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Some Algebraic Properties of Generalized Fuzzy Rough Approximations Derived by Fuzzy Set-Valued Homomorphism of LA-Γ-Semigroups

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Abstract: In this paper we define the concept of fuzzy set valued homomorphism of LA-Γ-semigroups and mention some features of them. We also investigate the approximations of a generalized fuzzy approximation space constructed on LA-Γ-semigroups and derived by fuzzy set valued homomorphisms of LA-Γ-semigroups. Especially, we focus on some algebraic properties of fuzzy subsets in terms of protection of some properties under these approximations.

Bulanık Küme Değerli LA-Γ-Yarıgrup Homomorfileri ile Türetilmiş Genelleştirilmiş Bulanık Kaba Yaklaşımların Bazı Cebirsel Özellikleri

Anahtar Kelimeler
Bulanık Küme değerli homomorfi,
Bulanık *LA*-Γ-semigoup,
Genelleştirilmiş bulanık yaklaşım uzayı

Özet: Bu çalışmada bulanık küme değerli *LA*-Γ-yarıgrup homomorfisi kavramını tanımlayacağız ve onların bazı özelliklerine değineceğiz. Ayrıca *LA*-Γ-yarıgruplar üzerine inşa edilmiş ve bulanık küme değerli *LA*-Γ-yarıgrup homomorfisi ile üretilmiş genelleştirilmiş bulanık yaklaşım uzayının yaklaşımlarını araştıracağız. Özellikle, bu yaklaşımlar altında bazı özelliklerin korunması açısından bulanık alt kümelerin bazı cebirsel özelliklerine odaklanacağız.

1. Introduction

Rough set theory is asserted by Pawlak [22] initially. It has been applied to algebra by Biswas and Nanda [4] proposing the rough subgroups. Some properties of rough approximations are investigated on semigroups via the congruence relations by Kuroki [19]. Using arbitrary relations, Yao [37] generalized the rough sets. Davvaz propose the concept of set-valued homomorphisms to construct a generalized approximation space on groups [5]. Also see [1, 2, 11, 33–36].
Fuzzy and rough sets (see [38]) are two distinct but complementary theories. Considering fuzzy relations instead of crisp binary relations, Dubois and Prade [6] introduce fuzzy rough sets. Using fuzzy similarity relation Radzikowska and Kerre [23] define $(\mathscr{I},\mathscr{T})$-fuzzy rough set. Li et al. [21] study fuzzy rough approximations constructed with t-norms and implications on a ring. Moreover in [20], Li and Yin discuss some properties of these approximations constructed on a semigroup through a $\mathscr{T}$-congruence L-fuzzy relation.
Wu et al. [29] generalize fuzzy rough sets. Through $\mathscr{T}L$-fuzzy relational morphism [12], generalized $(\mathscr{I},\mathscr{T})$-$L$-fuzzy rough sets are applied by Ekiz et al. to the rings and semigroups [8, 9]. Since they are fuzzy expansion of the set valued homomorphisms it is sensible to use $\mathscr{T}L$-fuzzy set valued homomorphism even though a $\mathscr{T}L$-fuzzy set valued homomorphism can be defined by a $\mathscr{T}L$-fuzzy relational morphism and vice versa. Recently, Ekiz et al. [7] have introduced the $\mathscr{T}L$-fuzzy set valued homomorphisms of groups.
Jun and Lee [13] introduce the fuzzy ideals in Γ-ring. The notion of an LA-semigroup (also known as AG-groupoids) is defined by Kazim and Naseeruddin [14]. Sen [24, 27] introduce the notion of Γ-semigroup. Recently, Shah and Rehman [25] have proposed the concept of LA-Γ-semigroup (Γ-AG-groupoid) and investigated on Γ-ideals and Γ-bi-ideals. In [26], they have defined fuzzy Γ-ideals of a Γ-AG-groupoid and studied its properties.
In this paper, we define the notions of relational morphism, set-valued homomorphism, fuzzy relational morphism and fuzzy set-valued homomorphism of LA-Γ-semigroups and denote some relations between them. This is important for a better interpretation of the place of some studies in the literature. To study on the algebraic features of generalized fuzzy rough sets is very pupular subject. In addition, the notion of LA-Γ-semigroup is a new and quite interesting algebraic structure. Hence our motivation in this paper is to combine this two topics. Thus this paper

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focuses on some algebraic features of approximations of generalized fuzzy rough sets constructed via fuzzy set valued homomorphisms of LA-Γ-semigroups. Especially we investigate preserving of algebraic properties of a fuzzy subset of an LA-Γ-semigroup under the fuzzy rough approximations. More specifically, we deal with fuzzy lower approximations in some their particular cases. The following informations are required in our study.

1.1. Fuzzy Subsets

Let J and Q be universes of discourse. From J into the unit interval $[0,1]$, a function is referred as a fuzzy subset of J (see [38]). $\mathscr{P}(J)(\mathscr{F}(J))$ will denote the set of all (fuzzy) subsets of J. A fuzzy binary relation from J to Q is a fuzzy subset of $J\times Q$. Let $\nu,\mu\in\mathscr{F}(J)$. $\nu\diamond\mu$ is a binary relation on $\mathscr{F}(J)$ defined by $(\nu\diamond\mu)(j)=\nu(j)\diamond\mu(j)$ for all $j\in J$, where "$\diamond$" is a binary relation on $[0,1]$. Let $\rho\in[0,1]$. Then the set $\nu_\rho=\{j\in J\mid \nu(j)\geq\rho\}$ is called ρ- cut (or level) subset of ν. $\nu\subseteq\mu$ if $\nu(x)\leq\mu(x)$ for all $x\in J$, and $\nu_\rho\subseteq\mu_\rho$ if $\nu\subseteq\mu$.

Let J,Q,D be non-empty sets. Then the compositions of $A:J\to\mathscr{F}(Q)$ and $B:Q\to\mathscr{F}(D)$ is the function $B*A:J\to\mathscr{F}(D)$ defined by $(B*A)(j)(d)=\bigvee_{q\in Q}A(j)(q)\wedge B(q)(d)$ for all $j\in J, d\in D$ (see [7, 17, 28]).

1.2. Fuzzy Logical Operators

A mapping $\mathscr{T}:[0,1]\times[0,1]\to[0,1]$ which is increasing, associative, commutative and providing the boundary condition $\mathscr{T}(j,1)=j$ for all $j\in[0,1]$ is called a t-norm on $[0,1]$. On $[0,1]$, the largest t-norm is the standard minimum operator $\mathscr{T}_M(j,q)=min\{j,q\}=j\wedge q$. A mapping $\mathscr{S}:[0,1]\times[0,1]\to[0,1]$ which is increasing, associative, commutative and providing the boundary condition: $\mathscr{S}(\alpha,0)=\alpha$ for all $j\in[0,1]$ is called a t-conorm on $[0,1]$. On $[0,1]$, the maximum operator $\mathscr{S}_M(\beta,\alpha)=max\{\beta,\alpha\}=\beta\vee\alpha$ is the smallest t-conorm. In this paper we use $\beta\wedge\alpha$ instead of $\mathscr{T}_M(\beta,\alpha)$ and we use $\beta\vee\alpha$ instead of $\mathscr{S}_M(\beta,\alpha)$, and especially, we consider $\mathscr{T}_M$ as t-norm and $\mathscr{S}_M$ as t-conorm. A mapping $\mathscr{N}:[0,1]\to[0,1]$ which is decreasing and providing the conditions $\mathscr{N}(1)=0,\mathscr{N}(0)=1$ is referred as a negator $\mathscr{N}$ on $[0,1]$. The negator $\mathscr{N}_s(\alpha)=1-\alpha$ for all $\alpha\in[0,1]$ is called standard negator. An implication on $[0,1]$ is a mapping $\mathscr{I}:[0,1]\times[0,1]\to[0,1]$ providing the conditions $\mathscr{I}(1,1)=\mathscr{I}(0,1)=\mathscr{I}(0,0)=1,\mathscr{I}(1,0)=0$. An $\mathscr{S}$-implication based on $\mathscr{S}$ and $\mathscr{N}$ is an implication defined by $\mathscr{I}(j,q)=\mathscr{S}(\mathscr{N}(j),q)$ for all $j,q\in[0,1]$, and R-implication (residual implication) based on a t-norm $\mathscr{T}$ is an implication defined by $\mathscr{I}(j,q)=\bigvee_{\mathscr{T}(j,\alpha)\leq q}\alpha$ for all $j,q\in[0,1]$ (See [3, 10, 18, 20]).

1.3. Generalized Rough Approximations

Let J and Q be non-empty sets. A generalized approximation space is the triple (J,Q,T) with a mapping $T:J\to\mathscr{P}(Q)$. T defines a binary relation $\varphi_T=\{(j,q)\mid q\in T(j)\}$ and T can defined by a binary relation $\varphi\subseteq J\times Q$ as $T_\varphi(j)=\{q\in Q\mid (j,q)\in\varphi\}$ for all $j\in J$. Let $D\subseteq Q$. A generalized rough set is the pair $(\underline{T}(D),\overline{T}(D))$ where the lower and upper approximations of D, respectively, are $\underline{T}(D)$ and $\overline{T}(D)$ defined by $\underline{T}(D)=\{j\in J\mid T(j)\subseteq D\}$ and $\overline{T}(D)=\{j\in J\mid T(j)\cap D\neq\emptyset\}$ (see [1, 37]).

A generalized fuzzy approximation space is the triple (J,Q,T) with a mapping $T:J\to\mathscr{F}(Q)$. T defines a fuzzy relation $R_T:J\times Q\to[0,1]$ by $R_T(j,q)=T(j)(q)$ for all $(j,q)\in J\times Q$ and T can defined by a fuzzy relation $R:J\times Q\to[0,1]$ as $T_R(j)(q)=R(j,q)$ for all $j\in J$ and $q\in Q$. Let ν be a fuzzy subset of Q, $\mathscr{T}$ be a t-norm and $\mathscr{I}$ be an implication on $[0,1]$. A pair of the lower and upper fuzzy rough approximations of ν is $(\underline{T}(\nu),\overline{T}(\nu))$ and it is referred as fuzzy rough set of ν with respect to (J,Q,T) where the approximations are fuzzy sets of J defined by

$\underline{T}(\nu)(l)=\bigwedge_{i\in Q}(T(l)(i)\mathscr{I}\nu(i))$,

$\overline{T}(\nu)(l)=\bigvee_{i\in Q}(T(l)(i)\mathscr{T}\nu(i))$

for all $l\in J$ (see [7, 20, 29–32]).

1.4. LA-Γ-Semigroups

Let Γ and S be nonempty sets. A mapping $S\times\Gamma\times S\to S$ which satisfies the identity $(j\gamma q)\alpha d=(d\gamma q)\alpha j$ for all $j,q,d\in S$ and $\gamma,\alpha\in\Gamma$ is called an LA-Γ-semigroup, where $j\gamma q$ is image of (j,γ,q). Let S be an LA-Γ-semigroup and $J,Q\subseteq S$. Then $J\Gamma Q:=\{j\gamma q\mid j\in J,q\in Q \text{ and } \gamma\in\Gamma\}$. By an LA-Γ-semigroups homomorphism, we mean a function $f:J\to Q$ which satisfies $f(s\gamma i)=f(s)\beta f(i)$ for all $s,i\in J$ and $\gamma,\beta\in\Gamma$. If $e\gamma j=j(j\gamma e=j)$ for all $j\in S$ and $\gamma\in\Gamma$, then an element e of S is called right (left) identity. The right (left) identity in an LA-Γ-semigroup is unique if it exists. An LA-Γ-semigroup is a commutative Γ-semigroup if it has a right identity. Let J be a nonempty subset of S. If $j\gamma q\in J$ for all $j,q\in J$ and $\gamma\in\Gamma$, then J is called an LA-Γ-subsemigroup of S. If $S\Gamma J\subseteq J$ ($J\Gamma S\subseteq J$), J is called a right (left) Γ-ideal of S. If J is left and right Γ-ideal of S, then it is referred as Γ-ideal of S. If $e\in S$, then a right Γ-ideal of S become a left Γ-ideal of S. If $(J\Gamma S)\Gamma J\subseteq J$, then J is called a generalized Γ-bi-ideal of S. If J is an LA-Γ-subsemigroup and $(J\Gamma S)\Gamma J\subseteq J$, then it is referred as Γ-bi-ideal of S. If $(S\Gamma J)\Gamma S\subseteq J$, then J is referred as Γ-interior ideal of S (See [14, 16, 24–27]).

1.5. Fuzzy LA-Γ-Subsemigroups

Let $\Omega\subseteq\Gamma$, ν,μ be fuzzy subsets of the LA-Γ-semigroup S and $d\in S$. The fuzzy subset $\nu\cdot_\Omega\mu$ is defined by

$$(\nu\cdot_\Omega\mu)(d)=\bigvee_{\substack{d=k\beta t\\ k,t\in S,\beta\in\Omega}}\nu(k)\wedge\mu(t)$$

and it is referred as Ω-product of ν and μ. If $\nu(j\gamma q)\geq\nu(j)\wedge\nu(q)$, then ν is called a fuzzy LA-Γ-subsemigroup of S. If $\nu(j\gamma q)\geq\nu(q)$ ($\nu(j\gamma q)\geq\nu(j)$), then ν is called a fuzzy right (left) Γ-ideal of S. It is referred as fuzzy Γ-two-sided ideal of S if ν is a fuzzy left and right ideal of S. If $\nu((j\gamma q)\alpha d)\geq\nu(j)\wedge\nu(d)$, then ν is called a fuzzy generalized Γ-bi-ideal of S. If $\nu((j\gamma q)\alpha d)\geq\nu(j)\wedge\nu(d)$, then a fuzzy LA-Γ-subsemigroup ν is referred a fuzzy Γ-bi-ideal of S. If $\nu((j\gamma d)\alpha q)\geq\nu(d)$, ν is referred as fuzzy Γ-interior ideal of S (See [13, 25, 26]).

2. Results

In this section we present the concepts of (fuzzy) set valued homomorphism and (fuzzy) relational morphism of LA-Γ-Semigroups and give a theorem which demonstrate the relation between of them. In terms of compliance with the literature, we construct the fuzzy rough approximations with fuzzy set valued homomorphism instead of fuzzy relational morphism since the set valued homomorphisms are more accepted in crisp literature on this topic. For this reason, it is not mentioned (fuzzy) relational morphism except the connection between (fuzzy) set valued homomorphisms. Throughout this section J,Q and D are LA-Γ-semigroups.

2.1. (Fuzzy) Relational Morphisms

The definitions of the fuzzy relational and the relational morphisms of LA-Γ-semigroups are given in the following [8, 9, 12].

Definition 1. *A relation $\varphi \subseteq J\times Q$ is referred as a relational morphism of LA-Γ-semigroups if it satisfies $(j_1,q_1),(j_2,q_2)\in\varphi$ imply $(j_1\gamma j_2, q_1\alpha q_2)\in\varphi$ for all $j_1,j_2\in J, q_1,q_2\in Q$ and $\gamma,\alpha\in\Gamma$.*

Definition 2. *A fuzzy relation $R: J\times Q\to[0,1]$ is referred as a fuzzy relational morphism of LA-Γ-semigroups if it satisfies $R(j_1,q_1)\wedge R(j_2,q_2)\le R(j_1\gamma j_2,q_1\alpha q_2)$ for all $j_1,j_2\in J, q_1,q_2\in Q$ and $\gamma,\alpha\in\Gamma$.*

2.2. (Fuzzy) Set Valued Homomorphisms

The definitions of the fuzzy set valued and the set valued homomorphism of LA-Γ-semigroups are given in the following [7].

Definition 3. *Let J,Q be LA-Γ-semigroups. A function $T: J\to\mathscr{P}(Q)$ is referred as a set valued homomorphism of LA-Γ-semigroups if it satisfies $T(j)\gamma T(q)\subseteq T(j\alpha q)$ for all $j,q\in J$ and $\alpha,\gamma\in\Gamma$ (see [2, 5, 11, 34–36]).*

Definition 4. *A mapping $T: J\to\mathscr{F}(Q)$ satisfying $T(j)\cdot_\Gamma T(q)\le T(j\gamma q)$ for all $j,q\in J$ and $\gamma\in\Gamma$, is called fuzzy set valued homomorphism of LA-Γ-semigroups. $Hom(J,\mathscr{F}(Q))$ is denote the set of all the fuzzy set valued homomorphisms from J to Q. A fuzzy set valued homomorphism $T: J\to\mathscr{F}(Q)$ is called strong if $\exists i\in Q$ such that $T(j)(i)=1$ for all $j\in J$. A fuzzy set valued homomorphism $T: J\to\mathscr{F}(Q)$ is called complete if $T(j)(x)\mathscr{T}T(q)(y)=T(j\gamma q)(x\alpha y)$ for all $j,q\in J, x,y\in Q$ and $\gamma,\alpha\in\Gamma$.*

Theorem 5. *There is a connection between (fuzzy) set valued homomorphisms and (fuzzy) relational morphisms;*

(i) The function $T: J\to\mathscr{P}(Q)$ is a set valued homomorphism if and only if the related relation $\varphi\subseteq J\times Q$ is a relational morphism of LA-Γ-semigroups.

(ii) The function $T: J\to\mathscr{F}(Q)$ is a fuzzy set valued homomorphism if and only if the related fuzzy relation $R: J\times Q\to[0,1]$ is a fuzzy relational morphism of LA-Γ-semigroups.

Proof.

(i) Suppose that $T: J\to\mathscr{P}(Q)$ be a set valued homomorphism. Let $(j_1,q_1),(j_2,q_2)\in\varphi_T$ for any $j_1,j_2\in J$ and $q_1,q_2\in Q$. Hence $q_1\in T(j_1)$ and $q_2\in T(j_2)$. Therefore $q_1\gamma q_2\in T(j_1)\gamma T(j_2)\subseteq T(j_1\beta j_2)$, and thus $(j_1\beta j_2,q_1\gamma q_2)\in\varphi_T$ for all $\beta,\gamma\in\Gamma$. So φ_T is a relational morphism. Conversely, suppose that $\varphi\subseteq J\times Q$ is a relational morphism. Let $z\in T_\varphi(j_1)\gamma T_\varphi(j_2)$ for any $j_1,j_2\in J$. Hence there exists $q_1\in T_\varphi(j_1)$ and $q_2\in T_\varphi(j_2)$ such that $z=q_1\gamma q_2$. Thus $(j_1,q_1),(j_2,q_2)\in\varphi$, and $(j_1\beta j_2,z)\in\varphi$. Therefore $T_\varphi(j_1)\gamma T_\varphi(j_2)\subseteq T_\varphi(j_1\beta j_2)$ since $z\in T_\varphi(j_1\beta j_2)$ for all $j_1,j_2\in J$.

(ii) Suppose that $T: J\to\mathscr{F}(Q)$ is a fuzzy set valued homomorphism. Let $(j_1,q_1),(j_2,q_2)\in J\times Q$. Then $R_T(j_1,q_1)\wedge R_T(j_2,q_2)=T(j_1)(q_1)\wedge T(j_2)(q_2)\le(T(j_1)\cdot_\Gamma T(j_2))(q_1\gamma q_2)\le T(j_1\beta j_2)(q_1\gamma q_2)=R_T(j_1\beta j_2,q_1\gamma q_2)$. Therefore $R_T: J\times Q\to[0,1]$ is a fuzzy relational morphism. Conversely, suppose that $R: J\times Q\to[0,1]$ is a fuzzy relational morphism. Let $j_1,j_2\in J$. Then $(T_R(j_1)\cdot_\Gamma T_R(j_2))(z)=\bigvee_{z=q_1\gamma q_2}(T_R(j_1)(q_1)\wedge T_R(j_2)(q_2))=\bigvee_{z=q_1\gamma q_2}(R(j_1,q_1)\wedge R(j_2,q_2))\le\bigvee_{z=q_1\gamma q_2}R(j_1\beta j_2,q_1\gamma q_2)=R(j_1\beta j_2,z)=T_R(j_1\beta j_2)(z)$. Therefore the function $T_R: J\to\mathscr{F}(Q)$ is a fuzzy set valued homomorphism.

□

Example 6. *Let $f: J\to Q$ be LA-Γ-semigroups homomorphism. For an $j\in J$, let $T(j): Q\to[0,1]$ be defined by*

$$T(j)(l)=\begin{cases} s, & \text{if } f(j)\neq l;\\ i, & \text{if } f(j)=l.\end{cases}$$

for all $s\le i, s,i\in[0,1]$ and $l\in Q$. Then $T\in Hom(J,\mathscr{F}(Q))$.

Lemma 7. *Let $T\in Hom(J,\mathscr{F}(Q))$ and $\varphi\in[0,1]$. Then $T(l)_\varphi\Gamma T(q)_\varphi\subseteq(T(l)\cdot_\Gamma T(q))_\varphi$ for all $l,q\in J$.*

Proof. Let $l,q\in J$ and $j\in T(l)_\varphi\Gamma T(q)_\varphi$. Thus there exist $s_1\in T(l)_\varphi, s_2\in T(q)_\varphi,\gamma\in\Gamma$ such that $x=s_1\gamma s_2$. Hence $T(l)(s_1)\ge\varphi, T(q)(s_2)\ge\varphi$. Since $T(l)(s_1)\wedge T(q)(s_2)\ge\varphi\wedge\varphi$, then $\bigvee_{j=s_1\gamma s_2}T(l)(s_1)\wedge T(q)(s_2)\ge\varphi$. So $\left(T(l)\cdot_\Gamma T(q)\right)(j)\ge\varphi$. Thus $j\in\left(T(l)\cdot_\Gamma T(q)\right)_\varphi$. We have $T(l)_\varphi\Gamma T(q)_\varphi\subseteq(T(l)\cdot_\Gamma T(q))_\varphi$. □

Theorem 8. *$T\in Hom(J,\mathscr{F}(Q))$ if and only if $K: J\to\mathscr{P}(Q)$ defined by $K(x)=T(x)_\varphi$ for all $x\in J$ is a set valued homomorphism for all $\varphi\in[0,1]$.*

Proof. Let $T\in\mathrm{Hom}(J,\mathscr{F}(Q))$ and $\varphi\in[0,1]$. $K(j)\Gamma K(q)=T(j)_\varphi\Gamma T(q)_\varphi\subseteq(T(j)\cdot_\Gamma T(q))_\varphi\subseteq T(j\gamma q)_\varphi=K(j\gamma q)$ for all $j,q\in J$ and $\gamma\in\Gamma$ by Lemma 7. Conversely, let K be a set valued homomorphism for all $\varphi\in[0,1]$. Let $\varphi=T(j)(c)\wedge T(q)(d)$ for any $j,q\in J$ and $c,d\in Q$. Then $\varphi\le T(j)(c)$ and $\varphi\le T(q)(d)$. Hence $c\in T(j)_\varphi$ and $d\in T(q)_\varphi$. Thus $c\beta d\in T(j)_\varphi\beta T(q)_\varphi=K(j)\beta K(q)\subseteq K(j)\Gamma K(q)\subseteq K(j\gamma q)=T(j\gamma q)_\varphi$ for all $\beta,\gamma\in\Gamma$.
We obtain that $T(j)(c)\wedge T(q)(d)\le T(j\gamma q)(c\beta d)$ for all $j,q\in J, c,d\in Q$ and $\beta,\gamma\in\Gamma$. If $(T(j)\cdot_\Gamma T(q))(l)\neq0$, then $(T(j)\cdot_\Gamma T(q))(l)=\bigvee_{l=c\beta d}(T(j)(c)\wedge T(q)(d))\le\bigvee_{l=c\beta d}T(j\gamma q)(c\beta d)=T(j\gamma q)(l)$. Hence $(T(j)\cdot_\Gamma T(q))\le T(j\gamma q)$ for all $j,q\in J$ and $\gamma\in\Gamma$. □

Theorem 9. *Let $T_1: J\to\mathscr{F}(Q)$ and $T_2: Q\to\mathscr{F}(D)$. If $T_1\in Hom(J,\mathscr{F}(Q))$ and $T_2\in Hom(Q,\mathscr{F}(D))$, then $T_1*T_2\in Hom(J,\mathscr{F}(D))$.*

Proof. Let $j,q \in J$ and take any element $d \in D$. Thus

$$\begin{aligned}
((T_1*T_2)(j)\cdot_\Gamma(T_1*T_2)(q))(d) &= \bigvee_{d=k\delta t}\Big((T_1*T_2)(j)(k)\wedge(T_1*T_2)(q)(t)\Big)\\
&= \bigvee_{d=k\delta t}\Big(\bigvee_{y_1\in Q}(T_1(j)(y_1)\wedge T_2(y_1)(q))\wedge\bigvee_{y_2\in Q}(T_1(q)(y_2)\wedge T_2(y_2)(t))\Big)\\
&= \bigvee_{\substack{d=k\delta t\\ y_1,y_2\in Q}}\Big(T_1(j)(y_1)\wedge T_2(y_1)(k)\wedge T_1(q)(y_2)\wedge T_2(y_2)(t)\Big)\\
&\le \bigvee_{y_1,y_2\in Q}\Big((T_1(j)\cdot_\Gamma T_1(q))(y_1\delta y_2)\wedge(T_2(y_1)\cdot_\Gamma T_2(y_2))(d)\Big)\\
&\le \bigvee_{y_1,y_2\in Q}\Big(T_1(j\gamma q)(y_1\delta y_2)\wedge T_2(y_1\delta y_2)(d)\Big)\\
&= (T_1*T_2)(j\gamma q)(d)
\end{aligned}$$

for all $\gamma\in\Gamma$. Then $T_1*T_2\in \mathrm{Hom}(J,\mathscr{F}(D))$. □

2.3. Generalized Fuzzy Rough Approximations Derived by Fuzzy Set Valued Homomorphisms

In this part of the paper, we focus on some properties of the generalized fuzzy lower and upper rough approximations derived by set valued homomorphisms. We investigate maintaining of some certain algebraic features of fuzzy subsets under the fuzzy lower and upper rough approximations.

Proposition 10. *Let a mapping $\mathscr{F}(Q)\times\Gamma\times\mathscr{F}(Q)\to\mathscr{F}(Q)$ be defined by $\mu.\gamma.\nu=\mu\cdot_{\{\gamma\}}\nu$. Then $\mathscr{F}(Q)$ is an LA-Γ-semigroup.*

Proof. Let $\mu,\nu,\eta\in\mathscr{F}(Q)$ and $l\in Q,\gamma,\beta\in\Gamma$.

$$\begin{aligned}
\Big((\nu\beta\eta)\gamma\mu\Big)(l) &= \bigvee_{l=j\gamma d}(\nu\beta\eta)(j)\wedge\mu(d)\\
&= \bigvee_{l=j\gamma d}\Big(\big(\bigvee_{j=k\beta t}(\nu(k)\wedge\eta(t))\big)\wedge\mu(d)\Big)\\
&= \bigvee_{l=(k\beta t)\gamma d}\Big(\nu(k)\wedge\eta(t)\wedge\mu(d)\Big)\\
&= \bigvee_{l=(d\beta t)\gamma k}\Big(\mu(d)\wedge\eta(t)\wedge\nu(k)\Big)\\
&= \bigvee_{l=j\gamma k}\Big(\big(\bigvee_{j=d\beta t}(\mu(d)\wedge\eta(t))\big)\wedge\nu(k)\Big)\\
&= \bigvee_{l=j\gamma k}\Big(((\mu\beta\eta)(j))\wedge\nu(k)\Big)\\
&= \Big((\mu\beta\eta)\gamma\nu\Big)(l).
\end{aligned}$$

Thus $\mathscr{F}(Q)$ is an LA-Γ-semigroup. (See the proof of Proposition 1 in [15] for LA-semigroups (AG-groupoids).) □

Theorem 11. *Let $T\in Hom(J,\mathscr{F}(Q))$ and $\mathscr{F}(Q)$ be the LA-Γ-semigroup which is given in Proposition 10. Then $\overline{T}\in Hom(\mathscr{F}(Q),\mathscr{F}(J))$.*

Proof. Let $\nu,\mu\in\mathscr{F}(Q)$ and $\beta\in\Gamma$. Take $l\in J$, then

$$\begin{aligned}
\Big(\overline{T}(\nu)\cdot_\Gamma\overline{T}(\mu)\Big)(l) &= \bigvee_{l=j\gamma d}\overline{T}(\nu)(j)\wedge\overline{T}(\mu)(d)\\
&= \bigvee_{l=j\gamma d}\Big(\bigvee_{k\in Q}(T(j)(k)\wedge\nu(k))\wedge\bigvee_{t\in Q}(T(d)(t)\wedge\mu(t))\Big)\\
&= \bigvee_{l=j\gamma d}\bigvee_{k,t\in Q}\Big((T(j)(k)\wedge T(d)(t))\wedge(\nu(k)\wedge\mu(t))\Big)\\
&= \bigvee_{l=j\gamma d}\Big(\bigvee_{k,t\in Q}\Big(T(j)(k)\wedge T(d)(t)\Big)\wedge\bigvee_{k,t\in Q}\Big(\nu(k)\wedge\mu(t)\Big)\Big)\\
&\le \bigvee_{l=j\gamma d}\bigvee_{k,t\in Q}\Big((T(j)\beta T(d))(k\beta t)\wedge(\nu\beta\mu)(k\beta t)\Big)\\
&\le \bigvee_{l=j\gamma d}\bigvee_{k,t\in Q}\Big((T(j)\cdot_\Gamma T(d))(k\beta t)\wedge(\nu\beta\mu)(k\beta t)\Big)\\
&\le \bigvee_{l=j\gamma d}\bigvee_{i\in Q}\Big((T(j)\cdot_\Gamma T(d))(i)\wedge(\nu\beta\mu)(i)\Big)\\
&\le \bigvee_{l=j\gamma d}\bigvee_{i\in Q}\Big(T(j\gamma d)(i)\wedge(\nu\beta\mu)(i)\Big)\\
&= \bigvee_{i\in Q}\Big(T(l)(i)\wedge(\nu\beta\mu)(i)\Big)\\
&= \overline{T}(\nu\beta\mu)(l).
\end{aligned}$$

Thus we have $\Big(\overline{T}(\nu)\cdot_\Gamma\overline{T}(\mu)\Big)(l)\le\overline{T}(\nu\beta\mu)(l)$. Therefore $\overline{T}\in \mathrm{Hom}(\mathscr{F}(Q),\mathscr{F}(J))$. □

Corollary 12. *Let $\mu,\nu\in\mathscr{F}(Q)$. If $T\in Hom(J,\mathscr{F}(Q))$, then $\overline{T}(\mu)\cdot_\Gamma\overline{T}(\nu)\le\overline{T}(\mu\cdot_\Gamma\nu)$.*

Proof. Let $\gamma\in\Gamma$. Since $\mu\gamma\nu\le\mu\cdot_\Gamma\nu$, then $\overline{T}(\mu\gamma\nu)\le\overline{T}(\mu\cdot_\Gamma\nu)$ (See Theorem 4.1 (FH7) in [29]). Thus we have $\overline{T}(\mu)\cdot_\Gamma\overline{T}(\nu)\le\overline{T}(\mu\cdot_\Gamma\nu)$ by Theorem 11. □

Definition 13. *[7] Let $T\in Hom(J,\mathscr{F}(Q))$. Then the image of T is a mapping defined by $ImT(q)=\bigvee_{i\in J}T(i)(q)$ for all $q\in Q$ and denoted by ImT.*

Proposition 14. *Let $T\in Hom(J,\mathscr{F}(Q))$. Then ImT is a fuzzy LA-Γ-subsemigroup of Q.*

Proof. Let $q,d\in Q$ and $\gamma\in\Gamma$.

$$\begin{aligned}
\mathrm{Im}T(q\gamma d) &= \bigvee_{j\in J}T(j)(q\gamma d)\\
&\ge \bigvee_{j=t\beta l}(T(t)\cdot_\Gamma T(l))(q\gamma d)\\
&\ge \bigvee_{t,l\in J}\Big(T(j)(q)\wedge T(l)(d)\Big)\\
&= \bigvee_{t\in J}T(t)(q)\wedge\bigvee_{l\in J}T(l)(d)\\
&= \mathrm{Im}T(q)\wedge\mathrm{Im}T(d).
\end{aligned}$$

Thus $\mathrm{Im}T$ is a fuzzy LA-Γ-subsemigroup of Q. □

Theorem 15. *Let $T\in Hom(J,\mathscr{F}(Q))$. If ν is a fuzzy LA-Γ-subsemigroup of Q, then $\overline{T}(\nu)$ is a fuzzy LA-Γ-subsemigroup of Q.*

Proof. Let $j,q\in J$ and $\beta\in\Gamma$.

$$\begin{aligned}
\overline{T}(\nu)(j)\wedge\overline{T}(\nu)(q) &= \bigvee_{i\in Q}(T(j)(i)\wedge\nu(i))\wedge\bigvee_{i\in Q}(T(q)(l)\wedge\nu(l))\\
&= \bigvee_{i,l\in Q}\Big(T(j)(i)\wedge T(q)(l)\Big)\wedge\bigvee_{i,l\in Q}\Big(\nu(i)\wedge\nu(l)\Big)\\
&\le \bigvee_{i,l\in Q}\Big(\big(T(j)\cdot_\Gamma T(q)\big)(i\gamma l)\wedge\nu(i\gamma l)\Big)\\
&\le \bigvee_{u\in Q}\Big(\big(T(j)\cdot_\Gamma T(q)\big)(u)\wedge\nu(u)\Big)\\
&\le \bigvee_{u\in Q}\Big(\big(T(j\beta q)\big)(u)\wedge\nu(u)\Big)\\
&= \overline{T}(\nu)(j\beta q).
\end{aligned}$$

Thus $\overline{T}(\nu)$ is a fuzzy LA-Γ-subsemigroup of Q. □

Theorem 16. *Let $T\in Hom(J,\mathscr{F}(Q))$ be strong. If ν is a fuzzy LA-Γ-right (resp. left or two-sided) ideal of Q, then $\overline{T}(\nu)$ is a fuzzy LA-Γ-right (resp. left or two-sided) ideal of J.*

Proof. Let $j,q\in J$ and $\beta\in\Gamma$.

$$\begin{aligned}
\overline{T}(\nu)(j) &= \overline{T}(\nu)(j)\wedge 1\\
&= \overline{T}(\nu)(j)\wedge T(q)(i),\quad \exists i\in Q\\
&= \Big(\bigvee_{l\in Q}T(j)(l)\wedge\nu(l)\Big)\wedge T(q)(i)\\
&\le \Big(\bigvee_{l\in Q}T(j)(l)\wedge\nu(l\gamma i)\Big)\wedge T(q)(i)\\
&= \bigvee_{l,i\in Q}\Big(T(j)(l)\wedge T(q)(i)\wedge\nu(l\gamma i)\Big)\\
&\le \bigvee_{l,i\in Q}\Big(\big(T(j)\cdot_\Gamma T(q)\big)(l\gamma i)\wedge\nu(l\gamma i)\Big)\\
&\le \bigvee_{d\in Q}\Big(T(j\gamma q)(d)\wedge\nu(d)\Big)\\
&= \overline{T}(\nu)(j\beta q).
\end{aligned}$$

Similarly, if ν is a fuzzy LA-Γ-left ideal of Q, then $\overline{T}(\nu)$ is a fuzzy LA-Γ-left ideal of J. □

Theorem 17. *Let $T \in Hom(J, \mathscr{F}(Q))$ be strong. If ν is a fuzzy generalized LA-Γ-bi-ideal of Q, then $\overline{T}(\nu)$ is a fuzzy generalized LA-Γ-bi-ideal of J.*

Proof. Let $j, q, l \in J$ and $\alpha, \gamma \in \Gamma$.

$$\begin{aligned}
\overline{T}(\nu)(j) \wedge \overline{T}(\nu)(l) &= \overline{T}(\nu)(j) \wedge \overline{T}(\nu)(l) \wedge 1 \\
&= \overline{T}(\nu)(j) \wedge \overline{T}(\nu)(l) \wedge T(q)(i), \quad \exists i \in Q \\
&= \Big(\bigvee_{k \in Q} (T(j)(k) \wedge \nu(k))\Big) \wedge \Big(\bigvee_{r \in Q} (T(l)(r) \wedge \nu(r))\Big) \wedge T(q)(i) \\
&= \bigvee_{k,r \in Q} \Big(T(j)(k) \wedge T(q)(i) \wedge T(l)(r) \wedge \nu(k) \wedge \nu(r)\Big) \\
&= \bigvee_{k,r,i \in Q} \Big(T(j)(k) \wedge T(q)(i) \wedge T(l)(r) \wedge \nu(k) \wedge \nu(r)\Big) \\
&\leq \bigvee_{k,r,i \in Q} \Big(T(j) \cdot_\Gamma T(q)(k\gamma i) \wedge T(l)(r) \wedge \nu(k) \wedge \nu(r)\Big) \\
&\leq \bigvee_{k,r,i \in Q} \Big(T(j\gamma q)(k\gamma i) \wedge T(l)(r) \wedge \nu(k) \wedge \nu(r)\Big) \\
&\leq \bigvee_{k,r,i \in Q} \Big((T(j\gamma q) \cdot_\Gamma T(l))((k\gamma i)\alpha r) \wedge \nu(k) \wedge \nu(r)\Big) \\
&\leq \bigvee_{k,r,i \in Q} \Big(T((j\gamma q)\gamma l)((k\gamma i)\alpha r) \wedge \nu(k) \wedge \nu(r)\Big) \\
&\leq \bigvee_{k,r,i \in Q} \Big(T((j\gamma q)\gamma l)((k\gamma i)\alpha r) \wedge \nu((k\gamma i)\alpha r)\Big) \\
&\leq \bigvee_{s \in Q} \Big(T((j\gamma q)\gamma l)(s) \wedge \nu(s)\Big) \\
&= \overline{T}(\nu)((j\gamma q)\gamma l).
\end{aligned}$$

Let $T \in \mathrm{Hom}(J, \mathscr{F}(Q))$ be strong. If ν is a fuzzy LA-Γ-bi-ideal of Q, then $\overline{T}(\nu)$ is a fuzzy LA-Γ-bi-ideal of J by Theorem 15. □

Theorem 18. *Let $T \in Hom(J, \mathscr{F}(Q))$ be strong. If ν is a fuzzy LA-Γ-interior ideal of Q, then $\overline{T}(\nu)$ is a fuzzy LA-Γ-interior ideal of J.*

Proof. Let $j, q, l \in J$ and $\alpha, \gamma \in \Gamma$.

$$\begin{aligned}
\overline{T}(\nu)(l) &= 1 \wedge \overline{T}(\nu)(l) \wedge 1 \\
&= T(j)(k) \wedge \overline{T}(\nu)(l) \wedge T(q)(i), \quad \exists k, i \in Q \\
&= T(j)(k) \wedge (\bigvee_{r \in Q} (T(l)(r) \wedge \nu(r)) \wedge T(q)(i) \\
&= \bigvee_{r \in Q} T(j)(k) \wedge T(l)(r) \wedge T(q)(i) \wedge \nu(r)\Big) \\
&= \bigvee_{k,r,i \in Q} \Big(T(j)(k) \wedge T(l)(r) \wedge T(q)(i) \wedge \nu(r)\Big) \\
&\leq \bigvee_{k,r,i \in Q} \Big(T(j) \cdot_\Gamma T(l)(k\gamma r) \wedge T(q)(i) \wedge \nu(r)\Big) \\
&\leq \bigvee_{k,r,i \in Q} \Big((T(j\gamma l) \cdot_\Gamma T(q))((k\gamma r)\alpha i) \wedge \nu(r)\Big) \\
&\leq \bigvee_{k,r,i \in Q} \Big(T(j\gamma l)(k\gamma r) \wedge T(q)(i) \wedge \nu(r)\Big) \\
&\leq \bigvee_{k,r,i \in Q} \Big(T((j\gamma l)\gamma q)((k\gamma r)\alpha i) \wedge \nu(r)\Big) \\
&\leq \bigvee_{k,r,i \in Q} \Big(T((j\gamma l)\gamma q)((k\gamma r)\alpha i) \wedge \nu((k\gamma r)\alpha i)\Big) \\
&\leq \bigvee_{s \in Q} \Big(T((j\gamma l)\gamma q)(s) \wedge \nu(s)\Big) \\
&= \overline{T}(\nu)((j\gamma q)\gamma l).
\end{aligned}$$

Thus $\overline{T}(\nu)$ is a fuzzy LA-Γ-interior ideal of J. □

Theorem 19. *Let $T \in Hom(J, \mathscr{F}(Q))$ be complete and let $\mathscr{I}$ be R-implication based on $\mathscr{T}_M$. If ν is a fuzzy LA-Γ-subsemigroup of Q, then $\underline{T}(\nu)$ is a fuzzy LA-Γ-subsemigroup of J.*

Proof. Let $j, q \in J$.

$$\begin{aligned}
\underline{T}(\nu)(j) \wedge \underline{T}(\nu)(q) &= \bigwedge_{l \in Q} (T(j)(l) \mathscr{I} \nu(l)) \wedge \bigwedge_{s \in Q} (T(q)(s) \mathscr{I} \nu(s)) \\
&= \bigwedge_{l \in Q} (\bigvee_{T(j)(l) \wedge x_1 \leq \nu(l)} x_1) \wedge \bigwedge_{s \in Q} (\bigvee_{T(q)(s) \wedge x_2 \leq \nu(s)} x_2) \\
&\leq \bigwedge_{l,k \in Q} (\bigvee_{T(j)(l) \wedge x_1 \leq \nu(l)} x_1 \wedge \bigvee_{T(q)(s) \wedge x_2 \leq \nu(s)} x_2) \\
&= \bigwedge_{l,k \in Q} (\bigvee_{\substack{T(j)(l) \wedge x_1 \leq \nu(l) \\ T(q)(s) \wedge x_2 \leq \nu(s)}} x_1 \wedge x_2) \\
&\leq \bigwedge_{l\alpha s \in Q} (\bigvee_{T(j\gamma q)(l\alpha s) \wedge x_1 \wedge x_2 \leq \nu(i\alpha s)} x_1 \wedge x_2) \\
&\leq \bigwedge_{u \in Q} (\bigvee_{T(j\gamma q)(u) \wedge x \leq \nu(u)} x) \\
&= \bigwedge_{u \in Q} (T(j\gamma q)(u) \mathscr{I} \nu(u)) \\
&= \underline{T}(\nu)(j\gamma q).
\end{aligned}$$

Thus $\underline{T}(\nu)$ is a fuzzy LA-Γ-subsemigroup of J. □

Theorem 20. *Let $T \in Hom(J, \mathscr{F}(Q))$ be complete and strong, and let $\mathscr{I}$ be R-implication based on $\mathscr{T}_M$. If ν is a fuzzy LA-Γ-right (resp. left or two-sided) ideal of Q, then $\underline{T}(\nu)$ is a fuzzy LA-Γ-right (resp. left or two-sided) ideal of J.*

Proof. Let $j, q \in J$.

$$\begin{aligned}
\underline{T}(\nu)(j) &= \bigwedge_{i \in Q} T(j)(i) \mathscr{I} \nu(i) \\
&= \bigwedge_{i \in Q} (\bigvee_{T(j)(i) \wedge x \leq \nu(i)} x) \\
&= \bigwedge_{i \in Q} (\bigvee_{T(j)(i) \wedge 1 \wedge x \leq \nu(i)} x) \\
&= \bigwedge_{i \in Q} (\bigvee_{T(j)(i) \wedge T(q)(s) \wedge x \leq \nu(i)} x), \quad \exists s \in Q \\
&= \bigwedge_{i \in Q} (\bigvee_{T(j\gamma q)(i\alpha s) \wedge x \leq \nu(i)} x) \\
&\leq \bigwedge_{i \in Q} (\bigvee_{T(j\gamma q)(i\alpha s) \wedge x \leq \nu(i\alpha s)} x) \\
&= \bigwedge_{l \in Q} (\bigvee_{T(j\gamma q)(l) \wedge x \leq \nu(l)} x) \\
&= \bigwedge_{l \in Q} T(j\gamma q)(l) \mathscr{I} \nu(l) \\
&= \underline{T}(\nu)(j\gamma q).
\end{aligned}$$

Similarly, $\underline{T}(\nu)$ is a fuzzy LA-Γ-left ideal of J if ν is a fuzzy LA-Γ-left ideal of Q. □

Theorem 21. *Let $T \in Hom(J, \mathscr{F}(Q))$ be complete and strong, and let $\mathscr{I}$ be R-implication based on $\mathscr{T}_M$. If ν is a fuzzy generalized LA-Γ-bi-ideal of Q, then $\underline{T}(\nu)$ is a fuzzy generalized LA-Γ-bi-ideal of J.*

Proof. Let $j, q, l \in J$.

$$\begin{aligned}
\underline{T}(\nu)(j) \wedge \underline{T}(\nu)(l) &= \bigwedge_{i \in Q} (T(j)(i) \mathscr{I} \nu(i)) \wedge \bigwedge_{f \in Q} (T(l)(f) \mathscr{I} \nu(f)) \\
&= \bigwedge_{i \in Q} (\bigvee_{T(j)(i) \wedge x_1 \leq \nu(i)} x_1) \wedge \bigwedge_{f \in Q} (\bigvee_{T(l)(f) \wedge x_2 \leq \nu(f)} x_2) \\
&= \bigwedge_{i \in Q} (\bigvee_{T(j)(i) \wedge 1 \wedge x_1 \leq \nu(i)} x_1) \wedge \bigwedge_{f \in Q} (\bigvee_{T(l)(f) \wedge x_2 \leq \nu(f)} x_2) \\
&= \bigwedge_{i \in Q} (\bigvee_{T(j)(i) \wedge T(q)(s) \wedge x_1 \leq \nu(i)} x_1) \wedge \bigwedge_{f \in Q} (\bigvee_{T(l)(f) \wedge x_2 \leq \nu(f)} x_2), \quad \exists s \in Q \\
&= \bigwedge_{i \in Q} (\bigvee_{T(j\gamma q)(i\beta s) \wedge x_1 \leq \nu(i)} x_1) \wedge \bigwedge_{f \in Q} (\bigvee_{T(l)(f) \wedge x_2 \leq \nu(f)} x_2) \\
&= \bigwedge_{i,f \in Q} (\bigvee_{T(j\gamma q)(i\beta s) \wedge x_1 \leq \nu(i)} x_1 \wedge \bigvee_{T(l)(f) \wedge x_2 \leq \nu(f)} x_2) \\
&= \bigwedge_{i,f \in Q} \bigvee_{\substack{T(j\gamma q)(i\beta s) \wedge x_1 \leq \nu(i) \\ T(l)(f) \wedge x_2 \leq \nu(f)}} (x_1 \wedge x_2) \\
&\leq \bigwedge_{i,f \in Q} \bigvee_{T((j\gamma q)\alpha l)((i\beta s)\delta f) \wedge (x_1 \wedge x_2) \leq \nu(i) \wedge \nu(f)} (x_1 \wedge x_2) \\
&\leq \bigwedge_{i,f \in Q} \bigvee_{T((j\gamma q)\alpha l)((i\beta s)\delta f) \wedge (x_1 \wedge x_2) \leq \nu((i\beta s)\delta f)} (x_1 \wedge x_2) \\
&= \bigwedge_{u \in Q} \bigvee_{T((j\gamma q)\alpha l)(u) \wedge (x_1 \wedge x_2) \leq \nu(u)} (x_1 \wedge x_2) \\
&= \bigwedge_{u \in Q} (T((j\gamma q)\alpha l)(u) \mathscr{I} \nu(u)) \\
&= \underline{T}(\nu)((j\gamma q)\alpha l).
\end{aligned}$$

Let $T \in \mathrm{Hom}(J, \mathscr{F}(Q))$ be complete and strong. If ν is a fuzzy LA-Γ-bi-ideal of Q, then $\underline{T}(\nu)$ is a fuzzy LA-Γ-bi-ideal of J by Theorem 19. ☐

Theorem 22. *Let $T \in Hom(J, \mathscr{F}(Q))$ be complete and strong, and let $\mathscr{I}$ be R-implication based on $\mathscr{T}_M$. If ν is a fuzzy LA-Γ-interior ideal of Q, then $\underline{T}(\nu)$ is a fuzzy LA-Γ-interior ideal of J.*

Proof. Let $j, q, l \in J$.

$$\begin{aligned}
\underline{T}(\nu)(q) &= \bigwedge_{p\in Q} (T(q)(p)\mathscr{I}\nu(p)) \\
&= \bigwedge_{p\in Q} \Big(\bigvee_{T(q)(p)\wedge x_1 \le \nu(p)} x_1 \Big) \\
&= \bigwedge_{p\in Q} \Big(\bigvee_{1\wedge T(q)(p)\wedge 1\wedge x_1 \le \nu(p)} x_1 \Big) \\
&= \bigwedge_{p\in Q} \Big(\bigvee_{(T(j)(s)\wedge T(q)(p))\wedge T(l)(k)\wedge x_1 \le \nu(p)} x_1 \Big), \quad \exists s,k \in Q \\
&= \bigwedge_{p\in Q} \Big(\bigvee_{T(j\gamma q)(s\beta p)\wedge T(l)(k)\wedge x_1 \le \nu(p)} x_1 \Big) \\
&= \bigwedge_{p\in Q} \Big(\bigvee_{(T(j\gamma q)\alpha l)((s\beta p)\delta k)\wedge x_1 \le \nu(p)} x_1 \Big) \\
&\le \bigwedge_{p\in Q} \Big(\bigvee_{(T(j\gamma q)\alpha l)((s\beta p)\delta k)\wedge x_1 \le \nu((s\beta p)\delta k)} x_1 \Big) \\
&= \bigwedge_{u\in Q} \Big(\bigvee_{T((j\gamma q)\alpha l)(u)\wedge x_1 \le \nu(u)} x_1 \Big) \\
&= \bigwedge_{u\in Q} (T((j\gamma q)\alpha l)(u)\mathscr{I}\nu(u)) \\
&= \underline{T}(\nu)((j\gamma q)\alpha l).
\end{aligned}$$

Thus $\underline{T}(\nu)$ is a fuzzy LA-Γ-interior ideal of J. ☐

Theorem 23. *Let $T \in Hom(J, \mathscr{F}(Q))$ be complete and let $\mathscr{I}$ be S-implication based on $\mathscr{S}_M$ and standard negator $\mathscr{N}_s$. If ν is a fuzzy LA-Γ-two-sided ideal of Q, then $\underline{T}(\nu)$ is a fuzzy LA-Γ-subsemigroup of J.*

Proof. Let $j, q \in J$.

$$\begin{aligned}
\underline{T}(\nu)(j) \wedge \underline{T}(\nu)(q) &= \bigwedge_{d\in Q} (T(j)(d)\mathscr{I}\nu(d)) \wedge \bigwedge_{k\in Q} (T(q)(k)\mathscr{I}\nu(k)) \\
&= \bigwedge_{d\in Q} ((1-T(j)(d)) \vee \nu(d)) \wedge \bigwedge_{k\in Q} ((1-T(q)(k)) \vee \nu(k)) \\
&= \bigwedge_{d,k\in Q} \Big(((1-T(j)(d)) \vee \nu(d)) \wedge ((1-T(q)(k)) \vee \nu(k)) \Big) \\
&= \bigwedge_{d,k\in Q} \Big(((1-T(j)(d)) \wedge (1-T(q)(k))) \vee (\nu(k) \vee \nu(d)) \Big) \\
&\le \bigwedge_{d,k\in Q} \Big((1-(T(j)(d) \wedge T(q)(k))) \vee (\nu(k) \vee \nu(d)) \Big) \\
&= \bigwedge_{d,k\in Q} \Big((1-(T(j\gamma q)(d\alpha k))) \vee (\nu(k) \vee \nu(d)) \Big) \\
&\le \bigwedge_{d,k\in Q} ((1-(T(j\gamma q)(d\alpha k))) \vee \nu(d\alpha k)) \\
&= \bigwedge_{p\in Q} \Big((1-(T(j\gamma q)(p))) \vee \nu(p) \Big) \\
&= (\bigwedge_{p\in Q} T(j\gamma q)(p)\mathscr{I}\nu(p)) \\
&= \underline{T}(\nu)(j\gamma q).
\end{aligned}$$

Thus $\underline{T}(\nu)$ is a fuzzy LA-Γ-subsemigroup of J. ☐

Theorem 24. *Let $T \in Hom(J, \mathscr{F}(Q))$ be complete and strong, and let $\mathscr{I}$ be S-implication based on $\mathscr{S}_M$ and standard negator $\mathscr{N}_s$. If ν is a fuzzy LA-Γ-right (resp. left or two-sided) ideal of Q, then $\underline{T}(\nu)$ is a fuzzy LA-Γ-right (resp. left or two-sided) ideal of J.*

Proof. Let $j, q \in J$.

$$\begin{aligned}
\underline{T}(\nu)(j) &= \bigwedge_{d\in Q} T(j)(d)\mathscr{I}\nu(d) \\
&= \bigwedge_{d\in Q} ((1-T(j)(d)) \vee \nu(d)) \\
&= \bigwedge_{d\in Q} ((1-(T(j)(d) \wedge 1)) \vee \nu(d)) \\
&= \bigwedge_{d\in Q} ((1-(T(j)(d) \wedge T(q)(k))) \vee \nu(d)), \quad \exists k \in Q \\
&= \bigwedge_{d\in Q} ((1-T(j\gamma q)(d\alpha k)) \vee \nu(d)) \\
&\le \bigwedge_{d\in Q} ((1-T(j\gamma q)(d\alpha k)) \vee \nu(d\alpha k)) \\
&= \bigwedge_{p\in Q} ((1-(T(j\gamma q)(p))) \vee \nu(p)) \\
&= (\bigwedge_{p\in Q} T(j\gamma q)(p)\mathscr{I}\nu(p)) \\
&= \underline{T}(\nu)(j\gamma q).
\end{aligned}$$

Similarly, if ν is a fuzzy LA-Γ-left (resp. two-sided) ideal of Q, then $\underline{T}(\nu)$ is a fuzzy LA-Γ-left (resp. two-sided) ideal of J. ☐

Theorem 25. *Let $T \in Hom(J, \mathscr{F}(Q))$ be complete and strong, and let $\mathscr{I}$ be S-implication based on $\mathscr{S}_M$ and standard negator $\mathscr{N}_s$. If ν is a fuzzy LA-Γ-two-sided ideal of Q, then $\underline{T}(\nu)$ is a fuzzy generalized LA-Γ-bi-ideal of J.*

Proof. Let $j, q, l \in J$.

$$\begin{aligned}
\underline{T}(\nu)(j) \wedge \underline{T}(\nu)(l) &= \bigwedge_{d\in Q} (T(j)(d)\mathscr{I}\nu(d)) \wedge (\bigwedge_{k\in Q} T(l)(k)\mathscr{I}\nu(k)) \\
&= \bigwedge_{d\in Q} ((1-T(j)(d)) \vee \nu(d)) \wedge \bigwedge_{k\in Q} ((1-T(l)(k)) \vee \nu(k)) \\
&= \bigwedge_{d,k\in Q} \Big(((1-T(j)(d)) \vee \nu(d)) \wedge ((1-T(l)(k)) \vee \nu(k)) \Big) \\
&\le \bigwedge_{d,k\in Q} \Big(((1-T(j)(d)) \wedge (1-T(l)(k))) \vee (\nu(k) \vee \nu(d)) \Big) \\
&\le \bigwedge_{d,k\in Q} ((1-(T(j)(d) \wedge T(l)(k))) \vee (\nu(k) \vee \nu(d))) \\
&= \bigwedge_{d,k\in Q} ((1-(T(j)(d) \wedge 1 \wedge T(l)(k))) \vee (\nu(k) \vee \nu(d))) \\
&= \bigwedge_{d,k\in Q} ((1-(T(j)(d) \wedge T(q)(s) \wedge T(l)(k))) \vee (\nu(k) \vee \nu(d))), \quad \exists s \in Q \\
&= \bigwedge_{d,k\in Q} ((1-T((j\gamma q)\alpha l)((d\beta s)\delta k)) \vee (\nu(k) \vee \nu(d))) \\
&\le \bigwedge_{d,k\in Q} ((1-T((j\gamma q)\alpha l)((d\beta s)\delta k)) \vee (\nu(k) \vee \nu(s) \vee \nu(d))) \\
&\le \bigwedge_{d,k\in Q} ((1-T((j\gamma q)\alpha l)((d\beta s)\delta k) \vee \nu((d\beta s)\delta k))) \\
&= \bigwedge_{i\in Q} ((1-T((j\gamma q)\alpha l)(i)) \vee \nu(i)) \\
&= \bigwedge_{i\in Q} (T((j\gamma q)\alpha l)(i)\mathscr{I}\nu(i)) \\
&= \underline{T}(\nu)((j\gamma q)\alpha l).
\end{aligned}$$

If ν is a fuzzy LA-Γ-two-sided ideal of Q, then $\underline{T}(\nu)$ is a fuzzy LA-Γ-bi-ideal of J by Theorem 23. ☐

Theorem 26. *Let $T \in Hom(J, \mathscr{F}(Q))$ be complete and strong, and let $\mathscr{I}$ be S-implication based on $\mathscr{S}_M$ and standart negator $\mathscr{N}_s$. If ν is a fuzzy LA-Γ-interior ideal of Q, then $\underline{T}(\nu)$ is a fuzzy LA-Γ-interior ideal of J.*

Proof. Let $j, q, l \in J$.

$$\begin{aligned}
\underline{T}(\nu)(q) &= \bigwedge_{d\in Q} T(q)(d)\mathscr{I}\nu(d) \\
&= \bigwedge_{d\in Q} ((1-T(q)(d)) \vee \nu(d)) \\
&= \bigwedge_{d\in Q} ((1-(1 \wedge T(q)(d) \wedge 1)) \vee \nu(d)) \\
&= \bigwedge_{d\in Q} ((1-(T(j)(k) \wedge T(q)(d) \wedge T(l)(s))) \vee \nu(d)), \quad \exists k \in Q \\
&= \bigwedge_{d\in Q} ((1-(T((j\gamma q)\alpha l)((k\beta d)\delta s) \wedge \nu(d)) \\
&\le \bigwedge_{d\in Q} ((1-(T((j\gamma q)\alpha l)((k\beta d)\delta s) \wedge \nu((k\beta d)\delta s)) \\
&= \bigwedge_{u\in Q} ((1-(T((j\gamma q)\alpha l)(u) \wedge \nu(u)) \\
&= \bigwedge_{u\in Q} (T((j\gamma q)\alpha l)(u)\mathscr{I}\nu(u)) \\
&= \underline{T}(\nu)(j\gamma q)\alpha l).
\end{aligned}$$

Thus $\underline{T}(\nu)$ is a fuzzy LA-Γ-interior ideal of J. □

3. Discussion and Conclusion

Beside to being an applicable theory, the generalized rough set theory is also studied with some algebraic structures using the notion of set valued homomorphism [5, 33, 35, 36]. Set valued homomorphisms are very useful tools to study on rough set theory with the algebraic structures. However, to study on the generalized fuzzy rough set theory one can be need some other tools such as "fuzzy set valued homomorphism" which was introduced by Ekiz et al. [7]. Very closer notion "fuzzy relational morphisms" was introduced by Ignjatović [12] (See [8]). The notion fuzzy set valued homomorphism is more preferable than the notion fuzzy relational morphisms in the sense of the fuzzification of the notion "set valued homomorphism". Recently, studies on the notion of fuzzy *LA*-Γ-semigroups is increased among the researchers who are working on fuzzy algebraic structures. Thus it is sensible to consider the *LA*-Γ-semigroups as the algebraic structure in the generalized fuzzy rough set theory.

Our further work will be on fuzzy rough approximation space which is constructed with hypersemigroups and fuzzy set valued homomorphism.

References

[1] Ali, M.I., Davvaz, B. and Shabir, M. 2013. Some properties of generalized rough sets, Information Sciences 224 (2013), 170-179.

[2] Ali, M.I., Shabir, M. and Tanveer, S. 2012. Roughness in hemirings, Neural Comput and Applic 21 (2012), 171-180.

[3] Baczynski, M. and Jayaram, B. 2008. Fuzzy implications Studies in Fuzziness and Soft Computing, Vol.231, Springer Berlin Heidelberg.

[4] Biswas, R. and Nanda, S.1994. Rough groups and rough subgroups, Bulletin of the Polish Academy of Sciences mathematics 42 (1994), 251-254.

[5] Davvaz, B. 2008. A short note on algebraic T-rough sets, Information Sciences 178 (2008), 3247-3252.

[6] Dubois, D. and Prade, H. 1990. Rough fuzzy sets and fuzzy rough sets, International Journal of General Systems 17, (1990), 191-209.

[7] Ekiz, C., Ali, M.I. and Yamak, S. 2017 TL-fuzzy set valued homomorphisms and generalized (I,T)-L-fuzzy rough sets on groups, Filomat 31:13 (2017), 4153-4166.

[8] Ekiz, C., Çelik, Y. and Yamak, S. 2013 Generalized TL-fuzzy rough rings via $\mathscr{T}L$-fuzzy relational morphisms, Journal of Inequalities and Applications 2013.1 (2013), 279.

[9] Ekiz, C., Çelik, Y. and Yamak, S. 2014 Generalized (I,T)-L-fuzzy rough sets based on TL-fuzzy relational morphisms on semigroups, Annals of Fuzzy Mathematics and Informatics 8, (2014), 571-592.

[10] Fodor, J.C. and Roubens, M. 1994 Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer Dordrecht.

[11] Hooshmandasl, M.R., Karimi, A., Almbardar, M. and Davvaz, B. 2013. Axiomatic systems for rough set-valued homomorphisms of associative rings, International Journal of Approximate Reasoning 54 (2), (2013), 297-306.

[12] Ignjatović, J., Ćirić, M. and Bogdanović, S. 2009. Fuzzy homomorphisms of algebras, Fuzzy Sets and Systems 160, (2009), 2345-2365.

[13] Jun, Y.B. and Lee, C.Y. 1981. Fuzzy Γ-rings, Pusan Kyongnam Math. J., 84, (1981), 264-269.

[14] Kazıım, M.A. and Naseeruddin, M. 1972. On almost semigroups, The Alig. Bull. Math. 2, (1972),1-7.

[15] Khan, M., and Khan, N. A. 2009. 2009arXiv0904.0077K

[16] Khan, M., Smarandache, F. and Anis, S. 2015. Theory of Abel Grassmann's Groupoids, Educational Publisher, Columbus.

[17] Kim, J.P. and Bae, D.R. 1997. Fuzzy congruences in groups, Fuzzy Sets and Systems 85, (1997), 115-120.

[18] Klement, E.P., Mesiar R. and Pap, E. 2000. Triangular Norms, Kluwer Academic Publishers, Dordrecht

[19] Kuroki, N. 1997. Rough ideals in semigroups, Information Sciences 100, (1997), 139-163.

[20] Li, F. and Yin, Y. 2012. The υ-lower and T-upper fuzzy rough approximation operators on a semigroup, Information Sciences 195, (2012), 241-255.

[21] Li, F., Yin, Y. and Lu, L. 2007. $(\mathscr{I},\mathscr{T})$-fuzzy rough approximation operators and $\mathscr{T}L$-fuzzy rough ideals on a ring, Information Sciences 177, (2007), 4711-4726.

[22] Pawlak, Z. 1982. Rough sets, Int. J. Comput. Information Sciences 11, (1982), 341-356.

[23] Radzikowska, A.M. and Kerre, E.E. 2002. A comparative study of rough sets, Fuzzy Sets and Systems 126, (2002), 137-155.

[24] Sen, M. K. 1981 On Γ-semigroups. ss 1-8. Sen, M. K., ed. 1981. Proceeding of International Symposium on Algebra and Its Applications, Decker Publication, New York, 30s.

[25] Shah, T. and Rehman, I. 2010. On Γ-ideals and Γ-bi-ideals in Γ-AG-groupoids, International Journal of Algebra 4 (2010), no.6,267-276.

[26] Shah, T., Rehman, I. and Khan, A. 2014. Fuzzy Γ-ideals in Γ-AG-groupoids, Hacettepe Journal of Mathematics and Statistics 43(4), (2014),625-634.

[27] Sen, M. K. and Saha, N. K. 1986. On Γ-semigroups I, Bull. Cal. Math. Soc. 78, (1986),180-186.

[28] Wang, Z., Yu, Y. and Dai, F. 2001. On $\mathscr{T}$-congruence L-relations on groups and rings, Fuzzy Sets and Systems 119, (2001), 393-407.

[29] Wu, W.-Z., Leung, Y. and Mi, J.-S. 2005. On characterizations of $(\mathscr{I}, \mathscr{T})$-fuzzy rough approximation operators, Fuzzy Sets and Systems 154, (2005), 76-102.

[30] Wu, W.-Z., Leung, Y. and Shao, M.-W. 2013. Generalized fuzzy rough approximation operators determined by fuzzy implicators, International Journal of Approximate Reasoning 54, (2013), 1388-1409.

[31] Wu, W.-Z., Mi, J.-S. and Zhang, W.-X. 2003. Generalized fuzzy rough sets, Information Sciences 151, (2003), 263-282.

[32] Wu, W.-Z. and Zhang, W.-X. 2004. Constructive and axiomatic approaches of fuzzy approximation operators, Information Sciences 159, (2004), 233-254.

[33] Xiao, Q. 2011 T-roughness in semigroups. ss 391-394. Xiao, Q., ed. 2011. International Conference on Computer Science and Automation Engineering, IEEE, 4s.

[34] Xiao, Q. and Li, Q. 2012. Generalized Lower and Upper Approximations in Quantales, Journal of Applied Mathematics 2012, (2012) Article ID 648983, 11 pages, doi:10.1155/2012/648983.

[35] Yamak, S., Kazancı, O. and Davvaz, B. 2011. Approximations in a module by using set-valued homomorphisms, International Journal of Computer Mathematics 88, (2011), 2901-2911.

[36] Yamak, S., Kazancı, O. and Davvaz, B. 2010. Generalized lower and upper approximations in a ring, Information Sciences 180, (2010), 1759-1768.

[37] Yao, Y.Y. 1998. Constructive and algebraic methods of the theory of rough sets, Information Sciences 109, (1998), 21-47.

[38] Zadeh, L.A. 1965. Fuzzy Sets, Inform. and Control 8, (1965), 338-353.