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CERTAIN CONSTANT ANGLE SURFACES CONSTRUCTED ON CURVES

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Abstract. In this paper we classify certain ruled surfaces in $\mathbb{E}^3$. We study the tangent developable and conical surfaces from the point of view of the constant angle property. Moreover, the natural extension to normal and binormal constant angle surfaces is given.

1. Introduction

Developable ruled surfaces represent a special category of ruled surfaces and are defined as ruled surfaces with vanishing Gaussian curvature or, more generally, those ruled surfaces which have constant Gauss map along each ruling (see for details [7]). Since nineteenth century, the developable surfaces captured the attention of mathematicians and the main properties of these surfaces are mentioned in almost all monographs and books on classical differential geometry. Regarding the property of the Gauss map, in [1] it is presented an extended study on varieties with degenerate Gauss maps, meaning that the Gaussian curvature vanishes everywhere, which includes also the case of developable surfaces.

Due to their flatness, isometries with planes are allowed and, for this reason, recently it became interesting to discover more ways to use these surfaces in different practical applications. For example, in [11] it is proposed a way of constructing and displaying graphically the kinematic surfaces, a class of surfaces generated by mutual moving ruled surfaces that touch each other along their common ruling. A more suggestive application is described in [5], where some examples of using developable surfaces in contemporary architecture are discussed.

Motivated by their flatness property, in this paper we classify the developables and some other special surfaces constructed on curves from the point of view of the constancy angle property, i.e. the normal to the surface makes a constant angle with a fixed direction. Initially, the study of constant angle surfaces was proposed for the product space $\mathbb{S}^2 \times \mathbb{R}$ in [2]. After that, surfaces endowed with this property in other ambient spaces, namely $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{E}^3$ were investigated (see for details [3, 4, 8]).

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Moreover, in [8] it is also given a comparison between the results obtained so far in the above mentioned ambient spaces. We mention that in $\mathbb{S}^2\times\mathbb{R}$ one gets positive Gaussian curvature K, in $\mathbb{H}^2\times\mathbb{R}$ negative Gaussian curvature is obtained, while K vanishes identically when the ambient is the Euclidean $3-$space. Explicitly, the constant angle property of the Gauss map with a fixed direction in $\mathbb{E}^3$ is equivalent to the fact that the Gauss map lies on a circle in the $2-$sphere $\mathbb{S}^2$. As it has no interior points in $\mathbb{S}^2$, the Gaussian curvature of the surface vanishes identically. It follows that flatness is a consequence of the constancy angle property. At this point, noticing this common aspect between developable ruled surfaces and constant angle surfaces in $\mathbb{E}^3$, in the present paper we would like to see precisely which types of developable ruled surfaces satisfy the constancy angle property. Other results on a related topic involving curves on constant angle surfaces can be found also in [10].

It is well known the following classification of surfaces in $\mathbb{E}^3$ involving their degenerated Gauss map: planes, cylinders, cones and tangent surfaces – ruled surfaces generated by the tangent lines in every point of a curve in space. If we drop the flatness property, the construction of tangent developable surfaces can be used in order to obtain some other surfaces constructed on curves replacing the tangent line to the curve with the normal line or the binormal line respectively, rising therefore the so called normal surfaces or binormal surfaces.

In next section we mention some basic facts in the general theory of curves and surfaces useful for the rest of the paper. In *Section* 3 the main result says that *the only tangent developable constant angle surfaces are generated by generalized helices*. Very recent results implying generalized helices, also called *slope lines*, can be found in [9] (see also its references).

Section 4 consists in the extension of the study on the constancy angle property for the normal and binormal surfaces. The main result is that *the normal constant angle surfaces are pieces of planes* and *the binormal constant angle surfaces are pieces of cylinders*. In *Section* 5 we make some observations regarding the *conical constant angle surfaces*, motivated by their affiliation to the classification of flat surfaces in $\mathbb{E}^3$.

2. Preliminaries

Traditionally, the differential geometry of curves starts with a smooth map of s, let us call it $\alpha : I\subset\mathbb{R}\to\mathbb{E}^3$, that parameterizes a spatial curve denoted again with α. We say that the curve is parameterized by arc length if $|\alpha'(s)|=1$, where α' is the derivative of α with respect to s. Throughout this paper s is the arc length parameter. Let us denote $\mathbf{t}(s)=\alpha'(s)$ the (unit) tangent to the curve. By definition, the curvature of α is $\kappa(s)=|\alpha''(s)|$. If $\kappa\neq 0$, then the (unit) normal of α can be obtained from $\alpha''(s)=\kappa(s)\mathbf{n}(s)$. Moreover, $\mathbf{b}(s)=\mathbf{t}(s)\times\mathbf{n}(s)$ is called the (unit) binormal to α. With these considerations $\mathbf{t},\mathbf{n},\mathbf{b}$ define an orthonormal basis. Recall the Frenet-Serret formulae:

$$\begin{aligned}\mathbf{t}'(s)&=\kappa(s)\mathbf{n}(s)\\ \mathbf{n}'(s)&=-\kappa(s)\mathbf{t}(s)+\tau(s)\mathbf{b}(s)\\ \mathbf{b}'(s)&=-\tau(s)\mathbf{n}(s)\end{aligned}\tag{2.1}$$

where $\tau(s)$ is the torsion of α at s.

A natural extension from curves to the theory of surfaces constructed on curves can be made as follows. Given a curve α parameterized by arc length in Euclidean $3-$space, we can think of constructing ruled surfaces involving α and the tangent, normal or binormal lines to the curve. As a consequence, we have three well known types of surfaces of this kind, namely

- tangent developable surface: $r(s,v) = \alpha(s) + v\mathbf{t}(s)$
- normal surface: $r(s,v) = \alpha(s) + v\mathbf{n}(s)$
- binormal surface: $r(s,v) = \alpha(s) + v\mathbf{b}(s)$

(see for details [6]). The curve α is the generating curve and the rulings are respectively the tangent, the normal and the binormal lines to the curve.

The characterization of constant angle surfaces in $\mathbb{E}^3$ was given in [8], where the constant angle is denoted by θ and, without loss of generality, the fixed direction is taken to be the real axis, denoted by k. The main result of [8] is

Theorem A. ([8]) *A surface M in $\mathbb{E}^3$ is a constant angle surface if and only if it is locally isometric to one of the following surfaces:*

(i) a surface given by

$$r : M \to \mathbb{E}^3,\ (u_1, u_2) \mapsto (u_1 \cos\theta(\cos u_2, \sin u_2) + \gamma(u_2), u_1 \sin\theta) \tag{2.2}$$

with

$$\gamma(u_2) = \cos\theta \left(-\int_0^{u_2} \eta(t) \sin t \ dt, \int_0^{u_2} \eta(t) \cos t \ dt \right) \tag{2.3}$$

for η a smooth function on an interval $I \subset \mathbb{R}$,

(ii) an open part of the plane $x \sin\theta - z \cos\theta = 0$,

(iii) an open part of the cylinder $\beta \times \mathbb{R}$, where β is a smooth curve in $\mathbb{R}^2$.

In the next sections we deal with surfaces constructed on curves and we study their constancy angle property. We also show how one can retrieve this type of surfaces constructed on curves from the general theorem of characterization above mentioned.

3. Tangent developable surfaces

We start this section with some important properties of developable ruled surfaces. In [7] it is proved that the parametrization of every flat ruled surface generically written $(u,v) \mapsto r(u,v)$, in other words, an open and dense subset of every flat ruled surface can be subdivided into subintervals such that the parametrization corresponding to these subintervals can be included in one of the following types: plane, cylinder, cone, tangent developable surface.

Thinking now of the class of constant angle surfaces in $\mathbb{E}^3$, which are flat as we have mentioned in *Preliminaries*, we show how one can retrieve the case of tangent developable surfaces, that is not mentioned explicitly in Theorem A from [8], among the constant angle surfaces in $\mathbb{E}^3$.

First we state and prove the following result concerning which types of tangent developable surfaces satisfy the constancy angle property:

Theorem 3.1. *The tangent developable constant angle surfaces are generated by cylindrical helices.*

Proof. Let us consider a tangent developable surface M, oriented, immersed in $\mathbb{E}^3$, given by

$$r(s,v) = \alpha(s) + v\mathbf{t}(s) \tag{3.1}$$

where $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ is a spatial curve parameterized by arc length consisting of the edge of regression of M and $\mathbf{t}$ is the unit tangent to α. The surface M is smooth everywhere, except in the points of the curve α.

Let us determine the normal to the surface. To do this, we compute the partial derivatives of r with respect to s and v

$$r_s(s,v) = \alpha' + v\alpha'' \text{ and } r_v(s,v) = \alpha'.$$

Using now (2.1), the normal to the surface is given by

$$N = \pm \frac{r_s \times r_v}{|r_s \times r_v|} = \mp \, \mathbf{b}.$$

Choosing an orientation of the surface we take the normal to the surface equal to the binormal of the generating curve α. In the case of constant angle surfaces it follows that the binormal $\mathbf{b}$ of the curve α makes a constant angle θ with the fixed direction k, namely

$$\widehat{(\mathbf{b}, k)} = \widehat{(N, k)} = \theta,\ \theta \in [0, \pi). \tag{3.2}$$

It follows that α is a cylindrical helix.

□

In order to write the parametrization of the cylindrical helix α we proceed this way. In general, for a curve α with $|\alpha'(s)| = 1$ and satisfying (3.2) one can write

$$\alpha(s) = \left(\psi(s), \frac{b}{c}\ s\right) \tag{3.3}$$

where the curve $\psi : I \subset \mathbb{R} \to \mathbb{R}^2$ satisfies $|\psi'(s)| = \dfrac{a}{c}$, such that $a^2 + b^2 = c^2$, $a, b, c \in (0, \infty)$. It results that the derivative of ψ with respect to s can be written $\psi'(s) = \left(\dfrac{a}{c} \sin\lambda(s), \dfrac{a}{c}\cos\lambda(s)\right)$ for a certain function λ. Integrating, the expression of ψ is obtained, and substituting it in (3.3) one gets:

$$\alpha(s) = \left(\frac{a}{c}\int \sin\lambda(s)ds,\ \frac{a}{c}\int \cos\lambda(s)ds,\ \frac{b}{c}\ s\right), \tag{3.4}$$

where a, b, c satisfy the above condition and $\lambda : I \subset \mathbb{R} \to \mathbb{R}$ is a smooth function. The curve α parameterized by (3.4) is said to be a *cylindrical helix*.

We would like to see the direct connection between the tangent developable surfaces satisfying the constant angle property and Theorem A. We have

Theorem 3.2. *The tangent developable constant angle surfaces are obtained for* $\eta(t) = -\lambda^{-1}(\frac{\pi}{2} - t)$ *in Theorem A.*

Proof. We start the proof with the classical case of a circular helix, namely for $\lambda(s) = -s$, obtaining the parametrization:

$$\alpha(s) = \left(\frac{a}{c}\cos s, \frac{a}{c}\sin s, \frac{b}{c}s\right). \tag{3.5}$$

Substituting $\alpha'(s) = \left(-\dfrac{a}{c}\sin s, \dfrac{a}{c}\cos s, \dfrac{b}{c}\right)$ and expression (3.5) in parametrization (3.1) we get that the tangent developable of the cylindrical helix α has the form

$$r(s,v)=\left(\frac{a}{c}(\cos s-v\sin s),\frac{a}{c}(\sin s+v\cos s),\frac{b}{c}(s+v)\right). \tag{3.6}$$

We prove that this parametrization is a particular case of (i) in Theorem A. We determine the general function η starting with parametrization (3.6) and rewriting it in the form (2.2).

Let us look at the third component of the parameterizations (2.2) and (3.6). Recall that the fixed direction k can be decomposed into its normal and tangent parts and, developing the same technique as in [8], we get that

$$k=\sin\theta\alpha'+\cos\theta N.$$

Computing $\langle r_s,k\rangle$ in two ways, first $\langle r_s,k\rangle=\langle\alpha',\sin\theta\alpha'\rangle=\sin\theta$ and secondly $\langle r_s,k\rangle=\frac{b}{c}$, where $\langle\ ,\ \rangle$ denotes the Euclidean scalar product, one gets $\frac{b}{c}=\sin\theta$. We obtain also $\frac{a}{c}=\cos\theta$.

After the change of parameter $u_1:=s+v$ in (3.6), we get the equivalent parametrization

$$\begin{aligned} r(s,u_1)\;=\;&\Big(\cos\theta(\cos s-u_1\sin s+s\sin s),\\ &\cos\theta(\sin s+u_1\cos s-s\cos s),u_1\sin\theta\Big).\end{aligned} \tag{3.7}$$

A second reparametrization, namely $u_2:=s+\frac{\pi}{2}$, yields

$$r(u_1,u_2)=(u_1\cos\theta(\cos u_2,\sin u_2)+\gamma(u_2),u_1\sin\theta) \tag{3.8}$$

where

$$\gamma(u_2)=\cos\theta\left(\sin u_2-\left(u_2-\frac{\pi}{2}\right)\cos u_2,-\cos u_2-\left(u_2-\frac{\pi}{2}\right)\sin u_2\right). \tag{3.9}$$

Now, by comparison with Theorem A, (3.8) is identical with (2.2) and we only have to determine the smooth function η in order to write (3.9) in the same manner as (2.3).

We claim that $\eta(t)=\frac{\pi}{2}-t$.

In order to prove the claim we compute

$$-\int_0^{u_2}\eta(t)\sin t\,dt=\left(\frac{\pi}{2}-u_2\right)\cos u_2+\sin u_2-\frac{\pi}{2}$$

and

$$\int_0^{u_2}\eta(t)\cos t\,dt=\left(\frac{\pi}{2}-u_2\right)\sin u_2-\cos u_2+1.$$

We complete the proof in this case concluding that the expression (3.9) is equivalent with (2.3) for $\eta(t)=\frac{\pi}{2}-t$ and taking into account the integration limits, i.e. a translation in the $xy-$plane.

Let us return to the general case of a cylindrical helix.

Follow-on the same idea like in the previous case, already knowing that $\frac{a}{c}=\cos\theta$, $\frac{b}{c}=\sin\theta$ and taking the derivative of α with respect to s,

$\alpha'(s) = \left(\frac{a}{c}\sin\lambda(s),\ \frac{a}{c}\cos\lambda(s),\ \frac{b}{c}\right)$, we get the parametrization for the tangent developable corresponding to the generalized helix given by (3.4) in the form

$$r(s,v) = \left(\cos\theta\left(\int \sin\lambda(s)ds + v\sin\lambda(s)\right),\ \cos\theta\left(\int \cos\lambda(s)ds + v\cos\lambda(s)\right), (s+v)\sin\theta\right). \tag{3.10}$$

Making now the change of parameters $u_1 := s+v$ and $u_2 := \frac{\pi}{2} - \lambda(s)$ in (3.10), one gets the equivalent parametrization

$$r(u_1,u_2) = (u_1\cos\theta(\cos u_2, \sin u_2) + \gamma(u_2), u_1\sin\theta) \tag{3.11}$$

where

$$\gamma(u_2) = \cos\theta\left(\int \left(\lambda^{-1}\left(\frac{\pi}{2}-u_2\right)\right)'\cos u_2\ du_2 - \lambda^{-1}\left(\frac{\pi}{2}-u_2\right)\sin\left(\frac{\pi}{2}-u_2\right),\ \int \left(\lambda^{-1}\left(\frac{\pi}{2}-u_2\right)\right)'\sin u_2\ du_2 - \lambda^{-1}\left(\frac{\pi}{2}-u_2\right)\cos\left(\frac{\pi}{2}-u_2\right)\right). \tag{3.12}$$

Our attention is focused on the function η which should be determined. By straightforward computations,

$$\begin{aligned} -\int_0^{u_2} \eta(t)\sin t\ dt &= \int_0^{u_2} \eta(t)\left(\sin\left(\frac{\pi}{2}-t\right)\right)' dt \\ &= \eta(u_2)\sin\left(\frac{\pi}{2}-u_2\right) - \eta(0) - \int_0^{u_2} \eta(t)'\cos t\ dt. \end{aligned}$$

and

$$\begin{aligned} \int_0^{u_2} \eta(t)\cos t\ dt &= -\int_0^{u_2} \eta(t)\left(\cos\left(\frac{\pi}{2}-t\right)\right)' dt \\ &= -\eta(u_2)\cos\left(\frac{\pi}{2}-u_2\right) + \int_0^{u_2} \eta(t)'\sin t\ dt. \end{aligned}$$

Taking into account the integration limits we conclude the proof of the theorem with the fact that also in this general case for $\eta(t) = -\lambda^{-1}\left(\frac{\pi}{2}-t\right)$ the expressions (3.12) and (2.3) are equivalent.

□

4. Normal and Binormal Surfaces

In this section we deal with the other two types of surfaces constructed on a spatial curve α, *normal* and *binormal surfaces.* As we have seen in *Preliminaries*, these surfaces are constructed by using the same technique as the tangent developable ones, replacing in (3.1) the tangent line **t** with the normal line **n**, respectively the binormal line **b**.

In the first part of this section we study under which conditions the normal and the binormal surfaces can be retrieved form Theorem A under the property of constant angle surfaces.

Concerning this aspect, we state and prove the following result

Theorem 4.1.

(1) *The normal constant angle surfaces are pieces of planes.*
(2) *The binormal constant angle surfaces are pieces of cylindrical surfaces.*

Proof. Let us consider first the parametrization of a normal surface

$$r(s,v) = \alpha(s) + v\mathbf{n}(s). \tag{4.1}$$

Computing the normal to the above surface, one gets

$$N = \frac{(1-\kappa v)\mathbf{b} - \tau v \mathbf{t}}{\sqrt{\Delta}}, \text{ where } \Delta = (1-\kappa v)^2 + \tau^2 v^2. \tag{4.2}$$

We are interested in those normal surfaces for which the normal N makes a constant angle θ with the fixed direction k, namely $(\widehat{N,k}) = \theta$, equivalently, $\langle N, k\rangle = \cos\theta$. Substituting (4.2) in this expression we get a vanishing polynomial expression of second order in v. So, all the coefficients must be identically zero, that is, the following relations are satisfied:

$$\langle \mathbf{b}, k\rangle^2 - \cos^2\theta = 0 \tag{4.3}$$

$$\kappa \langle \mathbf{b}, k\rangle^2 + \tau \langle \mathbf{b}, k\rangle \langle \mathbf{t}, k\rangle - \kappa \cos^2\theta = 0 \tag{4.4}$$

$$(\kappa \langle \mathbf{b}, k\rangle + \tau \langle \mathbf{t}, k\rangle)^2 - (\kappa^2 + \tau^2)\cos^2\theta = 0. \tag{4.5}$$

From (4.3) one obtains

$$\langle \mathbf{b}, k\rangle = \pm\cos\theta. \tag{4.6}$$

Substituting (4.3) in (4.4) we get

$$\tau \langle \mathbf{b}, k\rangle \langle \mathbf{t}, k\rangle = 0.$$

We distinguish the following cases:

a) $\tau = 0$.

Both (4.4) and (4.5) reduce to (4.3) which is automatically fulfilled because α being a planar curve its binormal coincides with the normal of the plane. Thinking now the normal surface as a ruled surface for which the rulings are the normal lines to the generating plane curve α, we get that the normal constant angle surface is a portion of plane.

b) $\tau \neq 0$.

b.1) $\langle \mathbf{b}, k\rangle = 0$. From (4.3) follows $\cos\theta = 0$ and substituting it in (4.5) we get $\langle \mathbf{t}, k\rangle = 0$. Moreover, taking the derivative with respect to s we get also $\langle \mathbf{n}, k\rangle = 0$. We have a contradiction: k is orthogonal to all $\mathbf{t}, \mathbf{n}, \mathbf{b}$ which already define an orthonormal basis!

b.2) $\langle \mathbf{t}, k\rangle = 0$. Analogously, we get a contradiction similar to the previous subcase. Again this situation cannot occur.

So, in the case of normal constant angle surfaces we retrieve here the case (*ii*) from Theorem A.

In the same manner, consider the parametrization of a binormal surface

$$r(s,v) = \alpha(s) + v\mathbf{b}(s). \tag{4.7}$$

The normal to the surface is

$$N = \frac{-\mathbf{n} - \tau v \mathbf{t}}{\sqrt{\Delta}}, \text{ where } \Delta = 1 + \tau^2 v^2.$$

With the same technique as before, one gets that the only case which can occur is for $\tau = 0$ with the additional condition $\langle \mathbf{t}, k \rangle = 0$. Yet, the binormal to the planar curve α is parallel to the fixed direction and $\theta = \frac{\pi}{2}$. Thus, the binormal constant angle surfaces are cylindrical surfaces generated by the planar curve α. In this manner we reach item (iii) of Theorem A.

□

We conclude this section pointing out that studying these types of surfaces constructed on curves under the constancy angle property, all three cases in Theorem A are retrieved. First item (i) has as a particular case the tangent developable constant angle surfaces, the second item (ii) includes the case of normal surfaces and the last one, (iii) corresponds, in particular, for binormal surfaces satisfying the constancy angle property.

5. Conical constant angle surfaces

In this last section let us return to the classification of flat surfaces in $\mathbb{E}^3$. In previous sections we recovered the planes, the cylinders and the tangent developable surfaces among the constant angle surfaces. Considering now the case of conical surfaces thought from de point of view of constant angle surfaces, we state and prove

Proposition 5.1. *The only conical constant angle surfaces are circular cones.*

Proof. A conical surface with the vertex in the origin is given by

$$r(s, v) = v\alpha(s) \tag{5.1}$$

where we consider now s, v standard parameters. This means that any cone generated by a generic curve α can be reparameterized using standard parameters such that α lies on the unit $2-$sphere, $|\alpha(s)| = 1$. In these conditions, the normal to the surface is given by $N = \alpha \times \alpha'$. The constant angle property (3.2) is equivalent with $\langle \alpha \times \alpha', k \rangle = \theta$. Taking the derivative with respect to s, we get

$$\langle \alpha \times \alpha'', k \rangle = 0. \tag{5.2}$$

Now, α'' can be decomposed in the orthonormal basis $\{\alpha, \alpha', \alpha \times \alpha'\}$ as

$$\alpha'' = \langle \alpha'', \alpha \rangle \, \alpha + \langle \alpha'', \alpha' \rangle \, \alpha' + \langle \alpha'', \alpha \times \alpha' \rangle \, \alpha \times \alpha'.$$

From $|\alpha(s)| = 1$ and $|\alpha'(s)| = 1$ we get $\langle \alpha'', \alpha' \rangle = 0$ and $\langle \alpha'', \alpha \rangle = -1$. Substituting these expressions in the decomposition of α we get

$$\alpha'' = -\alpha + \kappa_g (\alpha \times \alpha') \tag{5.3}$$

where κ_g denotes the geodesic curvature of α.

Substituting (5.3) in (5.2) one obtains that a conical surface is constant angle surface if it fulfills

$$\kappa_g \, \langle \alpha' \times k \rangle = 0.$$

At this point we can conclude that α is a planar curve. Moreover, knowing that α is on the unit 2-sphere, it follows that α is a circle. So, (5.1) parameterizes a circular cone. □

In order to retrieve circular cones from Theorem A, it suffices to consider a general parametrization for a circular cone and after a change of frame such that the cone's axis is parallel with the $z-$axis, we get the corresponding parametrization replacing (2.3) for $\eta(t) = 0$ in (2.2).

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