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1-TYPE CURVES AND BIHARMONIC CURVES IN EUCLIDEAN 3-SPACE

HÜSEYİN KOCAYİĞİT AND HASAN HİLMİ HACISALİHOĞLU

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Abstract. We study 1-type curves by using the mean curvature vector field of the curve. We also study biharmonic curves whose mean curvature vector field is in the kernel of Laplacian. We give some theorems for them in the Euclidean 3-space. Moreover we give some characterizations and results for a Frenet curve in the same space.

1. Introduction and Preliminaries

Chen and Ishikawa [2] classified biharmonic curves in semi-Euclidean space E^n_v. In this paper we shall give the characterizations of 1-type curves and biharmonic curves in Euclidean 3-space in terms of curvature and torsion. More precisely, we shall show that every 1-type curve in E^3 is a circular helix and every biharmonic curve in the same space is a geodesic.

Let (M^3, g) be a Riemannian 3-manifold. Let $\gamma : I \to M$ be an arclengthed curve on M. Namely the velocity vector field γ' satisfies $g(\gamma', \gamma') = 1$. A unit speed curve γ is said to be a geodesic if $\nabla_{\gamma'}\gamma' = 0$, where ∇ is the Levi-Civita connection of (M^3, g). In particular an arclengthed curve γ is said to be a geodesic if $\kappa = 0$, where κ is the curvature of γ. Note that if $\kappa = 0$, then automatically $\tau = 0$, where τ is the torsion of γ.

We first assume that $g(\gamma'', \gamma'') \neq 0$. A unit speed curve γ is said to be a Frenet curve if $g(\gamma'', \gamma'') \neq 0$. Every Frenet curve γ on (M^3, g) admits an orthonormal Frenet frame field $\{V_1, V_2, V_3\}$ along γ such that $V_1 = \gamma'(s)$ and there is the following Frenet-Serret formulae :

$$\begin{bmatrix} \nabla_{\gamma'}V_1 \\ \nabla_{\gamma'}V_2 \\ \nabla_{\gamma'}V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{1.1}$$

The functions $\kappa \geq 0$ and τ are called the curvature and torsion, respectively. The vector fields V_1, V_2, V_3 are called tangent vector field, principal normal vector field and binormal vector field of γ, respectively. A Frenet curve with constant curvature and zero torsion is called a pseudo circle. A circular helix is a Frenet curve whose

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curvature and torsion are constants. Pseudo circles are regarded as degenerate helices. Helices, which are not circles, are frequently called proper helices.

More generally a curve with constant curvature and zero torsion is called a (Riemannian) circle. Geodesics are regarded as Riemannian circles of zero curvature.

Let us denote the Laplace-Beltrami operator by Δ of γ and the mean curvature vector field along γ by $\mathbb{H}$.

The Frenet-Serret formulae of γ imply that the mean curvature vector field $\mathbb{H}$ is given by

$$\mathbb{H} = \nabla_{\gamma'}\gamma' = \nabla_{\gamma'}V_1 = \kappa V_2, \tag{1.2}$$

where κ is the curvature of γ.

The Laplacian operator of γ is defined by

$$\Delta = -\nabla^2_{\gamma'} = -\nabla_{\gamma'}\nabla_{\gamma'} \tag{1.3}$$

(see [5], and [2]).

Definition 1.1. Let $M \subset E^{n+d}$ be a compact submanifold and $x : M \to E^{n+d}$ be an isometric immersion. In the case of

$$x = x_0 + \sum_{i=1}^{k} x_i$$

then M is called *finite type* where x_0 is the constant vector and $\Delta x_i = \lambda_i x_i$, in the other case M is called *infinite type and* $x_1, x_2, ..., x_k$ are non-constant functions [1].

Theorem 1.1. *The submanifold $M \subset E^{n+d}$ is k-type if and only if the mean vector field $\mathbb{H}$ of M satisfy*

$$\Delta^k\mathbb{H}+c_1\Delta^{k-1}\mathbb{H}+ ... + c_{k-1}\Delta\mathbb{H} + c_k\mathbb{H} = 0, \tag{1.4}$$

where

$$\begin{aligned} c_1 &= -\sum_{t=p}^{q}\lambda_t \\ c_2 &= \sum_{t<s}\lambda_t\lambda_s\,, ..., \; c_{q-p+1} = (-1)^{q-p+1}\lambda_p...\lambda_q \qquad (k = q-p+1), \end{aligned}$$

where $\Delta x_i = \lambda_i x_i$ $(1 \le i \le k)$ [3].

According to the above theorem, the following definition can be given.

Definition 1.2. A unit speed curve $\gamma : I \to M$ on a Riemann 3-manifold M is said to be *1-type* if

$$\Delta\mathbb{H} = \lambda\mathbb{H}. \tag{1.5}$$

Definition 1.3. A unit speed curve $\gamma : I \to M$ on a Riemann 3-manifold M is said to be *biharmonic* if

$$\Delta\mathbb{H} = 0. \tag{1.6}$$

Theorem 1.2. *If M is Euclidean 3-space, then γ is biharmonic if and only if* $\Delta(\Delta\gamma) = 0$ [1].

Definition 1.4. A unit speed curve $\gamma : I \to M$ on a Riemann 3-manifold M is said to be a *general helix* which means that its $\frac{\kappa}{\tau}$ is constant but κ and τ are not constant (see [6] and [7]).

Lemma 1.1. *The mean curvature vector field* $\mathbb{H}$ *is harmonic* ($\Delta\mathbb{H} = 0$) *if and only if*

$$\bigtriangledown_{\gamma'} \bigtriangledown_{\gamma'} \bigtriangledown_{\gamma'}\gamma' = 0 \text{ [1].} \tag{1.7}$$

Theorem 1.3. *If* M *is the Euclidean space* E^m *, then along the curve* γ, $\mathbb{H}$ *satisfies* $\Delta\mathbb{H} = 0$ *if and only if* γ *is biharmonic (i.e.,* $\Delta(\Delta\gamma) = 0$ *since* $\Delta\gamma = -\mathbb{H}$*)* [3].

2. 1-type curves and biharmonic curves in Euclidean 3-space

Theorem 2.1. *Let* γ *be an arclengthed parametrized Frenet curve on Riemannian* $3-$*manifold* (M^3, g)*. Then, along the curve* γ, $\Delta\mathbb{H} = \lambda\mathbb{H}$ *holds if and only if*

$$\kappa\kappa' = 0,\ \kappa\tau^2 + \kappa^3 - \kappa'' = \lambda\kappa,\ 2\kappa'\tau + \kappa\tau' = 0 \tag{2.1}$$

Proof. From (1.1), (1.2) and (1.3) we get

$$\Delta\mathbb{H} = 3\kappa\kappa' V_1 - (\kappa'' - \kappa^3 - \kappa\tau^2)V_2 - (2\kappa'\tau + \kappa\tau')V_3 \ . \tag{2.2}$$

By (1.2) and (1.5) we have

$$3\kappa\kappa' V_1 - (\kappa'' - \kappa^3 - \kappa\tau^2)V_2 - (2\kappa'\tau + \kappa\tau')V_3 = \lambda\kappa V_2 \ . \tag{2.3}$$

According to (2.3) the equations of (2.1) are obtained.

Conversely, the equations of (2.1) satisfy the equation (1.5). □

Kılıç obtained the similar result in a different way [9].

Theorem 2.2. *Let* γ *be an arclengthed parametrized Frenet curve on Riemannian* $3-$*manifold* (M^3, g)*. Then, along the curve* γ, $\Delta\mathbb{H} = \lambda\mathbb{H}$ *holds if and only if* γ *is a circular helix, where*

$$\lambda = \kappa^2 + \tau^2 \ . \tag{2.4}$$

Proof. From Theorem 2.1, we have (2.1). Since γ is a Frenet curve, $\kappa \neq 0$. Thus (2.1) shows that γ is a circular helix and we obtain (2.4).

Conversely, since γ is a circular helix and $\lambda = \kappa^2 + \tau^2$, κ and τ are constants. From that $\Delta\mathbb{H} = \lambda\mathbb{H}$ is satisfied. □

Theorem 2.3. *Let* γ *be an arclengthed curve on Riemannian* $3-$*manifold* (M^3, g)*. Then, along the curve* γ, $\Delta\mathbb{H} = 0$ *holds if and only if* γ *is a geodesic.*

Proof. Let I be an open interval and $\gamma : I \to M$ be an arclengthed curve where its arclength is s. Let $\{V_1, V_2, V_3\}$ be the Frenet frame field of γ. By (1.2), direct computation shows that

$$\bigtriangledown_{\gamma'}\mathbb{H} = -\kappa^2 V_1 + \kappa' V_2 + \kappa\tau V_3.$$

Let us compute the Laplacian of $\mathbb{H}$:

$$\begin{aligned} -\Delta\mathbb{H} &= \bigtriangledown_{\gamma'} \bigtriangledown_{\gamma'} \mathbb{H} \\ &= -3\kappa\kappa' V_1 + (\kappa'' - \kappa^3 - \kappa\tau^2)V_2 + (2\kappa'\tau + \kappa\tau')V_3. \end{aligned}$$

Thus along the curve γ, $\Delta\mathbb{H} = 0$ holds if and only if $\kappa = 0$. So, γ is a geodesic.

Conversely, every geodesic curve satisfies $\Delta\mathbb{H} = 0$. □

Dimitric obtained the similar result by a different way [4].

Corollary 2.1. *Let* γ *be a curve in Euclidean space* E^3*. Then* γ *is biharmonic if and only if* γ *is a straight line.*

3. A GENERAL CHARACTERIZATION FOR A FRENET CURVE AND SOME RESULTS

The next theorem gives a general characterization for the Frenet curves in Riemannian $3-$ manifold (M^3, g).

Theorem 3.1. *Let γ be an arclengthed Frenet curve on Riemannian $3-$ manifold (M^3, g). Then γ satisfies the following differential equation*

$$\nabla^3_{\gamma'} V_1 + \lambda_1 \nabla^2_{\gamma'} V_1 + \lambda_2 \nabla_{\gamma'} V_1 + \lambda_3 V_1 = 0, \tag{3.1}$$

where

$$\begin{aligned}
\lambda_1 &= -(2\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau}),\\
\lambda_2 &= -\frac{\kappa''}{\kappa} + \frac{\kappa'\tau'}{\kappa\tau} + 2(\frac{\kappa'}{\kappa})^2 + \kappa^2 + \tau^2,\\
\lambda_3 &= \kappa\tau(\frac{\kappa}{\tau})'.
\end{aligned}$$

Proof. By (1.1) we get

$$V_2 = \frac{1}{\kappa}\nabla_{\gamma'} V_1 \ . \tag{3.2}$$

Since $\nabla_{\gamma'} V_3 = -\tau V_2$, we have, by (3.2), that

$$\nabla_{\gamma'} V_3 = -\frac{\tau}{\kappa}\nabla_{\gamma'} V_1 \ . \tag{3.3}$$

Since $\nabla_{\gamma'} V_2 = -\kappa V_1 + \tau V_3$, we get

$$V_3 = \frac{1}{\tau}\nabla_{\gamma'} V_2 + \frac{\kappa}{\tau} V_1 \ . \tag{3.4}$$

Now combining (3.2) with (3.4) we may write

$$V_3 = -\frac{\kappa'}{\tau\kappa^2}\nabla_{\gamma'} V_1 + \frac{1}{\tau\kappa}\nabla^2_{\gamma'} V_1 + \frac{\kappa}{\tau} V_1. \tag{3.5}$$

Taking the covariant derivative of (3.5) and considering (3.3) we obtain the equation (3.1). □

Corollary 3.1. *Let γ be an arclengthed Frenet curve on Riemannian $3-$ manifold (M^3, g). Then γ is a general helix if and only if*

$$\nabla^3_{\gamma'} V_1 + \lambda \nabla^2_{\gamma'} V_1 + \mu \nabla_{\gamma'} V_1 = 0, \tag{3.6}$$

where

$$\begin{aligned}
\lambda &= -3\frac{\kappa'}{\kappa},\\
\mu &= -\frac{\kappa''}{\kappa} + 3(\frac{\kappa'}{\kappa})^2 + \kappa^2 + \tau^2.
\end{aligned}$$

Proof. Suppose that γ is a general helix, *i.e.*, $\frac{\kappa}{\tau}$ is constant, in other words $\kappa'\tau = \kappa\tau'$. If we replace the value $\frac{\kappa'}{\kappa} = \frac{\tau'}{\tau}$ in (3.1), then we get (3.6).

Conversely, assume that (3.6) holds. We show that the curve γ is a general helix. To obtain (3.1) from (3.6), $\frac{\kappa}{\tau}$ must be constant. Thus γ is a general helix. □

Note that K. Ilarslan showed this result in a different way [8].

Corollary 3.2. *Let γ be an arclengthed Frenet curve on Riemannian $3-$ manifold (M^3, g). Then γ is a general helix if and only if, along the curve γ,*

$$\Delta\mathbb{H} + \lambda \bigtriangledown_{\gamma'} \mathbb{H} + \mu\mathbb{H} = 0, \tag{3.7}$$

where

$$\begin{aligned} \lambda &= 3\frac{\kappa'}{\kappa}, \\ \mu &= \frac{\kappa''}{\kappa} - 3(\frac{\kappa'}{\kappa})^2 - \kappa^2 - \tau^2. \end{aligned}$$

Proof. According to (3.6), (1.2) and (1.3) we have (3.7). Sufficiency is clear. □

Corollary 3.3. *Let γ be an arclengthed Frenet curve on Riemannian $3-$manifold (M^3, g). Then γ is a circular helix if and only if*

$$\bigtriangledown^3_{\gamma'} V_1 + (\kappa^2 + \tau^2) \bigtriangledown_{\gamma'} V_1 = 0. \tag{3.8}$$

By (3.1), the proof can be easily seen.

We know that K. Ilarslan showed this result in a different way [8].

Corollary 3.4. *Let γ be an arclengthed Frenet curve in Riemannian $3-$manifold (M^3, g). Then γ is a circular helix if and only if*

$$\Delta\mathbb{H} + \lambda\mathbb{H} = 0, \tag{3.9}$$

where $\lambda = -(\kappa^2 + \tau^2)$.

The proof is clear by using (3.1), (1.2) and (1.3).

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Celal Bayar University, Faculty of Arts and Science, Department of Mathematics, Muradiye Campus, 45030, Manisa/TURKEY
E-mail address: huseyin.kocayigit@bayar.edu.tr

Ankara University, Faculty of Science, Department of Mathematics, Tandoğan, Ankara, TURKEY
E-mail address: hacisali@science.ankara.edu.tr