

A NOTE ON BALANCED INCOMPLETE BLOCK DESIGNS

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ABSTRACT. A $t - (v, k, \lambda; q)$ design over the finite field \mathbb{F}_q is a set B of k -dimensional subspaces of the v -dimensional vectorspace \mathbb{F}_q^v such that each t -dimensional subspace of \mathbb{F}_q^v is contained in exactly λ members of B . In this paper we describe the relationship between this kind of designs and ordinary 2-designs which are also called balanced incomplete block designs.

1. INTRODUCTION

Let $L_k(v, q) := \{S \leq \mathbb{F}_q^v \mid \dim(S) = k\}$ denote the set of k -subspaces of the v -dimensional vectorspace \mathbb{F}_q^v over the finite field \mathbb{F}_q with q elements. Its cardinality is

$$\begin{bmatrix} v \\ k \end{bmatrix}_q := |L_k(v, q)| = \frac{(q^v - 1)(q^{v-1} - 1) \cdots (q^{v-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

A t -design over \mathbb{F}_q or a $t - (v, k, \lambda; q)$ design is a collection B of k -subspaces of \mathbb{F}_q^v such that each t -subspace is contained in exactly λ members of B . More formally, B is a $t - (v, k, \lambda; q)$ design, if and only if

$$B \subseteq L_k(v, q) \quad \text{and} \quad \forall T \in L_t(v, q) : |\{K \in B \mid T \subseteq K\}| = \lambda.$$

Since the definition of a design over a finite field arises from the definition of an ordinary design on sets by replacing the sets by vectorspaces over \mathbb{F}_q and their order by dimension, a design over a finite field is also called the q -analog of a combinatorial t -design.

Designs over finite fields were introduced by S. Thomas in 1986. He constructed the first nontrivial 2-design over a finite field which is a design with parameters $2 - (v, 3, 7; 2)$ for $v \geq 7$ and $v \equiv \pm 1 \pmod{6}$. Thomas used a geometric construction in a projective plane (see [8]).

In 1989 H. Suzuki extended Thomas' family of 2-designs to a family of designs with parameters $2 - (v, 3, q^2 + q + 1; q)$ for $v \geq 7$ and $v \equiv \pm 1 \pmod{6}$ admitting a Singer cycle (see [6, 7]).

In 1992 D. K. Ray-Chaudhuri and E. Schram constructed some families of *non-simple* 2- and 3-designs over \mathbb{F}_q where the term nonsimple indicates that blocks of the design may occur with multiplicities (see [3]). But in this paper we focus our interests only on simple designs.

In 1995 M. Miyakawa, A. Munemasa and S. Yoshiara gave a classification of $2 - (7, 3, \lambda; q)$ designs for $q = 2, 3$ with small λ (see [5]).

Given a $2 - (\ell, 3, q^3(q^{\ell-5}-1)/(q-1); q)$ design for $\ell \equiv 5 \pmod{6(q-1)}$ which admits a Singer cycle of $GL(\ell, q)$ T. Itoh constructed a new family of $2 - (m\ell, 3, q^3(q^{\ell-5}-1)/(q-1); q)$ designs for an arbitrary $m \geq 3$ which admits the action of $SL(m, q^\ell)$. This construction applied to Suzuki's designs provided a new family of 2-designs over \mathbb{F}_q (see [4]).

Finally, in 2005 A. Kerber, R. Laue and M. Braun published the first 3-design over a finite field, a $3 - (8, 4, 11; 2)$ design admitting the normalizer of a Singer cycle, as well as the smallest 2-design known, a design with parameters $2 - (6, 3, 3; 2)$ (see [2]).

Now we switch from designs over finite fields to ordinary designs over finite sets: A *balanced incomplete block design* (=BIBD) is an ordinary $2 - (n, m, \lambda)$ design i.e. a set of m -subsets of an n -set such that each 2-subset is contained in exactly λ blocks.

In this paper we describe how balanced incomplete block designs can be derived from designs over finite fields.

2. BALANCED INCOMPLETE BLOCK DESIGNS IN PROJECTIVE GEOMETRIES

In vectorspaces the 1-subspaces are called the lines, 2-subspaces are the planes, etc. Now we consider the vectorspaces as projective spaces, i.e. the dimension is decreased by 1. To be more precise, the $(v-1)$ -dimensional projective space $PG(v-1, q)$ can be identified with the set of subspaces of \mathbb{F}_q^v , where a subspace $S \in L_k(v, q)$ has projective dimension $k-1$. Hence the points of $PG(v-1, q)$ which have projective dimension 0 are the 1-subspaces of \mathbb{F}_q^v and the lines of $PG(v-1, q)$ which have projective dimension 1 are the 2-subspaces of \mathbb{F}_q^v .

Some major remarks on the projective geometry we need for further considerations are the following ones.

Remark 1. Let X denote the set of points of $PG(v-1, q)$.

- The cardinality of X is $\begin{bmatrix} v \\ 1 \end{bmatrix}_q = \frac{(q^v-1)}{(q-1)}$.
- Each k -subspace of \mathbb{F}_q^v is a $\begin{bmatrix} k \\ 1 \end{bmatrix}_q = \frac{(q^k-1)}{(q-1)}$ -subset of X .
- Two different points are contained in exactly one line, i.e. 2-subsets of X are contained in exactly one 2-subspace of \mathbb{F}_q^v .

These remarks together with the fact that each 2-subspace of \mathbb{F}_q^v is contained in exactly $\begin{bmatrix} v-2 \\ k-2 \end{bmatrix}_q$ enable us to define a famous infinite series of balanced incomplete block designs, which we also call *canonic* BIBDs in $PG(v-1, q)$:

As finite set we take X the set of points of $PG(v-1, q)$ and as set of blocks we take B the set of k -subspaces of \mathbb{F}_q^v . Then B is a BIBD with parameters

$$2 - \left(\frac{(q^v-1)}{(q-1)}, \frac{(q^k-1)}{(q-1)}, \begin{bmatrix} v-2 \\ k-2 \end{bmatrix}_q \right).$$

This design corresponds to the trivial $2 - (v, k, \lambda_{\max}; q)$ design which is defined to be $B = L_k(v, q)$. Analogously, a non-trivial 2-design $B \subset L_k(v, q)$ over a finite field also defines a balanced incomplete block design, which is the major result of this paper:

Theorem 1. *Let X denote the set of all points of $PG(v - 1, q)$ and let B be a non-trivial $2 - (v, k, \lambda; q)$ design. Then, if the blocks are considered as subsets of X , the set B also defines a balanced incomplete block design with parameters $2 - \left(\frac{(q^v-1)}{(q-1)}, \frac{(q^k-1)}{(q-1)}, \lambda\right)$.*

Proof. Each 2-subset of X is contained in exactly one 2-subspace of \mathbb{F}_q^v . Furthermore each 2-subspace is contained in exactly λ k -subspaces of B . □

3. RESULTS

In this final section we list some parameters of balanced incomplete block designs which can be derived from existing designs over finite fields.

Thomas' design and the generalization of Suzuki induce the existence of infinite families of BIBDs with parameters $2 - \left(\frac{(q^v-1)}{(q-1)}, \frac{(q^3-1)}{(q-1)}, q^2 + q + 1\right)$ for all prime powers q and all $v \geq 7$ with $v \equiv \pm 1 \pmod 6$.

Table 1 shows 2-designs over finite fields which were constructed in [1]. Application of Theorem 1 to these designs over finite fields yields balanced incomplete block designs which are depicted in Table 2.

TABLE 1. 2-designs over finite fields from [1]

parameters	designs over finite fields for values of λ
$2 - (10, 3, \lambda; 2)$	15, 30, 120
$2 - (9, 4, \lambda; 2)$	441, 576, 630, 819, 1260
$2 - (8, 4, \lambda; 2)$	21, 91, 140, 210, 231, 301
$2 - (8, 3, \lambda; 2)$	21
$2 - (7, 3, \lambda; 3)$	5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60
$2 - (7, 3, \lambda; 2)$	3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15
$2 - (6, 3, \lambda; 3)$	20
$2 - (6, 3, \lambda; 2)$	3, 6

TABLE 2. BIBDs derived from Table 1

parameters	BIBDs for values of λ
$2 - (1023, 7, \lambda)$	15, 30, 120
$2 - (511, 15, \lambda)$	441, 576, 630, 819, 1260
$2 - (255, 15, \lambda)$	21, 91, 140, 210, 231, 301
$2 - (255, 7, \lambda)$	21
$2 - (1093, 13, \lambda)$	5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60
$2 - (127, 7, \lambda)$	3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15
$2 - (364, 13, \lambda)$	20
$2 - (63, 7, \lambda)$	3, 6

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