

A NEW PROOF OF THE ERDÖS-MORDELL INEQUALITY

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ABSTRACT. In this paper we give a new proof of the famous Erdős-Mordell inequality. Some related conjectures checked by the computer are also given.

1. INTRODUCTION

In 1932, P.Erdős conjectured the following beautiful geometric inequality:

Theorem 1.1. *Let P be an interior point of the triangle ABC . Denote by R_1, R_2, R_3 the distance of P from the vertices A, B, C , and r_1, r_2, r_3 the distances of P from the sides BC, CA, AB respectively. Then*

$$(1.1) \quad R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3).$$

Equality holds if and only if triangle ABC is equilateral and P is its center.

P.Erdős [1] formally published inequality (1.1) as a problem in 1935. L.J.Mordell [2] first proved the theorem. Since then, inequality (1.1) is known as the Erdős-Mordell's inequality. Later, some other proofs were given in succession ([3]-[11]), many of which were based on the following inequality:

$$(1.2) \quad R_1 \geq \frac{cr_2 + br_3}{a},$$

where a, b, c are the lengths of the sides BC, CA, AB respectively. G.R.Veldkamp [4] and V.Komornik [5] proved inequality (1.2) by using area method. D.K.Kazarinoff [6] used a theorem of Pappus. L.Bankoff [7] used orthogonal projections and similar triangles. A.Avez [8] and H.Lee [9] used the Ptolemy's theorem.

In this note we give a new proof which does not use inequality (1.2). We also propose some related conjectures which are checked by the computer.

2. NEW PROOF OF THEOREM 1.1

Our new proof is based on the following lemma:

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Lemma 2.1. *For an arbitrary interior point P of triangle ABC , we have*

$$(2.1) \quad \sqrt{a^2 + 4r_1^2} \geq \frac{cr_1 + ar_3}{b} + \frac{ar_2 + br_1}{c}.$$

Equality holds if and only if the line PO (O is the circumcenter of ABC) parallels the side BC .

Proof. Let S denote the area of triangle ABC . By Heron's formula:

$$(2.2) \quad S = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = (a + b + c)/2$, we easily get

$$(2.3) \quad 16S^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4.$$

Then using this we can verify the identity:

$$(2.4) \quad \begin{aligned} & a^2 + \frac{16x^2S^2}{(ax+by+cz)^2} - \frac{4S^2}{(ax+by+cz)^2} \left(\frac{cx+az}{b} + \frac{ay+bx}{c} \right)^2 \\ &= \frac{[(2b^2c^2 + a^2b^2 + a^2c^2 - b^4 - c^4)x - a(b^2 + c^2 - a^2)(yb + zc)]^2}{4b^2c^2(ax+by+cz)^2}, \end{aligned}$$

where x, y, z are real numbers and $ax + by + cz \neq 0$. Therefore, it follows that

$$(2.5) \quad a^2 + \frac{16x^2S^2}{(ax+by+cz)^2} - \frac{4S^2}{(ax+by+cz)^2} \left(\frac{cx+az}{b} + \frac{ay+bx}{c} \right)^2 \geq 0.$$

Putting $x = r_1, y = r_2, z = r_3$ in the above inequality, then using the following identity:

$$(2.6) \quad ar_1 + br_2 + cr_3 = 2S,$$

we obtain

$$a^2 + 4r_1^2 \geq \left(\frac{cr_1 + ar_3}{b} + \frac{ar_2 + br_1}{c} \right)^2,$$

hence inequality (2.1) is valid. Clearly, the equality of (2.1) holds if and only if

$$(2.7) \quad (2b^2c^2 + a^2b^2 + a^2c^2 - b^4 - c^4)r_1 - a(b^2 + c^2 - a^2)(br_2 + cr_3) = 0.$$

Now we denote the areas of triangle BPC, CPA, APB by S_a, S_b, S_c respectively, then $S_a = \frac{1}{2}ar_1, S_b = \frac{1}{2}br_2, S_c = \frac{1}{2}cr_3$. Applying (2.3), we know (2.7) is equivalent to

$$(2.8) \quad S_a [16S^2 - a^2(b^2 + c^2 - a^2)] = a^2(b^2 + c^2 - a^2)(S_b + S_c).$$

If $A = \frac{\pi}{2}$, then $r_1 = 0$ from (2.7), thus P lies on BC and the circumcenter of triangle ABC is the midpoint of the side BC . If $A \neq \frac{\pi}{2}$, then $S_a \neq 0$ and it follows from (2.8) that

$$\frac{16S^2 - a^2(b^2 + c^2 - a^2)}{a^2(b^2 + c^2 - a^2)} = \frac{S_b + S_c}{S_a},$$

using the fact $S_a + S_b + S_c = S$, we get

$$S_a = \frac{1}{16S} a^2(b^2 + c^2 - a^2) = \frac{1}{4} a^2 \cot A = \frac{1}{2} R^2 \sin 2A,$$

where R is the circumradius of triangle ABC . But $S_{\triangle OBC} = \frac{1}{2}R^2 \sin 2A$, hence $S_a = S_{\triangle BPC} = S_{\triangle BOC}$, therefore the line PO parallels BC . This completes the proof of the Lemma 2.1. \square

According to the Lemma 2.1, we also have

$$(2.9) \quad \sqrt{b^2 + 4r_2^2} \geq \frac{ar_2 + br_1}{c} + \frac{br_3 + cr_1}{a},$$

$$(2.10) \quad \sqrt{c^2 + 4r_3^2} \geq \frac{br_3 + cr_2}{a} + \frac{cr_1 + ar_3}{b}.$$

We now prove the Erdős-Mordell theorem.

Proof. Let h_a, h_b, h_c denote the altitudes corresponding to the sides BC, CA, AB of ABC respectively. Using $2S = ah_a$ and Heron's formula (2.2) one obtains

$$h_a = \frac{1}{2a} \sqrt{[(b+c)^2 - a^2][a^2 - (b-c)^2]} \leq \frac{1}{2} \sqrt{(b+c)^2 - a^2},$$

thus we have

$$(2.11) \quad b + c \geq \sqrt{a^2 + 4h_a^2},$$

with equality if and only if $b = c$.

Applying inequality (2.11) to triangle PBC , we get

$$(2.12) \quad R_2 + R_3 \geq \sqrt{a^2 + 4r_1^2},$$

and we also have two similar forms. Adding these inequalities we obtain

$$(2.13) \quad 2(R_1 + R_2 + R_3) \geq \sqrt{a^2 + 4r_1^2} + \sqrt{b^2 + 4r_2^2} + \sqrt{c^2 + 4r_3^2},$$

with equality if and only if P is the circumcenter of triangle ABC .

On the other hand, adding inequalities (2.1), (2.9) and (2.10) gives:

$$(2.14) \quad \begin{aligned} & \sqrt{a^2 + 4r_1^2} + \sqrt{b^2 + 4r_2^2} + \sqrt{c^2 + 4r_3^2} \\ & \geq 2 \left(\frac{c}{b} + \frac{b}{c} \right) r_1 + 2 \left(\frac{a}{c} + \frac{c}{a} \right) r_2 + 2 \left(\frac{b}{a} + \frac{a}{b} \right) r_3, \end{aligned}$$

and the equality is the same as (2.13).

Since $\frac{c}{b} + \frac{b}{c} \geq 2$, $\frac{a}{c} + \frac{c}{a} \geq 2$, $\frac{b}{a} + \frac{a}{b} \geq 2$, thus from inequalities (2.13) and (2.14), we see Erdős-Mordell inequality (1.1) holds and the equality in (1.1) occurs only when $a = b = c$ and P is its center. This completes the proof of Theorem 1.1. \square

Remark 2.1. N.Dergiades [10] extended the following inequality concerning an internal point of triangle ABC :

$$(2.15) \quad R_1 + R_2 + R_3 \geq \left(\frac{c}{b} + \frac{b}{c} \right) r_1 + \left(\frac{a}{c} + \frac{c}{a} \right) r_2 + \left(\frac{b}{a} + \frac{a}{b} \right) r_3$$

to the case involving any point in the plane. Our lemma obviously can be extended to the same case and so can inequality (2.13). Therefore, by using the Lemma 2.1 we can prove Dergiades' result in [10].

Remark 2.2. The author [12] has proved the following inequalities:

$$(2.16) \quad \begin{aligned} & \left(\frac{c}{b} + \frac{b}{c} \right) r_1 + \left(\frac{a}{c} + \frac{c}{a} \right) r_2 + \left(\frac{b}{a} + \frac{a}{b} \right) r_3 \geq 2\sqrt{h_a r_1 + h_b r_2 + h_c r_3} \\ & \geq 2(r_1 + r_2 + r_3). \end{aligned}$$

From (2.13),(2.14) and (2.16), we can get the following refinements of Erdős-Mordell inequality:

$$\begin{aligned}
 R_1 + R_2 + R_3 &\geq \frac{1}{2} \left(\sqrt{a^2 + 4r_1^2} + \sqrt{b^2 + 4r_2^2} + \sqrt{c^2 + 4r_3^2} \right) \\
 &\geq \left(\frac{c}{b} + \frac{b}{c} \right) r_1 + \left(\frac{a}{c} + \frac{c}{a} \right) r_2 + \left(\frac{b}{a} + \frac{a}{b} \right) r_3 \\
 &\geq 2\sqrt{h_a r_1 + h_b r_2 + h_c r_3} \\
 (2.17) \quad &\geq 2(r_1 + r_2 + r_3).
 \end{aligned}$$

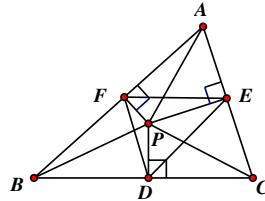


Figure 1

Remark 2.3. The Erdős-Mordell inequality can also be extended as the following:

$$(2.18) \quad 2(R + R_p) \geq R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3) \geq 2(r + r_p),$$

where R, r are the circumradius and inradius of ABC respectively, R_p, r_p the circumradius and inradius of the pedal triangle DEF (see Figure 1). The first inequality in (2.18) is one of the conjectures posed by the author in [13], which has been proved by Wang Zhen [14] recently. The last inequality will be published in one of my recent Chinese article.

3. SEVERAL CONJECTURES

In this section, we shall propose some related conjectures.

From the inequality chain (2.17) we see that

$$(3.1) \quad \sqrt{a^2 + 4r_1^2} + \sqrt{b^2 + 4r_2^2} + \sqrt{c^2 + 4r_3^2} \geq 4(r_1 + r_2 + r_3).$$

It seems not easy to prove this inequality directly. Considering its generalization we propose the following conjecture having been checked by the computer:

Conjecture 1. *If $k \geq 4$ be a real number, then*

$$(3.2) \quad \sqrt{a^2 + kr_1^2} + \sqrt{b^2 + kr_2^2} + \sqrt{c^2 + kr_3^2} \geq \sqrt{k + 12} (r_1 + r_2 + r_3).$$

A similar conjecture is the following:

Conjecture 2. *If $k \geq 4$ be a real number, then*

$$(3.3) \quad \sqrt{a^2 + kw_1^2} + \sqrt{b^2 + kw_2^2} + \sqrt{c^2 + kw_3^2} \geq \sqrt{k + 12} (w_1 + w_2 + w_3).$$

where w_1, w_2, w_3 are the angle-bisectors of $\angle BPC, \angle CPA, \angle APB$ respectively.

Remark 3.1. It is easy to prove that inequality (2.11) still holds after changing the altitude by the angle-bisector. So we actually have the following inequality:

$$(3.4) \quad 2(R_1 + R_2 + R_3) \geq \sqrt{a^2 + 4w_1^2} + \sqrt{b^2 + 4w_2^2} + \sqrt{c^2 + 4w_3^2},$$

which is stronger than (2.13). Therefore, if Conjecture 2 is true, then its special case $k = 4$ and (3.4) conclude the Borrow's inequality [3]:

$$(3.5) \quad R_1 + R_2 + R_3 \geq 2(w_1 + w_2 + w_3).$$

The first inequality in (2.18) is a reverse Erdős-Mordell inequality. It leads us to put forward the similar interesting conjecture:

Conjecture 3. *Let P be an interior point of triangle ABC . The lines AP, BP, CP cut the opposite sides BC, CA, AB at L, M, N respectively. then*

$$(3.6) \quad R_1 + R_2 + R_3 \leq 2(R + R_q),$$

where R, R_q are the circumradius of triangle ABC and the Cevian Triangle LMN respectively (see Figure 2).

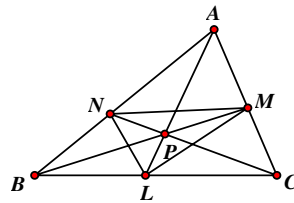


Figure 2

The more general conjecture is the following:

Conjecture 4. *Let P be an interior point of triangle ABC . If $0 < k \leq 1$, then*

$$(3.7) \quad R_1^k + R_2^k + R_3^k \leq 2R^k + (2R_q)^k.$$

If $k < 0$, then the reverse inequality holds.

Remark 3.2. For the pedal triangle (see Figure 1), the author [13] has proved the inequality:

$$(3.8) \quad \frac{1}{R_1^k} + \frac{1}{R_2^k} + \frac{1}{R_3^k} \geq \frac{2}{R^k} + \frac{1}{(2R_p)^k}.$$

where $k \geq 1$. We also supposed (3.8) is valid for $0 < k < 1$ and the reverse inequality holds for $0 < k \leq -1$ (the first inequality in (2.18) is the special case of this conjecture).

It is well known that Erdős-Mordell inequality can be generalized to the case with weights:

$$(3.9) \quad x^2R_1 + y^2R_2 + z^2R_3 \geq 2(yzr_1 + zxr_2 + xyr_3),$$

where x, y, z are arbitrary real numbers. The monograph [15,p.318,Theorem 15] states that inequality (3.9) holds only for positive real numbers x, y, z . In [16], the author showed that the inequality is valid for all real numbers x, y, z by using a simple method.

For Figure 2, we give a conjecture similar to (3.9):

Conjecture 5. *Let P be an interior point of triangle ABC and let r_a, r_b, r_c be the excircledradius of triangle ABC . Then*

$$(3.10) \quad x^2r_a + y^2r_b + z^2r_c \geq 2(yzh_l + zxh_m + xyh_n),$$

where h_l, h_m, h_n are the altitudes of Cevian triangle LMN .

Finally, we propose a conjecture which includes the Erdős-Mordell inequality as a special case.

Conjecture 6. *If $0 < k \leq 1.73$, then the following inequality holds:*

$$(3.11) \quad R_1^k + R_2^k + R_3^k \geq (r_2 + r_3)^k + (r_3 + r_1)^k + (r_1 + r_2)^k.$$

Clearly, the above inequality becomes the Erdős-Mordell inequality when $k = 1$.

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