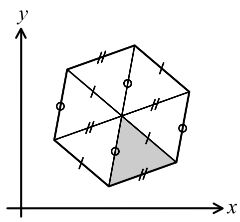

Figure 3

no Euclidean regular hexagon is maximum equilateral, either. Thus, no Euclidean regular hexagon is maximum regular. □

Proposition 3.2. *Every Euclidean regular quadrilateral (Euclidean square) is maximum, taxicab and CC regular.*

Proof. Since every side of the Euclidean square has the same Euclidean length and the angle between every two consecutive sides is a right angle, as shown in Figure 4, every side has the same maximum, taxicab and CC lengths by Corollary 2.1.

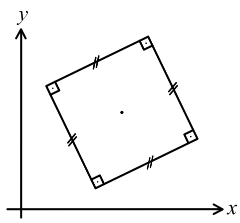

Figure 4

Thus, every Euclidean square is maximum, taxicab and CC equilateral, and therefore is maximum, taxicab and CC regular. □

Proposition 3.3. (i) *Every Euclidean regular octagon, one of whose RAOS's is parallel to a line in S_1, is maximum and taxicab regular.*

(ii) *Every Euclidean regular octagon and Euclidean regular 16-gon, one of whose RAOS's is parallel to a line in S_2, are CC regular.*

Proof. Clearly, every Euclidean regular octagon has four RAOS's, and if a RAOS of a Euclidean regular octagon is parallel to a line in S_1, then each of the other RAOS's is parallel to a line in S_1, too (see Figure 5). Since every two consecutive sides of such Euclidean regular octagons are symmetric about a line parallel to a line in S_1,

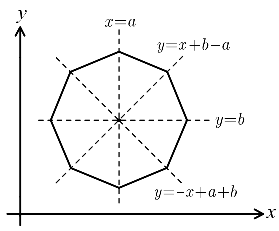

Figure 5

and every side has the same Euclidean length, by Corollary 2.1, these sides have the same maximum lengths. Thus, every Euclidean regular octagon, one of whose RAOS's is parallel to a line in S_1, is maximum equilateral, and therefore is maximum regular. $\square$

Theorem 3.1. *No Euclidean regular polygon, except the ones in Proposition 3.2 and Proposition 3.3, is maximum, taxicab or CC regular.*

Proof. Let us classify Euclidean regular polygons as $(2n-1)$-gons and $2n$-gons for $n \geq 2$ ($n \in \mathbb{N}$), and investigate them separately:

(i) Euclidean regular $(2n–1)$-gons: The case $n = 2$ is proved in Proposition 3.1. Let $n > 2$. It is clear that the number of AOS's of a Euclidean regular $(2n-1)$-gon is $2n-1$ (≥ 5), and each AOS passes through a vertex and the center of the polygon. Therefore, there exists at least one AOS that is not parallel to any line in S_1. Then, there are at least two consecutive sides symmetric about a line that is not parallel to any line in S_1. We know also that the angle between two consecutive sides of the Euclidean regular $(2n-1)$-gon is not a right angle. By Corollary 2.1, these consecutive sides do not have the same maximum lengths. Thus, for $n > 2$, the Euclidean regular $(2n-1)$-gon is not maximum equilateral, and therefore is not maximum regular. That is, no Euclidean regular $(2n-1)$-gon is maximum regular.

(ii) Euclidean regular $2n$-gons: The case $n = 2$ is included in Proposition 3.2. The case $n = 3$ is proved in Corollary 3.1. In order to exclude the case in Proposition 3.3-(i), let us consider a Euclidean regular octagon, none of whose RAOS's is parallel to any line in S_1, for the case $n = 4$. By Corollary 2.1, no two consecutive sides have the same maximum length. Thus, such Euclidean regular octagon is not maximum equilateral, and therefore is not maximum regular. Let $n > 4$. Clearly, the number of RAOS's of a Euclidean regular $2n$-gon is n. Therefore, there exists at least one RAOS that is not parallel to any line in S_1. Then, there are at least two consecutive sides symmetric about a line that is not parallel to any line in S_1. We know also that the angle between two consecutive sides of the Euclidean regular $2n$-gon is not a right angle for $n > 4$. By Corollary 2.1, these consecutive sides do not have the same maximum length. Thus, for $n > 4$, the Euclidean regular $2n$-gon is not maximum equilateral, and therefore is not maximum regular. □

4. Existences of Maximum, Taxicab and CC Regular $2n$-gons

We know now that which Euclidean regular polygons are maximum, taxicab or CC regular, and which are not. Furthermore, we also know existence of some maximum, taxicab and regular polygons. However, we do not have general knowledge about the existence of them. The next theorem shows the existence of maximum, taxicab and CC regular $2n$-gons by means of maximum, taxicab and CC circles. Recall that

(i) the maximum circle with center O and radius 1 is the set of all points whose maximum distance to O is 1. This locus of points is a Euclidean square, each side having length 2 parallel to a coordinate axis (see Figure 6),

(ii) the taxicab circle with center O and radius 1 is the set of all points whose taxicab distance to O is 1. This locus of points is a square with center O, each side having slope ± 1, and each diagonal having length 2 (see Figure 7),

(iii) the CC circle with center O and radius 1 is the set of all points whose CC distance to O is 1. This locus of points is a Euclidean regular octagon one of whose radial axes of symmetry has slope 0 (see Figure 8).

Just as for a Euclidean circle, the center and one point at a maximum, taxicab and CC distance r from the center completely determine the maximum, taxicab and CC circles, respectively.

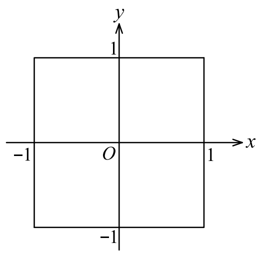

Figure 6

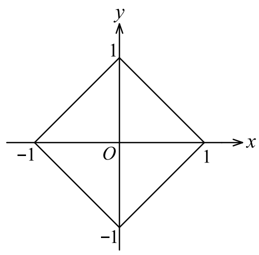

Figure 7

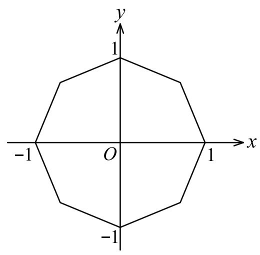

Figure 8

Theorem 4.1. *There exist two congruent maximum, taxicab and CC regular $2n$-gons ($n \geq 2$) having given any line segment as a side.*

Proof. Clearly, the measure of each interior angle of an equiangular $2n$-gon ($n \geq 2$) is $\pi(n-1)/n$ radians. Let us consider now any given line segment A_1A_2 in the maximum plane. It is obvious that $(n-1)$ line segments A_iA_{i+1}, $2 \leq i \leq n$, having the same maximum length $d_M(A_1, A_2)$, can be constructed using the maximum circles with center A_i and radius $d_M(A_1, A_2)$, such that the measure of the angle between every two consecutive segments is $\pi(n-1)/n$ radians (see Figure 9). Also it is not difficult to see that $\measuredangle A_2A_1A_{n+1} + \measuredangle A_nA_{n+1}A_1 = \pi(n-1)/n$. If we continue to construct line segments $A'_iA'_{i+1}$, $1 \leq i \leq n$, which are symmetric to A_iA_{i+1} about the midpoint of A_1A_{n+1}, respectively, we get a $2n$-gon (see Figure 10). Since the symmetry about a point (rotation of π radians

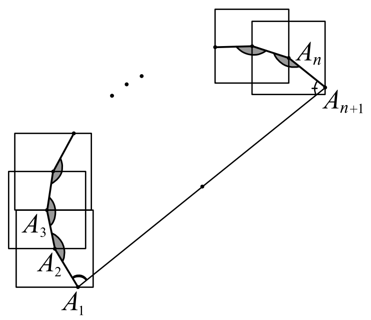

Figure 9

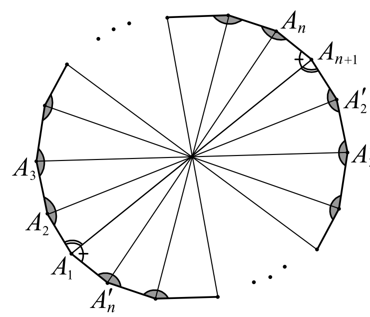

Figure 10

around a point) preserves both the maximum distances and the angle measures, we have $d_M(A_i, A_{i+1}) = d_M(A_i', A_{i+1}') = d_M(A_1, A_2)$ for $1 \leq i \leq n$, and $\measuredangle A_i = \measuredangle A_i' = \pi(n-1)/n$ for $2 \leq i \leq n$. Also it is easy to see that $\measuredangle A_1 = \measuredangle A_{n+1} = \pi(n-1)/n$. Thus, the constructed $2n$-gon is maximum regular. Furthermore, on the other side of the line A_1A_2, one can construct another maximum regular $2n$-gon, having the same line segment A_1A_2 as a side, using the same procedure (see Figure 11). However, it is easy to see that these two maximum regular

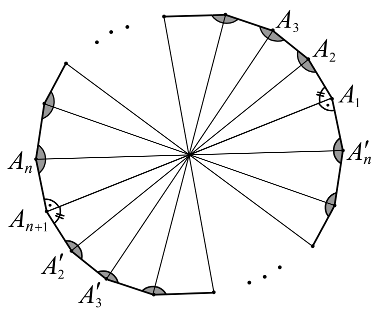

Figure 11

$2n$-gons are symmetric about the midpoint of the line segment A_1A_2, and congruent.

□

In every maximum, taxicab or CC regular $2n$-gon, there are n line segments joining the corresponding vertices of the $2n$-gon (A_iA_i', $1 \leq i \leq n$, for polygons in Figure 10 and Figure 11). We call each of these line segments an *axis* of the polygon. Clearly, axes of every maximum, taxicab or CC regular $2n$-gon intersect at one and only one point.

Example Using the procedure given in the proof of Theorem 4.1, one can easily construct maximum, taxicab or CC regular $2n$-gons having given any line segment as a side. To give examples, we construct one maximum, one taxicab and one CC regular hexagon, having given line segment A_1A_2 as a side, in Figure 12, 13 and 14, respectively:

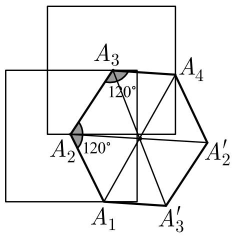

Figure 12

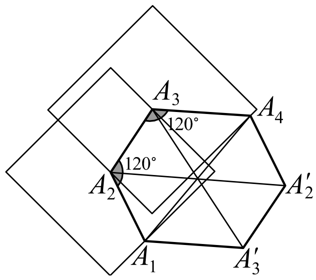

Figure 13

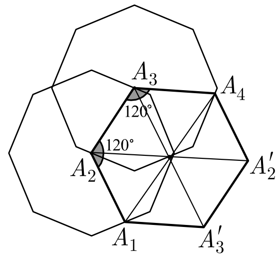

Figure 14

5. More About Maximum, Taxicab and CC Regular Polygons

The following corollary is another statement of Corollary 2.1:

Corollary 5.1. *Let A, B, and C be three non-collinear points in the Cartesian coordinate plane. Then*
(i) If $d_M(A,B) = d_M(B,C)$ then, $d_E(A,B) = d_E(B,C)$ if and only if $\measuredangle(ABC) = \pi/2$ or A and C are symmetric about the line passing through B, and parallel to one of lines in S_1.
(ii) If $d_T(A,B) = d_T(B,C)$ then, $d_E(A,B) = d_E(B,C)$ if and only if $\measuredangle(ABC) = \pi/2$ or A and C are symmetric about the line passing through B, and parallel to one of lines in S_1.
(iii) If $d_C(A,B) = d_C(B,C)$ then, $d_E(A,B) = d_E(B,C)$ if and only if $\measuredangle(ABC) \in \{\pi/4, \pi/2, 3\pi/4\}$ or A and C are symmetric about the line passing through B, and parallel to one of lines in S_2.

Proposition 5.1. *Every maximum, taxicab or CC square is Euclidean regular.*

Proof. Since every side of a maximum, taxicab and CC square has the same maximum, taxicab and CC length respectively, and the angle between every two consecutive sides is a right angle, every side has the same Euclidean length by Corollary 5.1. Thus, every maximum square taxicab or CC square is Euclidean equilateral, and therefore is Euclidean regular. □

We need a new notion to prove the next proposition: An equiangular polygon with an even number of vertices is called *equiangular semi-regular* if sides have the same Euclidean length alternately. There is always a Euclidean circle passing through all vertices of an equiangular semi-regular polygon (see [12]).

Proposition 5.2. (*i*) *Every maximum or taxicab regular octagon, one of whose axes is parallel to a line in S_1, is Euclidean regular.*
(*ii*) *Every CC regular octagon and CC regular 16-gon one, of whose axes is parallel to a line in S_2, are Euclidean regular.*

Proof. In every maximum regular octagon, sides have the same Euclidean length alternately since the measure of the angle between any two alternate sides is $\pi/2$ and sides have the same maximum length by Theorem 2.1 and Corollary 5.1.

Therefore, every maximum regular octagon is equiangular semi-regular. It is obvious that if any two consecutive sides of an equiangular semi-regular polygon have the same Euclidean length, then the polygon is Euclidean regular. Let us consider a maximum regular octagon, $A_1A_2...A_8$, one of whose axes, let us say A_1A_5, is parallel to $y = 0$, for one case (see Figure 15). Then there exist a

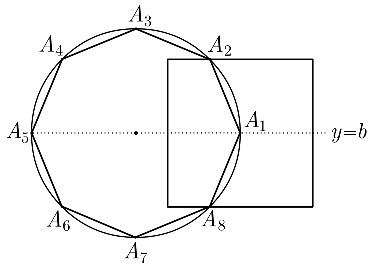

Figure 15

Euclidean circle with diameter A_1A_5, passing through points $A_1, A_2, ..., A_8$, and there exist a maximum circle with center A_1, passing through points A_2 and A_8. Since the Euclidean and the maximum circles are both symmetric about the line A_1A_5, the intersection points of them, A_2 and A_8, are also symmetric about the same line. Then two consecutive sides A_1A_2 and A_1A_8 have the same Euclidean length. Therefore, every maximum regular octagon, one of whose axes is parallel to the line $y = 0$, is Euclidean regular. □

Theorem 5.1. *No maximum, taxicab or CC regular polygon, except the ones in Proposition 5.1 and Proposition 5.2, is Euclidean regular.*

Proof. Assume that there exists a maximum regular polygon, except the ones in Proposition 5.1 and Proposition 5.2, that is also Euclidean regular. Then there exists a Euclidean regular polygon, except the ones in Proposition 3.2 and Proposition 3.3, that is also maximum regular. But this is in contradiction with Theorem 3.1. Therefore, no maximum regular polygon, except the ones in Proposition 5.1 and Proposition 5.2, is Euclidean regular. □

Consequently, the maximum, taxicab, CC and the Euclidean squares have the same shape in any position, and the only regular polygon having this property is the square.

6. On the Nonexistence of Maximum, Taxicab and CC (2n-1)-gons

The following proposition shows that there is no maximum, taxicab or CC regular triangle:

Proposition 6.1. *There is no maximum, taxicab or CC regular triangle.*

Proof. Every maximum equiangular triangle is also Euclidean regular. Since no Euclidean regular triangle is maximum equilateral by Proposition 3.1, no maximum equiangular triangle is maximum regular. Therefore, there is no maximum regular triangle. □

Unfortunately, we could not reach any conclusion by reasoning about the existence or nonexistence of maximum regular $(2n-1)$-gons for $n \geq 3$, as taxicab and CC regular $(2n-1)$-gons. However, we have seen that there is no maximum (also taxicab or CC) regular 5-gon, 9-gon and 15-gon using a computer program called *C.a.R* (*Compass and Ruler* [13]). Our conjecture is that there is no maximum, taxicab or CC regular $(2n-1)$ -gon since there is no center of symmetry of equiangular polygons. It seems interesting to study the open question: "Does there exist any maximum, taxicab or CC regular $(2n-1)$-gon?"

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