INDUCED STRUCTURES ON THE PRODUCT OF RIEMANNIAN MANIFOLDS

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ABSTRACT. The purpose of this paper is to determine some remarkable classes of the induced structures on the product of two Riemannian manifolds, these, being furnished with an almost complex or an almost contact structure.

1. INTRODUCTION

When studying the product of two almost contact metric manifolds, M. Capursi [3] established that this product is an almost Hermitian manifold. He characterized it for some classes of manifolds in the topic of cosymplectic geometry. He shows that this product is Hermitian, Kählerian, almost Kählerian or nearly Kählerian, if and only if, the two factors are normal, cosymplectic, almost cosymplectic or nearly cosymplectic respectively.

Regarding this result, one can ask if it is valid only in the case of cosymplectic geometry. In other words, what remarkable classes of structures can be induced on the product of two manifolds in Riemannian geometry?

This paper is organized in the following way. In §2 we examine the product of two almost contact metric manifolds. Since this product is an almost Hermitian manifold, we complete the study of Capursi. §3 is devoted to the case of the product of an almost Hermitian manifold with an almost contact metric manifold. This completes the work of Oubina [8] and [9]. In §4, we deal with the product of an almost quaternion manifold with an almost Hermitian almost contact metric manifold. This product has been used to construct other classes of almost contact metric manifolds with 3-structure in [12], and is a tool to construct some types of Riemannian submersions as reported in [13] and [14]. We end the study with some problems for which we have not found suitable response in the field of Riemannian geometry.

2. Review of needed notions

An almost Hermitian manifold is a Riemannian manifold (M, g) furnished with a tensor field J of type $(1, 1)$ satisfying:

(i) $J^2 D = -D$, and

(ii) $g(JD, JE) = g(D, E), \forall D, E \in \chi(M)$.

Any almost Hermitian manifold (M, g, J) is of even dimension $2m$ and admits a differential 2-form, Ω, defined by $\Omega(D, E) = g(D, JE)$. This form is called the fundamental form or the Kähler form. Let $\{E_1, \ldots, E_m, JE_1, \ldots, JE_m\}$ be a local J-basis of an open subset of M, then the coderivative δ of Ω is defined by

$$(2.1) \qquad \delta\Omega(D) = -\sum_{i=1}^{m} (\nabla_E \Omega)(E_i, D) + (\nabla_{JE_i}\Omega)(JE_i, D),$$

where ∇ denotes the Levi-Civita connection on M.

Almost Hermitian structures have been completely classified by A. Gray and L.M. Hervella [5]. We just recall the defining relations of some classes, which will be used in this study.

An almost Hermitian manifold (M^{2m}, g, J) is said to be :

(a) *Kählerian* if $d\Omega(D, E, G) = 0$ and $N_j = 0$, where N_J denotes the Nijenhuis tensor of J;

(b) *almost Kählerian*(or W_2-manifold) if $d\Omega(D, E, G) = 0$;

(c) *nearly Kählerian* (or W_1-manifold) if $(\nabla_D \Omega)(D, E) = 0$;

(d) W_3-*manifold* if $(\nabla_D \Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G) = 0 = \delta\Omega$;

(e) *semi-Kählerian* (or $W_1 \oplus W_2 \oplus W_3 - manifold$) if $\delta\Omega = 0$;

(f) $W_1 \oplus W_3$-*manifold* if $(\nabla_D \Omega)(D, E) - (\nabla JD\Omega)(JD, E) = 0 = \delta\Omega$;

(g) G_1-*manifold* if $(\nabla_D \Omega)(D, E) - (\nabla_{JD}\Omega)(JD, E) = 0$;

(h) *Hermitian* or ($W_3 \oplus W_4$-manifold) if $N_J = 0$ or equivalently

$$(\nabla_D \Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G) = 0.$$

By an almost contact metric manifold, one understands a quintuple $(M, g, \varphi, \xi, \eta)$ satisfying the following conditions:

(i) (M, g) is a Riemannian manifold;

(ii) ξ is a distinguished vector field;

(iii) η is a differential 1-form such that $\eta(\xi) = 1$;

(iv) φ is a tensor field of type (1,1) such that $\varphi^2 = -I + \eta \otimes \xi$;

(v) $g(\varphi D, \varphi E) = g(D, E) - \eta(D)\eta(E)$.

The last condition means that g is a compatible metric with the almost contact structure (φ, ξ, η). Almost contact metric manifolds are of odd dimension, $2m + 1$. The fundamental 2-form, ϕ, of an almost contact metric manifold is defined by

$$\phi(D, E) = g(D, \varphi E).$$

Let $\{E_1, \ldots, E_m, \varphi E_1, \ldots, \varphi E_m, \xi\}$ be a local φ-basis of an open subset of M, then the coderivative, δ, is given by

$$\delta\phi(D) = -\sum_{i=1}^{m} \left\{ (\nabla_{E_i}\phi)(E_i, D) + (\nabla_{\varphi E_i}\phi)(\varphi E_i, D) \right\} - (\nabla_\xi \phi)(\xi, D);$$

$$\delta\eta = -\sum_{i=1}^{m} \left\{ (\nabla_{E_i}\eta)E_i + (\nabla_{\varphi E_i}\eta)\varphi E_i \right\}.$$

Following Gray and Hervella [5], in the classification of almost Hermitian structures, D. Chinea an C. Gonzalez [4], have obtained a classification of almost contact metric manifolds.

Note that, among the 4,096 classes of these structures, only a few of them has been identified. We recall those which will be used in this paper.

An almost contact metric manifold is said to be :

(a) *cosymplectic* if $\nabla\varphi = 0$;

(b) *almost cosymplectic* if $d\phi = 0$ and $d\eta = O$;

(c) *semi-cosymplectic normal* if $\delta\phi = 0 = \delta\eta = N^{(1)}$;

(d) G_1 *semi-cosymplectic* if $\delta\phi = O = \delta\eta$ and

$$(\nabla_D\varphi)D - (\nabla_{\varphi D}\varphi)\varphi D + \eta(D)(\nabla_{\varphi D}\xi) = O;$$

(e) *Sasakian* if $(\nabla_D\varphi)E = g(D, E)\xi - \eta(E)D$;

(f) *quasi-Sasakian* if $d\phi = O$ and $N^{(1)} = O$;

(g) *semi-Sasakian* if $\eta = \frac{1}{2m}\delta\phi$;

(h) G_1-*Sasakian* if $(\nabla_D\varphi)D - (\nabla_{\varphi D}\varphi)\varphi D + \eta(D)(\nabla_{\varphi D}\xi) = O$;

(i) *Kenmotsu* if $(\nabla_D\varphi)E = g(\varphi D, E)\xi - \eta(E)\varphi D$.

Let $(\varphi_1, \xi_1, \eta_1)$, $(\varphi_2, \xi_2, \eta_2)$ and $(\varphi_3, \xi_3, \eta_3)$ be almost contact structures defined on a Riemannian manifold (M, g) such that each of them is compatible with the Riemannian metric g. Then $(M, g, (\varphi_1, \xi_1, \eta_1)_{i=1}^3)$ is called an almost contact metric manifold with 3-structure, [7], [15], if, for any cyclic permutation (i, j, k) of $\{1, 2, 3\}$, the following conditions are satisfied :

(a) $\eta_i(\xi_j) = \eta_j(\xi_i) = 0$;
(b) $\varphi_i(\xi_j) = -\varphi_j(\xi_i) = \xi_k$;
(c) $\eta_i \circ \varphi_j = -\eta_j \circ \varphi_i = \eta_k$;
(d) $\varphi_i \circ \varphi_j - \eta_j \otimes \xi_i = -\varphi_j \circ \varphi_i + \eta_i \otimes \xi_j = \varphi_k$.

Note that, for each i, the fundamental local 2-form, ϕ_i, is defined by

$$\phi_i(D, E) = g(D, \varphi_i E)$$

Almost contact metric manifolds with 3-structure are of odd dimension $4m + 3$.

3. Product of almost contact metric manifolds

Let $(M_1^{2m_1+1}, g_1, \varphi_1, \xi_1, \eta_1)$ and $(M_2^{2m_2+1}, g_2, \varphi_2, \xi_2, \eta_2)$ be almost contact metric manifolds. It is known that $M_1 \times M_2$ is a differentiable manifold with dimension

$$dim_{\mathbb{R}}(M_1 \times M_2) = 2m_1 + 2m_2 + 2 = 2(m_1 + m_2 + 1).$$

Putting $m_1 + m_2 + 1 = p$, we have $\dim_{\mathbb{R}}(M_1 \times M_2) = 2p$ which is even. We will denote by $\tilde{M} = M_1 \times M_2$. It is known that $(M_1, g_1) \times (M_2, g_2)$ is a Riemannian manifold furnished with the Riemannian metric defined by

$$(3.1) \qquad \tilde{g}((D_1, D_2), (E_1, E_2)) = g_1(D_1, E_1) + g_2(D_2, E_2).$$

Since $\tilde{M}$ is a Riemannian manifold of even dimension $2p$, one can suspect that it is equipped with an almost complex structure. We can put

$$(3.2) \qquad \tilde{J}(D_1, D_2) = (\varphi_1 D_1 - \eta_2(D_2)\xi_1, \varphi_2 D_2 + \eta_1(D_1)\xi_2).$$

The following is well known.

Proposition 3.1. *The triplet $(\tilde{M}, \tilde{g}, \tilde{J})$ constructed as above is an almost Hermitian manifold.*

Proof. Obvious.   □

The manifold $(\tilde{M}, \tilde{g}, \tilde{J})$ possesses a fundamental $2-$form , $\tilde{\Omega}$, the Kähler form defined by

$$\tilde{\Omega}((D_1, D_2), (E_1, E_2)) = \tilde{g}((D_1, D_2), \tilde{J}(E_1, E_2))$$

From definitions of $\tilde{g}$ and $\tilde{J}$, we get

$$(3.3) \qquad \tilde{\Omega}((D_1, D_2), (E_1, E_2)) = \phi_1(D_1, E_1) + \phi_2(D_2, E_2)$$
$$+ \eta_1(E_1)\eta_2(D_2) - \eta_2(E_2)\eta_1(D_1).$$

If each ϕ_i, and any of the contact forms η_i are closed, so is $\tilde{\Omega}$.

Following the same procedure, one can obtain other induced objects such as the induced covariant derivative, the covariant derivative of the Kähler form and many others. Concerning the exterior differential of the Kähler form, Capursi [3] has established that

$$(3.4) \qquad 3d\tilde{\Omega}((D_1, D_2), (E_1, E_2), (F_1, F_2)) = 3d\phi_1(D_1, E_1, F_1)$$
$$+ 3d\phi_2(D_2, E_2, F_2) - 2\eta_2(E_2)d\eta_1(F_1, D_1) - 2\eta_1(F_2)d\eta_1(D_1, E_1)$$
$$+ 2\eta_1(D_1)d\eta_2(E_2, F_2) + 2\eta_1(E_1)d\eta_2(F_2, D_2) + 2\eta_1(F_1)d\eta_2(D_2, E_2)$$
$$- 2\eta_2(D_2)d\eta_1(E_1, F_1).$$

Denoted by $N_{\varphi_1}^{(3)}(D_1) = (\mathcal{L}_{\xi_1}\varphi_1)D_1$, $N_{\varphi_1}^{(4)}(D_1) = (\mathcal{L}_{\eta_1}\varphi_1)D_1$ other Sasaki-Hatakeyama tensors [10], it is known that if $N_{\varphi_1}^{(1)}(D_1) = 0$, then $N_{\varphi_1}^{(i)}(D_1) = 0$ for all $i = 2, 3, 4$. The same holds for $N_{\varphi_2}^{(1)}$.

In [3], M. Capursi has shown that the Nijenhuis tensor $N_{\tilde{J}}$ is related to $N_{\varphi_1}^{(i)}$ and $N_{\varphi_2}^{(i)}$ as follows

$$N_{\tilde{J}}((D_1, 0), (E_1, 0)) = (N_{\varphi_1}^{(1)}(D_1, E_1), 0) + (N_{\varphi_1}^{(2)}(D_1, E_1))(0, \xi_2)$$

$$(3.5) \quad N_{\tilde{J}}((D_1, 0), (0, D_2)) = \eta_2(D_2)(N_{\varphi_1}^{(3)}(D_1, E_1), 0) - \eta_2(D_1)(N_{\varphi_2}^{(4)}(D_2))(\xi_1, 0)$$
$$+ \eta_1(D_1)(0, N_{\varphi_2}^{(3)}(D_2)) + \eta_2(D_2)(N_{\varphi_1}^{(4)}(D_1))(0, \xi_2).$$

$$(3.6) \qquad N_{\tilde{J}}((0, D_2), (0, E_2)) = (0, N_{\varphi_2}^{(1)}(D_2, E_2)) - (N_{\varphi_2}^{(2)}(D_2, E_2))(\xi_1, 0).$$

Proposition 3.2. *Let M_1 and M_2 be two almost contact metric manifolds such that their Sasaki-Hatakeyama tensors $N_{\varphi_i}^{(1)}$ vanish. If their fundamental forms ϕ_i and their contact forms η_i are closed, then the product $\tilde{M} = M_1 \times M_2$ is Kähler.*

Proof. Since $N_{\varphi_i}^{(1)} = 0$, the structure on the product is integrable. From equation (3.4), we have $d\tilde{\Omega} = d\phi_1 + d\phi_2$ because $d\eta_i = 0$. As each ϕ_i is closed, then $d\phi_i = 0$ so, $d\tilde{\Omega} = 0 = N_{\tilde{J}}$. This then defines on $\tilde{M}$ a Kähler structure. $\qquad\square$

The above proposition applies in the case when M_1 and M_2 are cosymplectic manifolds. It improves Proposition 3.4 of Capursi [3].

Proposition 3.3. *Let M_1 and M_2 be two almost contact metric manifolds. If M_1 and M_2 are almost cosymplectic, then the product $\tilde{M} = M_1 \times M_2$ is almost Kähler.*

Proof. Recall that an almost cosymplectic manifold is defined by $d\phi = 0 = d\eta$. Using (3.4), one gets $d\tilde{\Omega} = 0$ which is the defining relation of an almost Kähler structure. $\qquad\square$

4. PRODUCT $M'^{2m'} \times M^{2m+1}$

Let (M', g', J') be a $2m'$-dimensional almost Hermitian manifold and $(M, g, \varphi, \xi, \eta)$ be an almost contact metric manifold of dimension $2m + 1$. It is known that the product $\tilde{M} = M' \times M$ is a differentiable manifold of dimension $2(m' + m) + 1$. One can put $n = m' + m$ so that the dimension of $\tilde{M}$ is $2n + 1$.

On the product $\tilde{M} = M' \times M$, one defines an almost contact metric structure $(\tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$ by setting

$$(4.1) \qquad\qquad \tilde{\varphi}(D', D) = (J'D', \varphi D),$$

$$(4.2) \qquad\qquad \tilde{\eta}(D', D) = \frac{m}{n}\eta(D),$$

$$(4.3) \qquad\qquad \tilde{g}((D', D), (E', E)) = g'(D', E') + \frac{n^2}{m^2}g(D, E),$$

$$(4.4) \qquad\qquad \tilde{\xi} = \frac{n}{m}(0, \xi).$$

Looking to obtain the classification of this structure, one needs the fundamental form $\tilde{\phi}$, the Riemannian connection $\tilde{\nabla}$, the differential $d\tilde{\phi}$, the codifferential $\delta\tilde{\phi}$ and many others tensors on $\tilde{M}$. J.A. Oubina [8], [9] has defined some interesting identities to this classification; we recall those that will be needed in this study such as:

$$(4.5) \qquad\qquad \tilde{\phi}((D', D), (E', E)) = \Omega'(D', E') + \frac{m^2}{n^2}\phi(D, E);$$

$$(4.6) \qquad\qquad \left(\tilde{\nabla}_{(D', D)}\tilde{\varphi}\right)(E', E) = ((\nabla'_{D'}J')E', (\nabla_D\varphi)E);$$

$$(4.7) \qquad\qquad \left(\tilde{\nabla}_{(D', D)}\tilde{\phi}\right)((E', E), (G', G)) = (\nabla'_{D'}\Omega')(E', G') + \frac{m^2}{n^2}(\nabla_D\phi)(E, G);$$

$$(4.8) \qquad d\tilde{\phi}\left(\left(D',D\right),\left(E',E\right),\left(G',G\right)\right) = d\Omega'\left(D',E',G'\right) + \frac{m^2}{n^2}d\phi\left(D,E,G\right);$$

$$(4.9) \qquad d\tilde{\eta}\left(\left(D',D\right),\left(E',E\right)\right) = \frac{m}{n}d\eta\left(D,E\right);$$

$$(4.10) \qquad \delta\tilde{\eta} = \frac{n}{m}\delta\eta;$$

$$(4.11) \qquad \delta\tilde{\phi}\left(D',D\right) = \delta\Omega'\left(D'\right) + \delta\phi\left(D\right);$$

$$(4.12) \qquad \tilde{\nabla}_{(D',D)}\tilde{\xi} = \frac{n}{m}\left(0,\nabla_D\xi\right);$$

$$(4.13) \qquad \left(\tilde{\nabla}_{(D',D)}\tilde{\eta}\right)\left(E',E\right) = \frac{m}{n}\left(\nabla_D\eta\right)E;$$

$$(4.14) \qquad \tilde{N^{(1)}} = \left(N,N^{(1)}\right).$$

With these identities, Oubina established a result that we can complete by the following

Proposition 4.1. *Let (M',g',J') be an almost Hermitian manifold and (M,g,φ,ξ,η) an almost contact metric manifold. If $(M'\times M,\tilde{g},\tilde{\varphi},\tilde{\xi},\tilde{\eta})$ is an almost contact metric manifold obtained as above, then it is:*

(a) *semi-Sasakian if, and only if, M' is semi Kähler and M is semi-Sasakian;*
(b) *G_1-Sasakian if, and only if, M' is a G_1-manifold and M is G_1-Sasakian;*
(c) *Kenmotsu if, and only if, M' is Kähler and M is Kenmotsu;*
(d) *semi-cosymplectic normal if, and only if, M' is a W_3-manifold and M is semi-cosymplectic normal;*
(e) *G_1-semi-Sasakian if, and only if, M' is a $W_1 \oplus W_3$-manifold and M is G_1-semi-Sasakian.*

Proof. First, note that since $\tilde{M} = M'^{2m'} \times M^{2m+1}$, we have $\dim \tilde{M} = 2(m'+m)+1$. Suppose that $\tilde{M}$ is semi-Sasakian, we then have

$$(4.15) \qquad \tilde{\eta} = \frac{1}{2(m'+m)}\delta\tilde{\phi}.$$

From (4.2) and (4.11) we have respectively

$$(4.16) \qquad \tilde{\eta}(D',D) = \frac{m}{m'+m}\eta(D),$$

$$(4.17) \qquad \delta\tilde{\phi}(D',D) = \delta\Omega'(D') + \delta\phi(D).$$

Thus, combining (4.16) with (4.15) and (4.17) gives

$$\frac{m}{m'+m}\eta(D) = \frac{1}{2(m'+m)}(\delta\Omega'(D') + \delta\phi(D)),$$

which leads to

$$\frac{m}{m'+m}\eta(D) = \frac{1}{2(m'+m)}\delta\Omega'(D') + \frac{1}{2(m'+m)}\delta\phi(D).$$

Therefore

$$\eta(D) = \frac{m'+m}{2m(m'+m)}\delta\Omega'(D') + \frac{m'+m}{2(m'+m)}\delta\phi(D),$$

$$\eta(D) = \frac{1}{2m}\delta\Omega'(D') + \frac{1}{2m}\delta\phi,$$

and we deduce that $\eta = \frac{1}{2m}\delta\phi$ if and only if $\delta\Omega' = 0$. This means that $\tilde{M}$ is semi-Sasakian if and only if M' is semi-Kähler and M is semi-Sasakian.

Other statements are proved in the same way. $\qquad\qquad\square$

Some illustrations can be pointed out from [4] as follows.

$S^6 \times \mathbb{R}^{2m+1}$ is nearly-K-cosymplectic;

$S^2 \times \mathbb{R}^{2m+1}$ is quasi-K-cosymplectic;

$S^{2m+1} \times \mathbb{R}^{2p}$ is quasi Sasakian.

Looking through these examples, it is known that:

S^6 is nearly Kählerian and $\mathbb{R}^{2m+1}$ is cosymplectic;

S^2 is quasi Kählerian and $\mathbb{R}^{2m+1}$ is cosymplectic;

S^{2m+1} is Sasakian and $\mathbb{R}^{2p}$ is Kählerian.

It is known that there are 4096 classes of almost contact metric structures; thus the above proposition should take many pages; we then generalize it in the following

Theorem 4.1. *Let (M', g', J') be an almost Hermitian manifold and $(M, g, \varphi, \xi, \eta)$ an almost contact metric manifold. If $(\tilde{M}, \tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$ is an almost contact metric manifold obtained as above, then it is so that:*

(a)
$$d\tilde{\phi}\left((D', D), (E', E), (G', G)\right) = \frac{b}{3}\sigma\left\{\tilde{\eta}(D', D)\,\tilde{C}\right\}$$

if and only if,

$$d\Omega'(D', E', G') = 0 \quad and \quad d\phi(D, E, G) = \frac{b}{3}\sigma\left\{\eta(D)\,C\right\};$$

(b)
$$d\tilde{\phi}\left((D', D), (E', E), (G', G)\right) = \frac{b}{3}\sigma\left\{\tilde{\phi}\left((D', D), (E', E)\right)\tilde{C}\right\}$$

if and only if,

$$d\Omega'(D', E', G') = \frac{b}{3}\sigma\left\{\Omega'(D', E')\,C'\right\} \quad and$$

$$d\phi(D, E, G) = \frac{b}{3}\sigma\left\{\phi(D, E)\,C\right\};$$

(c)
$$\left(\tilde{\nabla}_{(D', D)}\tilde{\phi}\right)\left((D', D), (E', E)\right) = b.\tilde{\eta}(D', D)\,\tilde{\phi}\left((E', E), (D', D)\right)$$

if and only if,

$$\left(\nabla'_{D'}\Omega'\right)(D', E') = 0 \quad and \quad \left(\nabla_D\phi\right)(D, E) = b.\eta(D)\,\phi(E, D);$$

(d)
$$\left(\tilde{\nabla}_{(D',D)}\tilde{\phi}\right)\left(\left(E',E\right),\left(G',G\right)\right) + \left(\tilde{\nabla}_{\tilde{\varphi}(D',D)}\tilde{\phi}\right)\left(\tilde{\varphi}\left(E',E\right),\left(G',G\right)\right) =$$

$$b.\tilde{\eta}\left(D',D\right)\tilde{C}$$

if and only if,

$$\left(\nabla'_{D'}\Omega'\right)\left(E',G'\right) + \left(\nabla'_{J'D'}\Omega'\right)\left(J'E',G'\right) = 0 \quad and$$

$$\left(\nabla_D\phi\right)\left(E,G\right) + \left(\nabla_{\varphi D}\phi\right)\left(\varphi E,G\right) = b.\eta\left(D\right)C;$$

(e)
$$\sigma\left\{\left(\tilde{\nabla}_{(D',D)}\tilde{\phi}\right)\left(\left(E',E\right),\left(G',G\right)\right)\right\} -$$

$$\sigma\left\{\left(\tilde{\nabla}_{\tilde{\varphi}(D',D)}\tilde{\phi}\right)\left(\tilde{\varphi}\left(E',E\right),\left(G',G\right)\right) + b.\tilde{\eta}\left(D',D\right)\tilde{C}\right\} = 0$$

if and only if,

$$\sigma\left\{\left(\nabla'_{D'}\Omega'\right)\left(E',G'\right) - \left(\nabla'_{J'D'}\Omega'\right)\left(J'E',G'\right)\right\} = 0$$

and

$$\sigma\left\{\left(\nabla_D\phi\right)\left(E,G\right) - \left(\nabla_{\varphi D}\phi\right)\left(\varphi E,G\right) + b.\eta\left(D\right)C\right\} = 0;$$

(f)
$$\delta\tilde{\phi} = 0, \quad \delta\tilde{\eta} = 0, \quad d\tilde{\eta} = 0 \quad or \quad N^{(1)} = 0$$

if and only if,

$$\delta\tilde{\phi} = 0 = \delta\Omega'^{(1)} = 0 = N$$

respectively.

Proof.   (a) If

$$d\tilde{\phi}\left(\left(D',D\right),\left(E',E\right),\left(G',G\right)\right) = \frac{b}{3}\sigma\left\{\tilde{\eta}\left(D',D\right)\tilde{C}\right\},$$

then by (4.8) we have

$$d\Omega'\left(D',E',G'\right) + \frac{m^2}{n^2}d\phi\left(D,E,G\right) = \frac{b}{3}\sigma\left\{\frac{m}{n}\eta\left(D\right)C\right\};$$

this implies

$$d\Omega'\left(D',E',G'\right) = 0 \quad and \quad \frac{m}{n}d\phi\left(D,E,G\right) = \frac{b}{3}\sigma\left\{\eta\left(D\right)C\right\}.$$

Putting $a = \frac{b.n}{m}$, one gets $d\phi\left(D,E,G\right) = \frac{a}{3}\sigma\left\{\eta\left(D\right)C\right\}$.

Conversely, if $d\Omega'\left(D',E',G'\right) = 0$ and $d\phi\left(D,E,G\right) = \frac{a}{3}\sigma\left\{\eta\left(D\right)C\right\}$, then $d\Omega'\left(D',E',G'\right) + d\phi\left(D,E,G\right) = \frac{a}{3}\sigma\left\{\eta\left(D\right)C\right\}$.

Since $d\phi\left(D,E,G\right) = \frac{a}{3}\sigma\left\{\eta\left(D\right)C\right\}$, then $\frac{m}{n}d\phi\left(D,E,G\right) = \frac{a.m}{3n}\sigma\left\{\eta\left(D\right)C\right\}$.

On the other hand, taking $\frac{a.m}{n} = b$, one gets

$$\frac{m}{n}d\phi\left(D,E,G\right) = \frac{b}{3}\sigma\left\{\eta\left(D\right)C\right\},$$

from which we have

$$\frac{m^2}{n^2}d\phi\left(D,E,G\right) = \frac{b}{3}\sigma\left\{\frac{m}{n}\eta\left(D\right)C\right\}.$$

Since $\frac{m}{n}\eta(D) = \tilde{\eta}(D', D)$, then

$$d\Omega'(D', E', G') + \frac{m^2}{n^2}d\phi(D, E, G) = \frac{b}{3}\sigma\left\{\tilde{\eta}(D', D)\tilde{C}\right\},$$

which shows that

$$d\tilde{\phi}((D', D), (E', E), (G', G)) = \frac{b}{3}\sigma\left\{\tilde{\eta}(D', D)\tilde{C}\right\}.$$

(b) If

$$d\tilde{\phi}((D', D), (E', E), (G', G)) = \frac{b}{3}\sigma\{\tilde{\phi}((D', D), (E', E))\tilde{C}\},$$

then by (4.8) and (4.5) we get

$$d\Omega'(D', E', G') = \frac{b}{3}\sigma\{\Omega'(D', E', G')\}$$

and

$$d\phi(D, E, G) = \frac{b}{3}\sigma\{\phi(D, E)C\}.$$

The converse is established as in the above assertion (a).

Other statements are proved in the same procedure.   □

This theorem is important according to the following proposition, due again to Oubina [12, Propoition 2.1].

Proposition 4.2. *The manifold* $(\tilde{M}, \tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$ *defined as above can not be quasi-K-Sasakian.*

Proof. Recall that a quasi-K-Sasakian manifold is defined by the relation

$$(\nabla_D\phi)(E, G) + (\nabla_{\varphi D}\phi)(\varphi E, G) =$$

$$2g(D, E)\eta(G) - 2g(D, G)\eta(E) + g(\nabla_{\varphi D}\xi, G)\eta(E).$$

Therefore, if $\left(\tilde{M}, \tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}\right)$ is quasi-K-Sasakian, thus we get

$$(\nabla'_{D'}\Omega')(E', G') + (\nabla'_{J'D'}\Omega')(J'E', G') = 2g'(D', E') - 2g'(D', G') \qquad and$$

$$\frac{m^2}{n^2}(\nabla_D\phi)(E, G) + (\nabla_{\varphi D}\phi)(\varphi E, G) =$$

$$\frac{2n}{m}g(D, E)\eta(G) - g(D, G)\eta(E) + g(\nabla_{\varphi D}\xi, G)\eta(E),$$

which are absurd. Indeed, the first relation does not define a subclass in the classification of almost Hermitian structures from Gray and Hervella [5]. The second implies that $m = n$ from which we deduce $m' = 0$.   □

5. Product $M^{4m} \times M'^{4m'+2}$

Definition 5.1. Let (M, g, J) be an almost Hermitian manifold endowed with two almost contact structures $(\varphi_i, \xi_i, \eta_i)_{i=1,2}$ such that each of them is compatible with the metric tensor g. The manifold $(M^{4m+2}, g, J, (\varphi_i, \xi_i, \eta_i)_{i=1}^2)$ is said to be an *almost Hermitian almost contact metric manifold* [16] if

(a) $J\xi_1 = -\xi_2, J\xi_2 = \xi_1$;
(b) $\varphi_1 \circ \varphi_1 = \varphi_2 \circ \varphi_2 = -Id + \eta_1 \otimes \xi_1 + \eta_2 \otimes \xi_2$;
(c) $\varphi_1 \circ J = -J \circ \varphi_1 = \varphi_2$;
(d) $\varphi_1 \circ \varphi_2 = -\varphi_2 \circ \varphi_1 = J + \eta_1 \otimes \xi_2 - \eta_2 \otimes \xi_1$;

(e) $\varphi_2 \circ J = -J \circ \varphi_2 = -\varphi_1$

As in the case of almost contact metric manifolds and almost Hermitian ones, the fundamental forms are defined by

$$\phi_i(D, E) = g(D, \varphi_i E), \qquad i = 1, 2,$$

$$\Omega(D, E) = g(D, JE).$$

In [17] it is shown that the odd dimensional complex projective space $P_{2m+1}(\mathbb{C})$ is an example of almost Hermitian almost contact metric manifolds. It possesses a Kähler structure and two Sasakian ones. For this reason, it is called a Sasakian-Kähler manifold. These manifolds have been studied by Boothby [1], [2], Kobayashi [6], Shibuya [11], [16], [17], and Wolf [18]. The name almost Hermitian almost contact metric manifold is due to Watson [16] and [17]. As in §3, it can be shown that the product of an almost quaternion manifold with an almost Hermitian almost contact metric manifold is an almost Hermitian almost contact metric one.

Indeed, let $\left(M'^{4m'+2}, g', J', (\varphi'_i, \xi'_i, \eta'_i)_{i=1}^2\right)$ be an almost Hermitian almost contact metric manifold and $\left(M^{4m}, g, (J_i)_{i=1}^3\right)$ be an almost quaternion manifold. On the product $\tilde{M} = M \times M'$ one defines an almost Hermitian almost contact metric structure $\left(\tilde{g}, \tilde{J}, \left(\tilde{\varphi}_i, \tilde{\xi}_i, \tilde{\eta}_i\right)_{i=1}^2\right)$ by putting

$$\tilde{\varphi}_1(D, D') = (J_1 D, \varphi'_1 D');$$

$$\tilde{\varphi}_2(D, D') = (J_2 D, \varphi'_2 D');$$

$$\tilde{J}(D, D') = (J_3 D, J' D');$$

$$\tilde{\eta}_i(D, D') = \frac{m'}{m} \eta'_i(D');$$

$$\tilde{\xi}_i = \frac{m}{m'}(0, \xi'_i);$$

$$\tilde{g}((D, D'), (E, E')) = g(D, E) + \frac{m^2}{m'^2} g'(D', E').$$

Proposition 5.1. *Let* $\left(M^{4m}, g, (J_i)_{i=1}^3\right)$ *be an almost quaternion manifold and* $\left(M'^{4m'+2}, g', J', (\varphi'_i, \xi'_i, \eta'_i)_{i=1}^2\right)$ *be an almost Hermitian almost contact metric manifold. If* $\tilde{M} = M \times M'$ *is an almost Hermitian almost contact metric manifold, constructed as above, then it is such that:*

(a) *cosymplectic-Kähler if, and only if,* M' *is cosymplectic-Kähler and* M *is almost quaternion Kähler;*

(b) *Sasakian-Kähler if, and only if,* M' *is Sasakian-Kähler and* M *is almost quaternion Kähler.*

Proof. One can adapt Proposition 4.1.   □

6. Some problems

Let $(M^{4m+3}, g, (\varphi_i, \xi_i, \eta_i)_{i=1}^3)$ be an almost contact metric manifold with 3-structure, it is clear that the product $\tilde{M} = M^{4m+3} \times M'^{4m'+3}$, has dimension $4p + 2$, by putting $m + m' = p$. With this, one can suspect the existence of two almost contact metric structures on $\tilde{M}$. The problem is to find how to construct the two almost contact metric structures on this manifold $\tilde{M}$. The same problem occurs when we consider the product $\tilde{M} = M^{4m+3} \times M'^{4m'+2}$, which has dimension $4p + 1$; we suspect the existence of an almost contact metric structure that must be constructed on $\tilde{M}$. One can be interested by the same problem on the product $\tilde{M} = M^{4m+2} \times M'^{4m'+2}$, which has $4p$ as dimension and should possess an almost quaternion structure. We have to construct this structure on the product $\tilde{M}$.

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