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ON TIMELIKE PARALLEL p_i-EQUIDISTANT RULED SURFACES WITH A SPACELIKE BASE CURVE IN THE MINKOWSKI 3-SPACE R_1^3

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Abstract. The purpose of this paper is first, to give radii and curvature axes of osculator Lorentz spheres of the timelike parallel p_i-equidistant ruled surfaces by a spacelike base curve in the Minkowski 3-space R_1^3; second, to give arc lengths of indicatrix curves of base curves of these surfaces.

1. Introduction

I. E. Valeontis, [3] defined parallel p-equidistant ruled surfaces in E^3 and gave some results related with striction curves of ruled surfaces.
M. Masal, N. Kuruoğlu, [1] obtained arc lengths, curvature radii, curvature axes, spherical involut and areas of real closed spherical indicatrix curves of base curves of parallel p-equidistant ruled surfaces in E^3.
And also, M. Masal, N. Kuruoğlu, [2] defined timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in the Minkowski 3-space and obtained dralls, the shape operators, Gaussian curvatures, mean curvatures, shape tensor, q^{th} fundamental forms of these surfaces.
In this paper, radii and curvature axes of osculator Lorentz spheres, arc lengths of indicatrix curves of base curves of the timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve in the Minkowski 3-space are obtained.

2. Preliminaries

Let $\alpha : I \to R_1^3$, $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a differentiable spacelike curve parameterized by arc-length in the Minkowski 3-space, where I is an open interval in R containing the origin. Let V_1 be the tangent vector field of α, D be the Levi-Civita connection on R_1^3 and $D_{V_1}V_1$ be a timelike vector. If V_1 moves along α,then a timelike ruled surface with base curve α given by the parameterization

$$M : \varphi(t, v) = \alpha(t) + vV_1(t) \tag{2.1}$$

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can be obtained in the Minkowski 3-space. Let $\{V_1, V_2, V_3\}$ be an orthonormal frame field along α in R_1^3, where V_2 is a timelike vector and V_3 is a spacelike vector. If k_1 and k_2 are the natural curvature and torsion of $\alpha(t)$, respectively, then the Frenet formulas are, [4],

$$V_1{}' = k_1 V_2, \qquad V_2{}' = k_1 V_1 + k_2 V_3, \qquad V_3{}' = k_2 V_2. \tag{2.2}$$

Using $V_1 = \alpha'$ and $V_2 = \frac{\alpha''}{\|\alpha''\|}$, we have $k_1 = \|\alpha''\| > 0$, where "′" means derivate with respect to time t.

Definition 2.1. The planes which are corresponding to the subspaces $Sp\{V_1, V_2\}$, $Sp\{V_2, V_3\}$, $Sp\{V_3, V_1\}$; are called **asymptotic plane**, **polar plane** and **central plane**, respectively, [2].

Definition 2.2. Let M and M^* be two timelike ruled surfaces with a spacelike base curve in R_1^3 and p_1, p_2 and p_3 denote the distances between the polar planes, central planes and asymptotic planes, respectively. If
1) The generator vectors of M and M^* are parallel,
2) The distances p_i, $1 \le i \le 3$, at the corresponding points of α and α^* are constant,
then the pair of ruled surfaces M and M^* are called the **timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve** in R_1^3. If $p_i = 0$, then the pair of M and M^* are called the **timelike parallel p_i-equivalent ruled surfaces with a spacelike base curve** in R_1^3, [2].

From the definition 2.2, the timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve have the following parametric representations

$$M : \varphi(t, v) = \alpha(t) + vV_1(t), \qquad (t, v) \in I \times R$$

$$M^* : \varphi^*(t^*, v^*) = \alpha^*(t^*) + v^* V_1(t^*), \qquad (t^*, v^*) \in I \times R.$$

Throughout this paper, M and M^* will be used for the timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve.

Theorem 2.1. ***i)*** *The Frenet frames $\{V_1, V_2, V_3\}$ and $\{V_1^*, V_2^*, V_3^*\}$ are equivalent at the corresponding points in M and M^*, respectively. (For $\frac{dt^*}{dt}\rangle 0$.)*
ii) *If k_1 and k_1^* are the naturel curvatures of base curves of M and M^* and similarly, k_2 and k_2^* are the torsions of base curves of M and M^*, respectively, then we have,* [2].

$$k_i^* = k_i \frac{dt}{dt^*}, \ 1 \le i \le 2.$$

3. ON THE OSCULATOR LORENTZ SPHERES OF TIMELIKE PARALLEL p_i-EQUIDISTANT RULED SURFACES WITH A SPACELIKE BASE CURVE

In this Section, we find the locus of center of the osculator sphere S_1^2 which is fourth order contact with the base curve α of M.
Let us consider the function f denoted by

$$\begin{aligned} f &: I \to R \\ t &\to f(t) = \langle \alpha(t) - a, \alpha(t) - a \rangle - R^2, \end{aligned} \tag{3.1}$$

where a and R are the center and radius of S_1^2, respectively. Since S_1^2 is fourth order contact with the curve α, we can write

$$f(t) = f'(t) = f''(t) = f'''(t) = 0.$$

Using $f(t) = f'(t) = f''(t) = 0$ and (2.2) we have

$$\langle \alpha(t) - a, \alpha(t) - a \rangle = R^2, \tag{3.2}$$

$$\langle V_1(t), \alpha(t) - a \rangle = 0, \tag{3.3}$$

$$\langle V_2(t), \alpha(t) - a \rangle = \frac{1}{k_1(t)}. \tag{3.4}$$

Also, for the vector $\alpha(t) - a$, we can write

$$\alpha(t) - a = m_1(t)V_1(t) + m_2(t)V_2(t) + m_3(t)V_3(t), \quad m_i(t) \in R, \tag{3.5}$$

where $\{V_1, V_2, V_3\}$ is the orthonormal frame field of M. From here, we have

(3.6)
$$\langle \alpha(t) - a, V_1(t) \rangle = m_1(t), \quad \langle \alpha(t) - a, V_2(t) \rangle = -m_2(t), \quad \langle \alpha(t) - a, V_3(t) \rangle = m_3(t).$$

Since $f'(t) = f''(t) = 0$, we find

$$m_1(t) = 0, \quad m_2(t) = -\frac{1}{k_1(t)}. \tag{3.7}$$

Using (3.2), (3.5) and (3.7) we get

$$R = \sqrt{m_3^2 - m_2^2} \tag{3.8}$$

or

$$m_3 = \pm\sqrt{m_2^2 + R^2}. \tag{3.9}$$

From (3.5), for the center a of S_1^2, we can write

$$a = \alpha(t) + \frac{1}{k_1}V_2 - \lambda V_3, \quad \lambda = m_3(t) \in R. \tag{3.10}$$

Using $f'''(t) = 0$, we have

$$k_1{}' \langle V_2(t), \alpha(t) - a \rangle + k_1 \left\langle V_2{}'(t), \alpha(t) - a \right\rangle + k_1 \langle V_2(t), V_1(t) \rangle = 0.$$

Thus, from (2.2), (3.6) and (3.7), we find

$$m_3 = \frac{-k_1{}'}{k_1^2 k_2} = -\frac{m_2{}'}{k_2}. \tag{3.11}$$

Similarly, we compute the locus of center of osculator sphere S_1^{*2} which is fourth order contact with the spacelike base curve α^* of M^*. Let us consider the function f^* defined by

$$\begin{aligned} &f^* : I \to R \\ &t^* \to f^*(t^*) = \langle \alpha^*(t^*) - a^*, \alpha^*(t^*) - a^* \rangle - R^{*2}, \end{aligned} \tag{3.12}$$

where a^* and R^* are the center and the radius of S_1^{*2}. Since S_1^{*2} is fourth order contact with the curve α^*,we can write

$$f^*(t^*) = f^{*'}(t^*) = f^{*''}(t^*) = f^{*'''}(t^*) = 0.$$

From $f^*(t^*) = f^{*'}(t^*) = f^{*''}(t^*) = 0$ and (2.2), we have

$$\langle \alpha^*(t^*) - a^*, \alpha^*(t^*) - a^* \rangle = {R^*}^2, \tag{3.13}$$

$$\langle V_1^*(t^*), \alpha^*(t^*) - a^* \rangle = 0, \tag{3.14}$$

$$\langle V_2^*(t^*), \alpha^*(t^*) - a^* \rangle = \frac{1}{k_1^*(t^*)}. \tag{3.15}$$

Furthermore, for the vector $\alpha^*(t^*) - a^*$, we can write

$$\alpha^*(t^*) - a^* = m_1^*(t^*)V_1^*(t^*) + m_2^*(t^*)V_2^*(t^*) + m_3^*(t^*)V_3^*(t^*), \quad m_i^*(t^*) \in R, \tag{3.16}$$

where $\{V_1^*, V_2^*, V_3^*\}$ is orthonormal frame field of M^*. From here, we have

$$\begin{array}{l} \langle \alpha^*(t^*) - a^*, V_1^*(t^*) \rangle = m_1^*(t^*), \\ \langle \alpha^*(t^*) - a^*, V_2^*(t^*) \rangle = -m_2^*(t^*), \\ \langle \alpha^*(t^*) - a^*, V_3^*(t^*) \rangle = m_3^*(t^*). \end{array} \tag{3.17}$$

Since $f^{*'}(t^*) = f^{*''}(t^*) = 0$, we get

$$m_1^*(t^*) = 0, \quad m_2^*(t^*) = -\frac{1}{k_1^*(t^*)}. \tag{3.18}$$

Using (3.13), (3.16) and (3.18), we obtain

$$R^* = \sqrt{{m_3^*}^2 - {m_2^*}^2} \tag{3.19}$$

or

$$m_3^* = \pm\sqrt{{m_2^*}^2 + {R^*}^2} \tag{3.20}$$

Using (3.16), for the center a^* of ${S_1^*}^2$, we can write

$$a^* = \alpha^*(t^*) + \frac{1}{k_1^*}V_2^* - \lambda^* V_3^*, \quad \lambda^* = m_3^*(t^*) \in R. \tag{3.21}$$

From $f^{*'''}(t^*) = 0$ we have

$$k_1^{*'} \langle V_2^*(t^*), \alpha^*(t^*) - a^* \rangle + k_1^* \left\langle V_2^{*'}(t^*), \alpha^*(t^*) - a^* \right\rangle + k_1^* \langle V_2^*(t^*), V_1^*(t^*) \rangle = 0.$$

So from (2.2), (3.17) and (3.18), we get

$$m_3^* = \frac{-{k_1^*}'}{{k_1^*}^2 k_2^*} = -\frac{{m_2^*}'}{k_2^*}. \tag{3.22}$$

Now, we can compute the relations between the radii of osculator Lorentz spheres and curvature axes of the base curves of the timelike parallel p_i-equidistant ruled surfaces by a spacelike base curve M and M^*:
Using equation (3.7) and equation (3.18) from (ii) of Theorem 2.1 , we find,

$$m_1^*(t^*) = m_1(t) = 0, \quad m_2^*(t^*) = \frac{dt^*}{dt} m_2(t). \tag{3.23}$$

If $\frac{dt}{dt^*}$ is constant, then from (ii) of Theorem 2.1, we have

$$k_1^{*'} = {k_1}'\left(\frac{dt}{dt^*}\right)^2. \tag{3.24}$$

Thus, using (3.22), (3.24), (3.11) and (ii) of Theorem 2.1, we obtain

$$m_3^* = \frac{dt^*}{dt} m_3. \tag{3.25}$$

Combining (3.23), (3.25) and (ii) of Theorem 2.1, we have

$$\alpha^* - a^* = \frac{dt^*}{dt} (\alpha - a). \tag{3.26}$$

Similarly, combining (3.8), (3.9), (3.23) and (3.25), we get

$$R^{*2} = \left(\frac{dt^*}{dt}\right)^2 R^2$$

or

$$R^* = \left|\frac{dt^*}{dt}\right| R. \tag{3.27}$$

So, we may give the following Theorem without proof:

Theorem 3.1. *Let M and M^* be the timelike parallel p_i-equidistant ruled surfaces by a spacelike base curve.*
i) *If q_α and q_{α^*} are the curvature axes (the locus of center of osculator Lorentz spheres) of the base curves α and α^* of M and M^*, respectively, then we have*

$$q_{\alpha^*} - \alpha^* = \frac{dt^*}{dt} (q_\alpha - \alpha).$$

ii) *If R and R^* are the radii of osculator Lorentz spheres of base curves α and α^* of M and M^*, respectively, then we have*

$$R^* = \left|\frac{dt^*}{dt}\right| R.$$

4. ARC LENGTHS OF INDICATRIX CURVES OF THE TIMELIKE PARALEL p_i -EQUIDISTANT RULED SURFACES WITH A SPACELIKE BASE CURVE

In this section, we will investigate arc lengths of indicatrix curves of spacelike base curves of the timelike parallel p_i-equidistant ruled surfaces with a spacelike base curve.
Since V_1 and V_3 are spacelike vectors, the curves (V_1) and (V_3) generated by the spacelike vectors V_1 and V_3 on the pseudosphere S_1^2, are called the pseudo-spherical indicatrix curves. The curve (V_2) generated by the timelike vector V_2 on the pseudohyperbolic space H_1^2 is called indicatrix curve. Let us denote the arc lengths of indicatrix curves (V_i) and (V_i^*) generated by the vector fields V_i and V_i^* by S_{V_i} and $S_{V_i^*}$, respectively. Thus, we can write

$S_{V_i} = \int \left\|V_i'\right\| dt$ and $S_{V_i^*} = \int \left\|V_i^{*\prime}\right\| dt^*, \quad 1 \le i \le 3.$

Using the Frenet formulas and (ii) of Theorem 2.1, we have

$$S_{V_1^*} = \int -k_1 dt = S_{V_1}, \quad S_{V_2^*} = \int \sqrt{|k_1^2 + k_2^2|}.dt = S_{V_2}, \quad S_{V_3^*} = \int |k_2|\, .dt = S_{V_3},$$

where $\frac{dt}{dt^*} > 0$.
Similarly, for the arc lengths S_α and S_{α^*} of the indicatrix curves (α) and (α^*) generated by the spacelike curves α and α^* on the pseudosphere S_1^2, we can write

$S_\alpha = \int \|\alpha'\| \, dt = \int dt$ and $S_{\alpha^*} = \int \left\|\alpha^{*\prime}\right\| dt^* = \int dt^*$,
respectively. If $\frac{k_1}{k_1^*}$ is constant, then (ii) of Theorem 2.1, we get

$$S_{\alpha^*} = \frac{k_1}{k_1^*} S_\alpha.$$

So, we can give the following theorems without proofs:

Theorem 4.1. *If S_{V_i} and $S_{V_i^*}$, $1 \leq i \leq 3$, are the arc lengths of indicatrix curves of Frenet vectors V_i and V_i^* of spacelike base curves α and α^* of the timelike parallel p_i-equidistant ruled surfaces M and M^*, respectively, then we have*

$$S_{V_i^*} = S_{V_i}, \; 1 \leq i \leq 3.$$

Theorem 4.2. *Let S_α and S_{α^*} be the arc lengths of indicatrix curves of base curves α and α^* of the timelike parallel p_i-equidistant ruled surfaces M and M^*, respectively. If $\frac{k_1}{k_1^*}$ is constant, then we have $S_{\alpha^*} = \frac{k_1}{k_1^*} S_\alpha$.*

References

[1] Masal, M. and Kuruoğlu, N. , Some characteristics properties of the spherical indicatrices leading curves of parallel p-equidistant ruled surfaces, Bulletin of Pure and Applied Sciences, 19E(2000), no. 1, 405-410.

[2] Masal, M. and Kuruoğlu, N. , Timelike parallel p_i-equidistant ruled surfaces by a spacelike base curve in the Minkowski 3-space R_1^3, (In press).

[3] Valeontis, I. E. , Parallel p-Äquidistante regelflächen, Manuscripta math. , 54(1986), 391-404.

[4] Woestijne, I., Minimal surfaces of the 3-dimensional Minkowski space, World scientific publishing, Singapore, pp. 344-369, 1990.

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