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CONSTANT ANGLE SURFACES AND CURVES IN $\mathbb{E}^3$

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Abstract. In this paper, we study constant angle surfaces and curves in Euclidean 3–space. One of the results in this paper gives a classification of special developable surfaces and some conical surfaces from the point of view the constant angle property. Also we give some characterization for a curve lying on a surface for which the unit normal makes a constant angle with a fixed direction.

1. Introduction

A constant angle surface in Euclidean three-dimensional space $\mathbb{E}^3$ is a surface whose tangent planes make a constant angle with a fixed vector field of the ambient space. These surfaces generalize the concept of helix, that is, curves whose tangent lines make a constant angle with a fixed vector of $\mathbb{E}^3$. This kind of surfaces are models to describe some phenomena in physics of interfaces in liquids crystals and of layered fluids [1]. Constant angle surfaces were studied in product spaces $\mathbb{S}^2 \times \mathbb{R}$ in or $\mathbb{H}^2 \times \mathbb{R}$ in where $\mathbb{S}^2$ and $\mathbb{H}^2$ represent the unit 2-sphere in $\mathbb{R}^2$ and $\mathbb{R}^2_1$, respectively [2, 3]. The angle was considered between the unit normal of the surface M and the tangent direction to $\mathbb{R}$. Munteanu and Nistor obtained a classification of all surfaces in Euclidean 3-space for which the unit normal makes a constant angle with a fixed vector direction being the tangent direction to $\mathbb{R}$ [8]. Moreover in [9] it is also classified certain special ruled surfaces in $\mathbb{R}^3$ under the general theorem of characterization of constant angle surfaces.

In this paper we give a classification of special developable surfaces and some conical surfaces from the point of view the constant angle property. Also we give some characterization for a curve lying on a surface for which the unit normal makes a constant angle with a fixed direction.

But before this, we mention some basic facts in the general theory of curves and surfaces useful for the rest of the paper.

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2. Preliminary

The differential geometry of curves starts with smooth map of s, let's call it $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$, that parameterized a spatial curve denoted again with α. We say that the curve is parameterized by arc lenght if $\|\alpha'(s)\| = 1$, where α' is the derivative of w.r.t. s. Let us denote $t(s) = \alpha'(s)$ the (unit) tangent to the curve. By definition, the curvature of α is $\kappa(s) = \|\alpha''(s)\|$. If $\kappa \neq 0$, then the (unit) normal of α can be obtained from $\alpha''(s) = \kappa(s)n(s)$. Morever, $b(s) = t(s) \times n(s)$ is called the (unit) binormal to α.

With these considerations t, n, b define an orthonormal basis. Recall the Frenet-Serret formulae

$$
\begin{aligned}
t'(s) &= \kappa(s)n(s), \\
n'(s) &= -\kappa(s)t(s) + \tau(s)b(s), \\
b'(s) &= -\tau(s)n(s),
\end{aligned} \tag{2.1}
$$

where $\tau(s)$ is the torsion of α at s. For any unit speed curve $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ defined a vector field $w = \frac{1}{\sqrt{\tau^2+\kappa^2}}(\tau t + \kappa b)$ along α under the condition that $\kappa(s) \neq 0$ and called it the modified Darboux vector field of α.

We now recall some basic concepts on classical differential geometry of space curves and the definitions of general helix, slant helix and a curve of constant precession in Euclidean 3-space. A curve, $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ with unit speed, is a general helix if there is some constant vector u, so that $\langle t, u\rangle = \cos\theta$ is constant along the curve. It has been know that the curve is a general helix if and only if $\frac{\tau}{\kappa}(s)$ is constant. If both of $\kappa(s) \neq 0$ and $\tau(s)$ are constant, we call as a circular helix.

A curve α with, is called a slant helix if the principal normal line of α make a constant angle with a fixed direction.

A unit speed curve of constant precession is defined by the property that its (Frenet) Darboux vector revolves about a fixed line in space with angle and constant speed. A curve of constant precession is characterized by having

$$
\begin{aligned}
\kappa(s) &= c\sin(\mu s), \\
\tau(s) &= c\cos(\mu s),
\end{aligned} \tag{2.2}
$$

where $c > 0$, μ and are constant [10].

A natural extension from curves to the theory of surfaces constructed on curves can be made as follows. Given a curve parameterized by arc length in Euclidean 3-space, we can think of constructing ruled surfaces involving and the tangent, normal, binormal or Darboux lines to the curve. As a consequence, we have well known types of surfaces of this kind, namely

- tangent developable surface: $r(s, v) = \alpha(s) + vt(s)$
- normal surface: $r(s, v) = \alpha(s) + vn(s)$
- binormal surface: $r(s, v) = \alpha(s) + vb(s)$
- rectifying developable surface: $r(s, v) = \alpha(s) + vw(s)$
- Darboux developable surface: $r(s, v) = b(s) + vt(s)$
- tangential Darboux developable surface: $r(s, v) = w(s) + vn(s)$

(see for details [4, 5]).

The characterization of constant angle surface in $\mathbb{E}^3$ was given in [8], where the constant angle is denoted by θ and without loss of generality, the fixed direction is taken to be third real axis, denoted by k. The main result is the following:

TheoremA [8]: A surface M in $\mathbb{E}^3$ is a constant angle surface if and only if it is locally isometric to one of the following surfaces:

(i) a *surface given by*

$$r : M \longrightarrow \mathbb{R}^3, \quad r(u_1, u_2) = (u_1 \cos\theta\,(\cos u_2, \sin u_2) + \gamma(u_2), u_1 \sin\theta) \tag{2.3}$$

with

$$\gamma(u_2) = \cos\theta \left(-\int_0^{u_2} \eta(\tau) \sin\tau d\tau, \int_0^{u_2} \eta(\tau)\cos\tau d\tau \right) \tag{2.4}$$

for η a smooth function on an interval $I \subset \mathbb{R}$.

(ii) an open part of the plane $x \sin\theta - z\cos\theta = 0$,

(iii) an open part of the cylinder $\beta \times \mathbb{R}$, where β is a smooth curve in $\mathbb{E}^2$.

3. Developable Constant Angle Surfaces

In this section we consider three developable surfaces associated to a space curve. Developable surfaces are ruled surfaces. A ruled surface in $\mathbb{E}^3$ is (locally) the map $r : I \times \mathbb{R} \longrightarrow \mathbb{R}^3$ defined by $r(s, v) = \alpha(s) + v\delta(s)$, where $\alpha : I \longrightarrow \mathbb{E}^3$, $\delta : I \longrightarrow \mathbb{E}^3 \setminus \{0\}$ are smooth mappings and I is an open interval. α is called the base curve and δ is called the director curve.

Let α be a unit speed space curve with $\kappa(s) \neq 0$. A ruled surface $r(s, v) = \alpha(s) + vw(s)$ is called the rectifying developable of α. Izumiya defined a ruled surface $r(s, v) = b(s) + vt(s)$ which is called the Darboux developable of α. The Darboux developable of the unit tangent vector $n(s)$ of α which is given by $r(s, v) = w(s) + vn(s)$ is called the tangential Darboux developable of α [5].

The characterization of tangent, normal and binormal developable constant angle surfaces in $\mathbb{E}^3$ was given in [9] respectively:

(1) *The tangent developable constant angle surfaces are generated by cylindrical helices.*

(2) *The normal constant angle surfaces are pieces of planes.*

(3) *The binormal constant angle surfaces are pieces of cylindrical surfaces.*

Now, we state and prove the following result considering which types of rectifying developable surfaces satisfy the constanty angle property:

Theorem 3.1. *The rectifying developable constant angle surfaces are generated by slant helices.*

Proof. Let us consider a rectifying devolopable surface M oriented, immersed in $\mathbb{E}^3$ given by

$$r(s, v) = \alpha(s) + vw(s), \tag{3.1}$$

where $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ is a spatial curve parameterized by arc length consisting of the edge of regression of M and $w = \tau t + \kappa b$ is the Darboux vector to α. The surface M is smooth everywhere, except in points of the curve. Let us determine the normal to the surface. To do this, we compute the partial derivatives of r with respect to s and v.

$$r_s(s, v) = (1 + v\tau')\, t + \kappa' vb \text{ and } r_v(s, v) = \tau t + \kappa b. \tag{3.2}$$

Using now (3.2), the normal to the surface is given by

$$N = \pm \frac{r_s \times r_v}{\|r_s \times r_v\|} = \pm n. \tag{3.3}$$

Choosing an orientation of the surface we take the normal to the surface equal to the normal of the generating curve α. In the case of constant angle surfaces it follows that the normal n of the curve makes a constant angle with the fixed direction k, namely

$$\widehat{(n,k)} = \widehat{(N,k)} = \theta. \tag{3.4}$$

It follows that α is a slant helix. □

Theorem 3.2. *The Darboux developable constant angle surfaces are generated by binormal curves of cylindrical helices.*

Proof. Let us consider Darboux devolopable surface M oriented, immersed in $\mathbb{E}^3$ given by which is called the

$$r(s,v) = b(s) + vt(s), \tag{3.5}$$

where $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{E}^3$ is a spatial curve parameterized by arc length consisting of the edge of regression of M, t and b are the unit tangent and binormal to α recpectively. The surface M is smooth everywhere, except in points of the curve. Let us determine the normal to the surface. To do this, we compute the partial derivatives of r with respect to s and v.

$$r_s(s,v) = (-\tau + \kappa v)\, n \text{ and } r_v(s,v) = t. \tag{3.6}$$

Using now (2.3), the normal to the surface is given by

$$N = \pm \frac{r_s \times r_v}{\|r_s \times r_v\|} = \pm b. \tag{3.7}$$

Choosing an orientation of the surface we take the normal to the surface equal to the binormal of the generating curve α. In the case of constant angle surfaces it follows that the binormal b of the curve makes a constant angle with the fixed direction k, namely

$$\widehat{(b,k)} = \widehat{(N,k)} = \theta. \tag{3.8}$$

It follows that α is a cylindrical helix. □

The curve is said to be a cylindrical helix if it can be parameterized by

$$\alpha(s) = \left(\frac{a}{c}\int \sin\lambda(s)ds, \frac{a}{c}\int \cos\lambda(s)ds, \frac{b}{c}s\right), \tag{3.9}$$

where a, b, c are real constant satisfy the condition $a^2 + b^2 = c^2$ and $\lambda : I \subset \mathbb{R} \to \mathbb{R}$ is a smooth function.

We would like to see the direct connection between the Darboux devolopable surface satisfying the constant angle property and Theorem A. We want to determine the η function in TheoremA. Without loss of generality, we will take the classical case of a circular helix, namely for $\lambda(s) = -s$, obtaining the parametrization:

$$\alpha(s) = \left(\frac{a}{c}\cos s, \frac{a}{c}\sin s, \frac{b}{c}s\right). \tag{3.10}$$

Obtaining Frenet vector field of α we get that the Darboux devolopable of the cylindrical helix has the form

$$r(s,v) = \left(\left(\frac{b}{c} - v\frac{a}{c}\right)\sin s, \left(-\frac{b}{c} + v\frac{a}{c}\right)\cos s, \frac{a}{c} + v\frac{b}{c}\right). \tag{3.11}$$

We prove that this parametrization is a particular case of item (i) in TheoremA. We determine the function η starting with parametrization (3.11) and rewriting it in the form (2.3). Recall that the fixed direction k can be decomposed into its normal and tangent parts and developing the same technique as in [8], we get that

$$k = \sin\theta\alpha' + \cos\theta N. \tag{3.12}$$

Computing $< r_v, k >$ in two ways, first $< r_v, k >=< \alpha', \sin\theta\alpha' >= \sin\theta$ and secondly $< r_v, k >= \frac{b}{c}$, where $<, >$ denotes the Euclidean scalar product, one gets $\frac{b}{c} = \sin\theta$. We obtain also $\frac{a}{c} = \cos\theta$. If we look at the third component of the parameterizations (2.3) and (3.11) we should change of parameter $u_1 := v + \cot\theta$ in (3.11), we get the equivalent parametrization

$$r(s,u_1) = \left(\left(\frac{1}{\sin\theta} - u_1\cos\theta\right)\sin s, \left(-\frac{1}{\sin\theta} + u_1\cos\theta\right)\cos s, u_1\sin\theta\right). \tag{3.13}$$

A second reparametrization, namely $u_2 := s + \frac{\pi}{2}$, yields

$$r(s,u_1) = \left(u_1\cos\theta\left(\cos u_2, \sin u_2\right) + \gamma(u_2), u_1\sin\theta\right), \tag{3.14}$$

where $\gamma(u_2) = \left(-\frac{1}{\sin\theta}\cos u_2, -\frac{1}{\sin\theta}\sin u_2\right)$.

4. Conical Constant Angle Surfaces

A cone is a ruled surface that can be parameterized by $r(s,v) = v\alpha(s)$, where α is a regular curve. The vertex of the cone is the origin and the surface is regular wherever $t\left(\alpha(s)\times\alpha'(s)\right) \neq 0$. The characterization of conical constant angle surface in $\mathbb{E}^3$ was given in [9] *"the only conical constant angle surfaces are circular cones"*.

Now we consider the case of some conical surfaces regarded from de point of view of constant angle surfaces.

Theorem 4.1. *A tangent conical constant angle surfaces are generated by tangent curves of cylindrical helices.*

Proof. A tangent conical surface with the vertex in the origin is given by

$$r(s,v) = vt(s), \tag{4.1}$$

where we consider now s, v standard parameters and t is the unit tangent to α. Computing the normal to the above surface, one gets

$$N = \pm\frac{r_s \times r_v}{\|r_s \times r_v\|} = \pm b. \tag{4.2}$$

Choosing an orientation of the surface we take the normal to the surface equal to the binormal of the generating curve α. In the case of constant angle surfaces it follows that the binormal b of the curve makes a constant angle with the fixed direction k, namely

$$\widehat{(b,k)} = \widehat{(N,k)} = \theta. \tag{4.3}$$

It follows that α is a cylindrical helix. □

Definition 4.1. Let α be a curve in $\mathbb{E}^3$ with $\frac{\tau}{\kappa} \neq 0$ everywhere. A curve $\alpha(s)$ is said to be a Darboux helix if there is some constant unit vector k such that $< w, k >$ is constant along the curve α where $w(s)$ is a unit Darboux vector of α at s. The direction of the vector k is axis of the Darboux helix.[12]

We can identify Darboux helices by condition torsion and curvature.

Lemma 4.1. *α is a Darboux helix if and only if*

$$\frac{\left(\tau^2+\kappa^2\right)^{\frac{3}{2}}}{\kappa^2}\frac{1}{\left(\frac{\tau}{\kappa}\right)'} \tag{4.4}$$

is constant.[12]

Theorem 4.2. *A normal conical constant angle surfaces are generated by normal curves of Darboux helices.*

Proof. A normal conical surface with the vertex in the origin is given by

$$r(s,v)=vn(s), \tag{4.5}$$

where we consider now s,v standard parameters and n is the unit normal to α. Computing the normal to the above surface, one gets

$$N=\pm\frac{r_s\times r_v}{\|r_s\times r_v\|}=\pm\frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau t+\kappa b\right) \tag{4.6}$$

Choosing an orientation of the surface we take the normal to the surface equal to the Darboux vector of the generating curve α. In the case of constant angle surfaces it follows that Darboux vector w of the curve makes a constant angle with the fixed direction k, namely

$$\widehat{(w,k)}=\widehat{(N,k)}=\theta. \tag{4.7}$$

It follows that α is a Darboux helix. □

Theorem 4.3. *Let α be a curve constant precession. If the conical surfaces construct involving the normal lines to the curve α, then the surface is a constant angle surface with the axis of $k=w+\mu n$.*

Proof. A normal conical surface with the vertex in the origin is given by

$$r(s,v)=vn(s), \tag{4.8}$$

where we consider now s,v standard parameters and n is the unit normal to α. Computing the normal to the above surface, one gets

$$N=\pm\frac{r_s\times r_v}{\|r_s\times r_v\|}=\pm\frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau t+\kappa b\right)$$

Choosing an orientation of the surface we take the normal to the surface equal to the Darboux vector of the generating curve α. In the case of curves constant precession it follows that the unit normal N of the surface makes a constant angle with the fixed direction k, namely

$$\widehat{(w,k)}=\widehat{(N,k)}=\theta. \tag{4.9}$$

Since α is a curve constant precession, the axis of constant angle surface is $k=w+\mu n$ [10]. □

5. Constant Angle Surfaces and Curves

In the last section we give some characterization for a curve lying on a surface for which the unit normal makes a constant angle with a fixed direction.

The geodesic curvature plays a much more important role in the differential geometry of surfaces than the geodesic torsion does, so we start by focusing on it.

Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a unit speed curve on a surface. Take a frame along the curve $\{T,\ Y = T \times N,\ N\}$, where N is the unit normal M. The Darboux equations for this frame are

$$\begin{aligned} T' &= k_g Y + k_n N, \\ Y' &= -k_g T + \tau_g N, \\ N' &= -k_n T - \tau_g Y. \end{aligned} \tag{5.1}$$

Here k_n is the normal curvature of T on M, k_g is the geodesic curvature of α and τ_g is called geodesic torsion of α. Recall that a curve α is a line of curvature of M if T is always an eigenvector of the shape operator of M. Using the Darboux equations we can easily show that $\tau_g = 0$ if and only if α is a line of curvature. With the above notations, denote by $\gamma := \widehat{(N, b)}$, where γ, the angle function is between the unit normal and binormal to α. From Eq(5.1) one obtains

$$\begin{aligned} \tau_g &= \tau - \gamma', \\ k_n &= \kappa \sin\gamma, \\ k_g &= \kappa \cos\gamma. \end{aligned} \tag{5.2}$$

Theorem 5.1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a curve on a constant angle surface M with unit normal N and the fixed direction k. Take a frame along the curve $\{T,\ Y = T \times N,\ N\}$.*

(1) If a curve α on M is a geodesic then α is a slant helix with the axis k in $\mathbb{E}^3$.

(2) If a curve α on M is a asymptotic curve then α is a general helix with the axis k in $\mathbb{E}^3$.

(3) If α is a line of curvature, then the fixed direction k is in plane spanned by the vectors Y, N.

Proof. (1) Since α is a geodesic on M, the normal of the surface coincides with the principal normal of the curve. In the case of constant angle surfaces it follows that the principal normal of the curve makes a constant angle with the fixed direction k, namely

$$\widehat{(n, k)} = \widehat{(N, k)} = \theta. \tag{5.3}$$

It follows that α is a slant helix.

(2) Since α is a asymptotic curve on M, we have

$$k_n = 0. \tag{5.4}$$

From Eq(5.2) one obtains $\gamma = 0$ where γ, the angle function is between the unit normal and binormal to α. This actually means that the normal of the surface coincides with the binormal of the curve. In the case of constant angle surfaces it follows that the principal normal of the curve makes a constant angle with the fixed direction k, namely

$$\widehat{(b, k)} = \widehat{(N, k)} = \theta. \tag{5.5}$$

It follows that α is a general helix.

(3) Since α is a line of curvature, it follows that

$$N' = S(T) = cT, \tag{5.6}$$

for some constant c, where S is shape operator of M. In the case of constant angle surfaces

$$\langle N, k\rangle = \text{constant}.$$

By taking the derivative of this equation, we get

$$\langle N', k\rangle = 0.$$

Thus, one can write $k = \lambda Y + \mu N$. This completes the proof. □

Theorem 5.2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a line of curvature on a constant angle surface M with unit normal N and the fixed direction k. Take a frame along the curve $\{T,\ Y = T \times N,\ N\}$. Then the ruled surface*

$$r(s, v) = \alpha(s) + vN(\alpha(s)) \tag{5.7}$$

is a constant angle surface.

Proof. Let us determine the normal to the surface. To do this, we compute the partial derivatives of r with respect to s and v.

$$r_s(s, v) = (1 + vc)\,T \text{ and } r_v(s, v) = N \tag{5.8}$$

for some constant c. Using now (5.8), the normal to the ruled surface is given by

$$\widetilde{N} = \pm\frac{r_s \times r_v}{\|r_s \times r_v\|} = \pm Y. \tag{5.9}$$

Choosing an orientation of the ruled surface we take the normal to the surface equal to Y of the generating curve α. From the Theorem 6(3) we can write

$$k = \cos\theta N + \sin\theta Y.$$

Thus

$$\widehat{(\widetilde{N}, k)} = \frac{\pi}{2} - \theta.$$

This completes the proof. □

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