Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (4) (2019), 973 – 984
DOI : 10.15672/HJMS.2018.586

Research Article

Rings for which every cosingular module is projective

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Abstract

Let R be a ring and M be an R-module. In this paper we investigate modules M such that every (simple) cosingular R-module is M-projective. We prove that every simple cosingular module is M-projective if and only if for $N \leq T \leq M$, whenever T/N is simple cosingular, then N is a direct summand of T. We show that every simple cosingular right R-module is projective if and only if R is a right GV-ring. It is also shown that for a right perfect ring R, every cosingular right R-module is projective if and only if R is a right GV-ring. In addition, we prove that if every δ-cosingular right R-module is semisimple, then $\overline{Z}(M)$ is a direct summand of M for every right R-module M if and only if $\overline{Z}_\delta(M)$ is a direct summand of M for every right R-module M.

Mathematics Subject Classification (2010). 16D10, 16D40, 16D80

Keywords. projective module, cosingular module, δ-cosingular module, GV-ring

1. Introduction

All rings considered in this paper will be associative with an identity element and all modules will be unitary right modules unless otherwise stated. Let R be a ring and M an R-module. An R-module N is *generated by* M (or M*-generated*) if there exists an epimorphism $f: M^{(A)} \to N$ for some index set A. An R-module N is said to be *subgenerated by* M if N is isomorphic to a submodule of an M-generated module. We denote by $\sigma[M]$ the full subcategory of the right R-modules whose objects are all right R-modules subgenerated by M (see [15]). A submodule L of M is *essential in* M denoted by $L \leq_e M$, if for every nonzero submodule K of M, $L \cap K \neq 0$. As a dual concept, a submodule N of a module M is called *small* in M (denoted by $N \ll M$), if for every proper submodule L of M, $N + L \neq M$. As a generalization of small submodules, a submodule K of M is δ*-small* in M, in case $M = K + L$ with M/L singular implies that $M = L$. A module M is called *hollow* if every proper submodule of M is small in M.

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Received: 13.09.2017; Accepted: 05.02.2018

A module N is said to be (δ-)M*-small* if there exists a module $L \in \sigma[M]$ such that ($N \ll_\delta L$) $N \ll L$. It is well-known that N is (δ-)M-small if and only if ($N \ll_\delta \hat{N}$) $N \ll \hat{N}$, where $\hat{N}$ is injective envelope of N in $\sigma[M]$ (for the δ-case see [10]). Note that "(δ-)R-small" means "(δ-)small". Let N and L be submodules of M. N is called a *supplement of* L *in* M if it is minimal with respect to the property $M = N+L$, equivalently, $M = N + L$ and $N \cap L \ll N$. M is called *supplemented* (resp., *weakly supplemented*) if for each submodule A of M, there exists a submodule B of M such that $M = A + B$ and $A\cap B \ll B$ (resp., $A\cap B \ll M$). Any module M is called *amply supplemented* if for any two submodules A and B with $M = A + B$, A contains a supplement of B in M. Recall that M is called H*-supplemented* provided for every submodule N of M, there exists a direct summand D of M such that $\frac{N+D}{N} \ll \frac{M}{N}$ and $\frac{N+D}{D} \ll \frac{M}{D}$. Also M is called $\oplus$*-supplemented* in case for every $N \leq M$, there exists a direct summand K of M such that $M = N + K$ and $N \cap K \ll K$. Let us call an R-module N *small projective* if $\mathrm{Hom}(N, -)$ is exact with respect to the exact sequences $0 \to K \to L \to M \to 0$ in Mod-R with K small in L and for each R-module M (see [15, 19.10(8) and 23.9 Exercises]). Also N is *small* M*-projective* if $\mathrm{Hom}(N, -)$ is exact with respect to the exact sequences $0 \to K \to M \to L \to 0$ in Mod-R with K small in M.

The singular submodule $Z(M)$ of a module M is the set of $m \in M$ such that, $mI = 0$ for some essential right ideal I of R. Let M and N be two R-modules. In [13], Talebi and Vanaja defined $\overline{Z}_M(N)$ as a dual of singular submodule as follows: $\overline{Z}_M(N) = \bigcap\{Kerf \mid f: N \to U, U \in \mathcal{S}\}$ where $\mathcal{S}$ denotes the class of all M-small modules. They called N an M*-cosingular* (*non-*M*-cosingular*) module if $\overline{Z}_M(N) = 0$ ($\overline{Z}_M(N) = N$). Clearly every M-small module is M-cosingular. We should note that "cosingular and noncosingular" means "R-cosingular and non-R-cosingular". In [10], the author defined a new submodule of a module N as: $\overline{Z}_{\delta_M}(N) = \bigcap\{Kerg \mid g: N \to D, D \in \delta - \mathcal{S}\}$. Here δ-$\mathcal{S}$ shows the class of all δ-M small modules. Following [10], N is called δ*-*M*-cosingular* (*non-*δ*-*M*-cosingular*) provided that $\overline{Z}_{\delta_M}(N) = 0$ ($\overline{Z}_{\delta_M}(N) = N$). It is not hard to check that $\overline{Z}_{\delta_M}(N) \subseteq \overline{Z}_M(N)$. So, every M-cosingular R-module is δ-M-cosingular and every non-δ-M-cosingular R-module is non-M-cosingular. It is obvious that last statements hold for (non)cosingular and (non-)δ-cosingular modules.

$Rad(M)$, $Soc(M)$ and $E(M)$ denote the radical, the socle and the injective envelope of a module M, respectively, and $J(R)$ denotes the Jacobson radical of a ring R. Let M be a module. The notations $N \leq M$ and $N \leq_\oplus M$ will denote a submodule and a direct summand of M, respectively.

Keskin and Tribak in [6], introduced and studied modules M such that every M-cosingular module is projective in $\sigma[M]$. They called such modules $COSP$. They investigated some general properties of $COSP$-modules. They also characterized $COSP$-modules when every injective module in $\sigma[M]$ is amply supplemented. Finally they obtained that a $COSP$-module is Artinian if and only if every submodule has finite hollow dimension.

In a recent work [5], the authors defined and studied rings for which the cosingular submodule of every module is a direct summand. They called this property as (P). It is shown that a commutative perfect ring R has (P) if and only if R is semisimple.

Inspiring by [5] and [6], in this paper we study modules M such that every (simple) cosingular R-module is M-projective. We investigate rings for which every (simple) cosingular R-module is projective. We realize that these concepts are closely related to known rings, namely, *Generalized* V*-rings* (GV-rings for short).

In Section 2, we investigate modules M such that every (simple) cosingular R-module is M-projective. We investigate some of their properties. It is shown that the class of these modules is closed under submodules, factor modules and finite direct sums. It is proved that any locally injective module M such that every cosingular module is M-projective is noncosingular (Theorem 2.7). We also give an equivalent condition for a module M having the property that every simple cosingular module is M-projective (Theorem 2.9).

Sections 3 is devoted to study rings for which every (simple) cosingular module is projective. We show that for a ring R, every simple cosingular R-module is projective if and only if every simple δ-cosingular R-module is projective if and only if R is a GV-ring (Theorem 3.1). It is proved that for a ring R with all δ-cosingular R-modules semisimple, the following are equivalent:

(1) Every δ-cosingular R-module is projective;
(2) Every simple δ-cosingular R-module is projective;
(3) R is a right GV-ring;
(4) Every cosingular R-module is projective;
(5) For every R-module M, $\overline{Z}_\delta(M)$ is a direct summand of M;
(6) R has (P). (Theorem 3.19).

We also consider some assumptions for an Artinian serial ring with $J(R)^2 = 0$ having the property that every cosingular R-module is projective.

2. Modules M such that every cosingular module is M-projective

In this section we investigate modules M such that every (simple) cosingular module is M-projective. It is clear that any simple module has the stated property. Hence by the next proposition, every finitely generated semisimple module has the property, too.

Proposition 2.1. *The following hold.*

(1) *Let M be a module and $N \leq M$ such that every cosingular R-module is M-projective. Then every cosingular R-module is N-projective and M/N-projective.*
(2) *Let $M = \bigoplus_{i=1}^n M_i$ be a module. Then every cosingular R-module is M-projective if and only if every cosingular R-module is M_i-projective for each $i \in \{1, \ldots, n\}$.*

Proof. (1) is clear from [8, Proposition 4.31] and (2) holds by [8, Proposition 4.33]. □

Proposition 2.2. *Let M be a module such that every cosingular module is M-projective. Then the following hold.*

(1) *Every small submodule of M is semisimple.*
(2) *$Rad(M) \subseteq Soc(M)$.*
(3) *$Rad(M) \ll M$.*

Proof. (1) Let $N \ll M$ and L be an arbitrary submodule of N. To prove that N is semisimple, we observe that L is a direct summand of N. Since $N/L \ll M/L$, it is cosingular. Now, by assumption, N/L is M-projective and so N-projective by Proposition 2.1(1). It follows that L is a direct summand of N.
(2) It is known that $Rad(M)$ is the sum of all small submodules of M. By (1), each small submodule is semisimple. So $Rad(M)$ is a semisimple submodule of M, which must be contained in $Soc(M)$.
(3) Suppose that $Rad(M)$ is not small in M. So, there exists a proper submodule L of M such that $Rad(M) + L = M$. Now by (2), we have $Soc(M) + L = M$. Since $M/L \cong Soc(M)/(Soc(M) \cap L)$ is semisimple, M/L has at least one maximal submodule N/L. Therefore, N is a maximal submodule of M containing L. It follows that, $M = Rad(M) + L \subseteq N$, a contradiction. □

Corollary 2.3. *If every cosingular module is R-projective, then $J(R)$ is nilpotent with nilpotency index 2.*

Proof. It is known that $Soc(R_R)J(R) = 0$. By Proposition 2.2, $J(R) \subseteq Soc(R_R)$. This implies that $J(R)^2 = 0$. □

The following example introduces some modules M such that not every cosingular module is M-projective.

Example 2.4. By Proposition 2.2(3), every radical module M can not have the property that every cosingular module is M-projective. In particular, $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Z}_{p^\infty}$ as $\mathbb{Z}$-modules do not have the stated property.

The following is one of the useful results to characterize cosingular modules which are M-projective for a module M.

Lemma 2.5. *Let M be a module such that every cosingular module is M-projective. Then $\overline{Z}(M)$ is a direct summand of M. In this case $\overline{Z}(M)$ is the largest noncosingular submodule of M.*

Proof. Since $M/\overline{Z}(M)$ is a cosingular module, it is M-projective. This implies that M has a decomposition $M = \overline{Z}(M) \oplus L$ for some submodule L of M. Note that L is cosingular. □

Proposition 2.6. *Let M be a module such that every cosingular module is M-projective. If M is amply supplemented and cosingular, then the following hold.*

(1) *Every homomorphic image of M is cosingular.*
(2) *M is semisimple.*

Proof. (1) Let M be amply supplemented cosingular and $N \leq M$. Consider the natural epimorphism $\pi\colon M \longrightarrow M/N$. By [13, Theorem 3.5], $\pi(\overline{Z}^2(M)) = \overline{Z}^2(M/N)$. By Proposition 2.1(1), every cosingular module is M/N-projective. Now, by Lemma 2.5, $\overline{Z}^2(M/N) = \overline{Z}(M/N)$ and $\overline{Z}^2(M) = \overline{Z}(M) = 0$. So, $\overline{Z}(M/N) = 0$. It follows that M/N is cosingular.
(2) Let N be a submodule of M. Then M/N is cosingular by (1). Also, the hypothesis implies that M/N is M-projective. Hence N is a direct summand of M. Therefore M is semisimple. □

Recall from [4] that, a module M is *locally injective* if, for every submodule N of M, which is not essential in M, there exists a nonzero injective submodule K of M with $N \cap K = 0$. Every direct summand of a locally injective module is locally injective. Note that for a module M with every nonzero homomorphic image of M non-small, all homomorphisms from M to a small module is zero. In this case $\overline{Z}(M) = M$.

Theorem 2.7. *Let M be a module such that every cosingular module is M-projective. If M is locally injective, then M is noncosingular.*

Proof. It is enough to show that every nonzero homomorphic image of M is non-small. Let $X < M$ and $\frac{M}{X}$ be a small module. By assumption $\frac{M}{X}$ is M-projective. So X is a direct summand of M. Let $M = X \oplus X'$ where $X' \leq M$. It follows that X is non-essential. Since M is locally injective, there exists a nonzero injective direct summand Q of M such that $Q \cap X = 0$. Let $M = Q \oplus Q'$ for some $Q' \leq M$. Since $\frac{M}{X} = \frac{Q+X}{X} + \frac{Q'+X}{X}$ and $\frac{Q+X}{X} \cong \frac{Q}{Q\cap X} \cong \frac{Q}{0} \cong Q$, we get that $\frac{Q+X}{X}$ is a direct summand of M/X. On the other hand, $\frac{Q+X}{X}$ is small as a submodule of the small module M/X. Therefore $Q + X = X$, so $Q \subseteq X$. It implies that $Q = 0$. This is a contradiction. Thus for every $X < M$, the module M/X can not be small. It follows that M is noncosingular. □

In the sequel we give some conditions under which the converse statement of Proposition 2.7 holds.

Theorem 2.8. *Let M be a noncosingular weakly supplemented R-module such that $Rad(M)$ is semisimple. If the class of cosingular R-modules is closed under taking homomorphic images (e.g. R is right perfect with (P) (see [5, Lemma 3.1])), then every cosingular R-module is M-projective.*

Proof. Let L be a cosingular R-module. We show that L is small M-projective. Let N be a small submodule of M. Let $f\colon L \longrightarrow M/N$ be an R-homomorphism and $\pi\colon M \longrightarrow M/N$ be the natural epimorphism. Consider the following diagram

$$\begin{array}{ccccc} & & L & & \\ & & \big\downarrow{\scriptstyle f} & & \\ M & \xrightarrow{\ \pi\ } & \frac{M}{N} & \longrightarrow & 0. \end{array}$$

Suppose $Imf = K/N$ for some $K \leq M$. Since L is cosingular, by assumption K/N is cosingular. We show that $K/N \ll M/N$. Let $K/N + T/N = M/N$. Since $\frac{K/N}{K/N \cap T/N} \cong \frac{M/N}{T/N} \cong \frac{M}{T}$, M is noncosingular and K/N is cosingular, we have $T = M$. So $K/N \ll M/N$. Since $N \ll M$, we conclude that $K \ll M$. By assumption K is semisimple. Hence $K = N \oplus N'$ and so there exists a natural isomorphism $h\colon K/N \longrightarrow N'$. Consider the sequence $L \overset{f}{\to} K/N \overset{h}{\to} N' \overset{j}{\to} M$. Then $\pi o j o h o f = f$.

$$\begin{array}{ccccc} & {\scriptstyle johof} & L & & \\ & \swarrow & \big\downarrow{\scriptstyle f} & & \\ M & \xrightarrow{\ \pi\ } & \frac{M}{N} & \longrightarrow & 0. \end{array}$$

So the diagram commutes. It follows that L is small M-projective. Since M is weakly supplemented, L is M-projective by [2, 17.14]. The proof is completed. □

The following theorem gives an equivalent condition for a module M such that every simple cosingular module is M-projective.

Theorem 2.9. *Let M be a module. Then every simple cosingular module is M-projective if and only if for every simple cosingular submodule T/N of M/N, N is a direct summand of T.*

Proof. ($\Longrightarrow$) Clear.
($\Longleftarrow$) Let K be a simple cosingular module. We show that K is M-projective. Let N be a submodule of M. Let $g\colon K \longrightarrow M/N$ be an R-homomorphism and $\pi\colon M \longrightarrow M/N$ be the natural epimorphism. Consider the following diagram.

$$\begin{array}{ccccc} & & K & & \\ & & \big\downarrow{\scriptstyle g} & & \\ M & \xrightarrow{\ \pi\ } & \frac{M}{N} & \longrightarrow & 0. \end{array}$$

Suppose $Img = T/N$ for some $T \leq M$. Since K is simple cosingular, by assumption $N \leq_{\oplus} T$. Set $T = N \oplus L$ for some $L \leq T$. Consider the sequence $K \overset{g}{\to} T/N \overset{h}{\to} L \overset{j}{\to} M$, where h is the isomorphism between T/N and L induced by the decomposition of T. Let $\overline{h} = j o h o g$. It is easy to see that $\pi o \overline{h} = g$. Now, we have the following diagram.

$$\begin{array}{ccccc} & {\scriptstyle \overline{h}} & K & & \\ & \swarrow & \big\downarrow{\scriptstyle g} & & \\ M & \xrightarrow{\ \pi\ } & \frac{M}{N} & \longrightarrow & 0. \end{array}$$

So the diagram commutes. It follows that K is M-projective. □

Corollary 2.10. *Let M be a module. If for every submodule T of M, $Soc(T) + \overline{Z}(T) = T$, then every simple cosingular module is M-projective.*

Proof. Let $N \leq T \leq M$ with T/N simple cosingular. It follows that $\overline{Z}(T) \subseteq N$. So, by assumption, $Soc(T) + N = T$. Now, $\frac{T}{N} \cong \frac{Soc(T)}{Soc(N)}$. Hence, $Soc(N) \oplus L = Soc(T)$ for some simple submodule L of T. It follows that $L + N = T$. Consider the submodule $L \cap N$ of L. Since L is simple, $L \cap N = 0$ or $L \cap N = L$. If $N \cap L = L$, then $L \subseteq N$. It follows that $N = T$, a contradiction. So $L \oplus N = T$. Therefore, by Theorem 2.9, the result follows. □

Corollary 2.11. *Let R be a ring with every homomorphic image of R cosingular. Then the following are equivalent.*

(1) *Every simple module is projective;*
(2) *Every simple module is R-projective;*
(3) *R is semisimple.*

Proof. (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (1) are obvious.
(2) $\Longrightarrow$ (3) Let I be a maximal right ideal of R. Then R/I is simple. By hypothesis, R/I is cosingular. Note that if N is a simple module, then it is also cosingular. By Theorem 2.9, I is a direct summand of R. Thus R is semisimple. □

3. Rings for which every (simple) cosingular module is projective

Recall from [7] that a ring R is a *right V-ring* provided every simple R-module is injective, equivalently R is a right V-ring if and only if for every R-module M, $Rad(M) = 0$ (see [7, Theorem 2.1]). Since the only cosingular module over a right V-ring is zero, every cosingular module over a right V-ring is projective. Also R is a *right GV-ring* if every simple R-module is either projective or injective. It is known that R is a right GV-ring if and only if every simple singular R-module is injective. For more information about V-rings and GV-rings we refer the readers to [7] and [11].

In this section we study rings R for which every (simple) cosingular R-module is projective. We prove that R is a right GV-ring if and only if every simple cosingular R-module is projective. We also show that over a right perfect ring R, every cosingular R-module is projective if and only if R is right GV if and only if every simple δ-cosingular R-module is projective.

We start this section by investigating rings over which every simple cosingular module is projective.

Theorem 3.1. *Let R be a ring. Then the following statements are equivalent.*

(1) *Every simple δ-cosingular R-module is projective;*
(2) *Every simple cosingular R-module is projective;*
(3) *R is a right GV-ring.*

Proof. (1) $\Longrightarrow$ (2) It is obvious since every cosingular R-module is δ-cosingular.
(2) $\Longrightarrow$ (3) Let M be a simple singular R-module. Then M is either small or injective. If M is small, then M is projective by assumption (2). This yields that $M = 0$, a contradiction. So M must be injective. It follows that R is right GV.
(3) $\Longrightarrow$ (1) Let M be a simple δ-cosingular R-module. Then M is either singular or projective. If M is singular, then by assumption (3) and [10, Theorem 4.1], M is non-δ-cosingular. Hence $M = 0$. Now, M must be projective. □

Corollary 3.2. *If R is a semisimple ring, then it is right GV. The converse holds if every simple module is δ-cosingular.*

Proof. The first assertion is obvious. Let R be a right GV-ring. Assume that every simple module is δ-cosingular. Let I be a maximal right ideal of R. Then R/I is simple, and so it is δ-cosingular. By Theorem 3.1, R/I is projective. Hence I is a direct summand of R. Thus R is semisimple. □

The following result is an immediate consequence of [9, Corollaries 1.10 and 2.9], [10, Theorem 4.1] and Theorem 3.1.

Corollary 3.3. *The following statements are equivalent for a ring R.*

(1) *R is a right GV-ring;*
(2) *Every (δ-)small R-module is projective;*
(3) *Every singular R-module is non(-δ-)cosingular;*
(4) *Every simple (δ-)cosingular R-module is projective.*

We next show that if every cosingular R-module is projective, then for a cosingular module M, being lifting, discrete, H-supplemented, $\oplus$-supplemented, amply supplemented and supplemented are all equivalent.

Proposition 3.4. *Let R be a ring such that every cosingular R-module is projective. Then the following statements are equivalent.*

(1) *Every cosingular R-module is discrete;*
(2) *Every cosingular R-module is lifting;*
(3) *Every cosingular R-module is H-supplemented;*
(4) *Every cosingular R-module is $\oplus$-supplemented;*
(5) *Every cosingular R-module is amply supplemented;*
(6) *Every cosingular R-module is supplemented.*

Proof. The result follows from the fact that for a projective module M, M is lifting if and only if M is H-supplemented if and only if M is $\oplus$-supplemented if and only if M is amply supplemented if and only if M is supplemented (see [8, Proposition 4.39]). □

In [13, Theorem 3.5 and Corollary 3.9], it is shown that if every M-cosingular module in $\sigma[M]$ is projective in $\sigma[M]$ and every injective module in $\sigma[M]$ is amply supplemented, then the class of M-cosingular modules is closed under homomorphic images.

Proposition 3.5. *Let R be a right GV-ring such that every cosingular R-module is amply supplemented. Then the class of cosingular R-modules is closed under homomorphic images. In particular over a right perfect right GV-ring, every homomorphic image of a cosingular module is cosingular.*

Proof. Let $0 \neq M$ be a cosingular R-module, $0 \neq x \in M$ and K be a maximal submodule of xR. Then xR/K is simple. If xR/K is singular, then it is noncosingular by Corollary 3.3(3). Consider the natural epimorphism $\pi\colon xR \to xR/K$. By assumption, xR is amply supplemented. Then, by [13, Theorem 3.5], $0 = \pi(\overline{Z}^2(xR)) = \overline{Z}^2(xR/K) = \overline{Z}(xR/K) = xR/K$, a contradiction. Hence the simple module xR/K must be projective. Thus $K \leq_\oplus xR$, and so xR is semisimple. Therefore M is semisimple. It follows that every homomorphic image of M is isomorphic to a submodule of M. This completes the proof. □

Let R be a ring. It is known by Proposition 3.8 that every cosingular R-module is projective if and only if every cosingular R-module is projective relative to every injective R-module. If a ring R has a radical module, then R can not have the property that every cosingular module is projective. Since $\mathbb{Q}$ is radical as a $\mathbb{Z}$-module, $\mathbb{Z}$ can not have the property (since $\mathbb{Z}$ is not a field (see Proposition 3.8)). It is known by Theorem 3.1 that if every cosingular R-module is projective, then R is a right GV-ring.

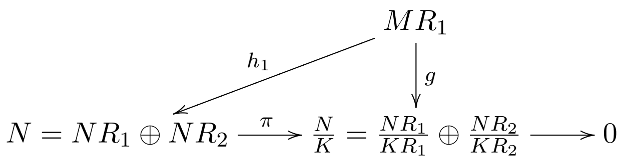
Proposition 3.6. *Let $f\colon R \longrightarrow S$ be a ring epimorphism. If every cosingular R-module is projective, then every cosingular S-module is projective.*

Proof. Let M be a cosingular S-module. Since $\overline{Z}_R(M) \subseteq \overline{Z}_S(M)$, then M is a cosingular R-module. So by assumption M is a projective R-module. It is not hard to check that M is a projective S-module, as required. □

The following is an analogue of [5, Proposition 2.8], for the rings for which every cosingular module is projective.

Proposition 3.7. *Let $R = R_1 \oplus R_2$ be a ring decomposition. Then every cosingular R-module is projective if and only if every cosingular R_i-module M_i is projective for $i = 1, 2$.*

Proof. The necessity follows from Proposition 3.6. For the sufficiency, let R_1 and R_2 have the stated property and M be a cosingular R-module. By [5, Lemma 2.7(1)], $M = MR_1 \oplus MR_2$, where MR_i can be regarded as an R_i-module for $i = 1, 2$. We also have by [5, Lemma 2.7(3b)], $\overline{Z}_{R_i}(MR_i) = \overline{Z}_R(MR_i)$ for $i = 1, 2$. It follows that MR_1 is a cosingular R_1-module and MR_2 is a cosingular R_2-module. By assumption, MR_i is a projective R_i-module for $i = 1, 2$. Note that MR_i is also an R-module with the multiplication $m_i(r_1 + r_2) = m_i r_i$, where $r_j \in R_j$ $(j = 1, 2)$ and $m_i \in MR_i$ $(i = 1, 2)$. Now, we prove that MR_i is a projective R-module for $i = 1, 2$. Consider the following diagram of R-modules where $K \leq N$ and π is the canonical R-epimorphism from N onto N/K and g is any R-homomorphism.

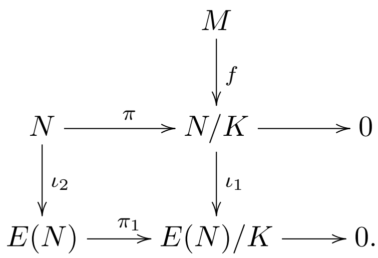


The R-module N is an R_1-module by $nr_1 = n_1 r_1$ for $n = n_1 + n_2 \in N$ and $r = r_1 + r_2 \in R = R_1 \oplus R_2$. Then $\pi(n) = \pi(n_1) + \pi(n_2) = \pi_1(n_1) + \pi_2(n_2)$, it follows that $\pi = \pi_1 \oplus \pi_2$ with π_1 is an epimorphism from NR_1 onto NR_1/KR_1 and π_2 is an epimorphism from NR_2 onto NR_2/KR_2. Since g is also an R_1-homomorphism, we have $g(MR_1) \subseteq NR_1/KR_1$. By hypothesis, there exists an R_1-homomorphism $h_1 \colon MR_1 \to NR_1$ such that $g = \pi_1 h_1 = \pi h_1$. Hence MR_1 is a projective R-module. A similar proof reveals also that MR_2 is a projective R-module. Therefore $M = MR_1 \oplus MR_2$ is a projective R-module. □

Proposition 3.8. *Let R be a Dedekind domain. Then the following are equivalent.*

(1) *Every cosingular R-module is projective;*
(2) *R is a field;*
(3) *Every cosingular R-module is projective relative to every injective R-module.*

Proof. (1) $\Longleftrightarrow$ (2) Similar to the proof of [5, Proposition 2.6].
(1) $\Longrightarrow$ (3) Obvious.
(3) $\Longrightarrow$ (1) Let M be a cosingular module. Consider the following diagram for a module N and $K \leq N$:



Since M is projective relative to $E(N)$, there exists a homomorphism $g \colon M \to E(N)$ such that $\pi_1 g = \iota_1 f$. For any $m \in M$, we have $g(m) + K = \pi_1 g(m) = \iota_1 f(m) = f(m) \in N/K$. This implies that $g(m) \in N$. Hence $g(M) \subseteq N$. Therefore M is N-projective. □

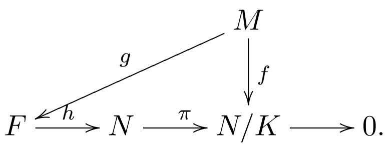
Proposition 3.9. *Let R be a ring and consider the following conditions.*

(1) *Every cosingular R-module is projective relative to every free R-module;*
(2) *Every cosingular R-module is projective relative to every projective R-module;*
(3) *Every cosingular R-module is projective relative to every flat R-module;*

(4) *Every cosingular R-module is R-projective.*

Then (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (4). *Also, all of them are equivalent for finitely generated modules.*

Proof. (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) $\Longrightarrow$ (4) Obvious.
(1) $\Longrightarrow$ (3) Let M be a cosingular module and N a flat module. Then N is a homomorphic image of a free module F, say $h\colon F \to N$ is an epimorphism. For any submodule K of N, consider the following diagram:



By (1), M is F-projective, and so there exists a homomorphism $g\colon M \to F$ such that $\pi hg = f$. Thus M is N-projective due to the homomorphism $hg\colon M \to N$.
(4) $\Longrightarrow$ (1) Let M be a finitely generated cosingular module and F be a free module. We may assume that $F = \bigoplus_{i\in I} R_i$ where $R_i = R$ for all $i \in I$. Since M is R-projective, by [8, Proposition 4.35], M is also F-projective. □

The following is a consequence of [5, Theorem 3.3].

Proposition 3.10. *Let R be a commutative perfect ring. Then the following are equivalent.*

(1) *Every cosingular R-module is projective;*
(2) *R is GV;*
(3) *R is semisimple.*

As a consequence, every cosingular $\mathbb{Z}_n$-module is projective if and only if $\mathbb{Z}_n$ is *GV* if and only if n is square-free.

The following theorem, which presents an equivalent condition for a ring R such that every cosingular R-module is projective, is taken from [13, Corollary 3.9]. We bring it here for the sake of completeness (Note that corresponded results in [13] are in $\sigma[M]$ and we bring it here in the category of right R-modules).

Theorem 3.11. *Let R be a ring. If every R-module is a direct sum of a noncosingular module and a semisimple module, then every cosingular R-module is projective. The converse holds, if every cosingular R-module is amply supplemented.*

Proof. ($\Longrightarrow$) Let M be a cosingular R-module. By hypothesis $M = U \oplus V$ where U is noncosingular and V is semisimple. Being M cosingular implies $U = 0$. So M is semisimple. Let $f\colon N \to M$ be an epimorphism where N is a projective R-module. Then, by hypothesis $N = K \oplus T$ where K is noncosingular and T is semisimple. Then, $f(K) = f(\overline{Z}(K)) = f(\overline{Z}(N)) \subseteq \overline{Z}(M) = 0$. It follows that $K \subseteq Kerf$. Hence, $Kerf = K \oplus (T \cap Kerf)$. Since T is semisimple, $T = S \oplus (T \cap Kerf)$ for a submodule S of T. Therefore, $N = K \oplus (T \cap Kerf) \oplus S = Kerf \oplus S$. So $Kerf \leq_\oplus N$. Hence, M is projective.
($\Longleftarrow$) Let M be an R-module. Since $M/\overline{Z}(M)$ is cosingular, by hypothesis $M/\overline{Z}(M)$ is projective. Then $M = \overline{Z}(M) \oplus L$, where L is cosingular and $\overline{Z}(M)$ is noncosingular. We show that L is semisimple. To prove this, we show that every submodule H of L, is a direct summand of L. Consider natural epimorphism $\pi\colon L \to L/H$. Since L is amply supplemented, by [13, Theorem 3.5], $\pi(\overline{Z}^2(L)) = \overline{Z}^2(L/H)$. Hence $\overline{Z}^2(L/H) = 0$ (because L is cosingular). By [13, Proposition 2.1(3)], $(L/H)/(\overline{Z}(L/H))$ is cosingular. Now by assumption and [8, Lemma 4.30], $\overline{Z}(L/H)$ is a direct summand of L/H. This yields that $\overline{Z}(L/H) = \overline{Z}^2(L/H) = 0$. It follows that L/H is cosingular. Therefore, $H \leq_\oplus L$ by the fact that every cosingular R-module is projective and [8, Lemma 4.30]. Hence L is semisimple. □

Recall that a ring R is *semilocal* in case $R/J(R)$ is semisimple. Now let R be a semilocal ring such that $J(R) \subseteq Soc({}_RR)$. By [14, Corollary 2.7(1)], $Soc({}_RR) = \overline{Z}(R_R)$. Then the ring $\frac{R}{Soc({}_RR)} = \frac{R}{\overline{Z}(R_R)}$ is semisimple. If M is a cosingular R-module, it is not hard to check that M is a cosingular $\frac{R}{\overline{Z}(R_R)}$-module. So M is semisimple as both an R and $\frac{R}{\overline{Z}(R_R)}$-module.

Corollary 3.12. *Let R be a ring such that every cosingular R-module is semisimple (for example, a semilocal ring R with $J(R) \subseteq Soc({}_RR)$). Then R has (P) if and only if every cosingular R-module is projective.*

Proof. $(\Longrightarrow)$ Let M be an R-module. Then $M = \overline{Z}(M) \oplus K$ for a submodule K of M by the property (P). It is clear that $\overline{Z}(M)$ is noncosingular and K is cosingular and hence semisimple by assumption. Therefore, Theorem 3.11 yields us the result.
$(\Longleftarrow)$ Let M be an R-module. Since $M/\overline{Z}(M)$ is cosingular, by hypothesis, $M/\overline{Z}(M)$ is projective. Hence $\overline{Z}(M)$ is a direct summand of M. □

Lemma 3.13. *If R is a right GV-ring, then every injective R-module is noncosingular.*

Proof. Let E be an injective R-module and $f\colon E \to U$ be an R-module homomorphism where U is a small R-module. Then $E/Kerf$ is a small R-module and hence by Corollary 3.3, $E/Kerf$ is projective. It follows that $E = Kerf \oplus L$ where L is small injective. Clearly L must be zero. So E is noncosingular. □

Proposition 3.14. *Let R be an Artinian serial ring with $J(R)^2 = 0$. If every injective R-module is noncosingular, then every cosingular R-module is projective.*

Proof. By assumption, every R-module is a direct sum of an injective module and a semisimple module. Since every injective R-module is noncosingular, the result follows from Theorem 3.11. □

The following example shows that if R is a ring such that every cosingular R-module is projective, then R need not be a V-ring.

Example 3.15. Let F be a field and $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ the ring of 2×2 upper triangular matrices over F. By [3, Example 13.6], every singular (left and right) R-module is injective. Hence R is a left and right GV-ring. Since $J(R) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$, R can not be a (left and right) V-ring. Also R is (left and right) hereditary Artinian serial from [3, Example 13.6]. It is easy to check that $J(R)^2 = 0$. Therefore, every cosingular R-module is projective by Proposition 3.14.

Proposition 3.16. *Let R be a right perfect ring such that every noncosingular R-module is injective. If every cosingular R-module is projective, then R is an Artinian serial ring with $J(R)^2 = 0$.*

Proof. Let M be an R-module. By hypothesis $M/\overline{Z}(M)$ is projective. There exists a submodule C of M such that $M = \overline{Z}(M) \oplus C$, where $\overline{Z}(M)$ is noncosingular and C is cosingular. Since every cosingular R-module is projective and R is right perfect, every cosingular R-module is semisimple (see Proposition 3.5). It follows that M is a direct sum of an injective and a semisimple module. Hence, by [3, 13.5], R is an Artinian serial ring with $J(R)^2 = 0$. □

Abyzov [1] defined a module to be *weakly regular* if, whenever N is a submodule of M which is not contained in $Rad(M)$, then N contains a nonzero direct summand of M.

Corollary 3.17. *Let R be a ring such that an R-module M is injective if and only if it is noncosingular. If R is right perfect, then the following statements are equivalent.*

(1) *Every cosingular R-module is projective;*
(2) *Every R-module is weakly regular;*
(3) *R is an Artinian serial ring with $J(R)^2 = 0$.*

Proof. It follows from Propositions 3.14, 3.16 and [1, Theorem 4]. □

Theorem 3.18. *Let R be a right perfect ring or a ring such that every δ-cosingular R-module is semisimple. Then the following statements are equivalent.*

(1) *Every δ-cosingular R-module is projective;*
(2) *Every simple δ-cosingular R-module is projective;*
(3) *R is a right GV-ring;*
(4) *Every cosingular R-module is projective.*

Proof. We prove the theorem in perfect case. The latter case is similar.
(1) $\Longrightarrow$ (2) It is obvious.
(2) $\Longleftrightarrow$ (3) Follows from Theorem 3.1.
(3) $\Longrightarrow$ (4) Let M be a cosingular R-module. Since R is a right GV-ring, it follows from Proposition 3.5 that M is semisimple. Set $M = \bigoplus_{i\in I} M_i$ where each M_i is simple. Since R is right GV, each M_i is projective (because each of them is simple cosingular). Therefore, M is projective.
(4) $\Longrightarrow$ (3) By Theorem 3.1.
(3) $\Longrightarrow$ (1) Let M be a δ-cosingular R-module. By a similar argument to Proposition 3.5, it can be shown that M is semisimple. We set $M = \bigoplus_{i\in I} M_i$ a direct sum of simple δ-cosingular R-modules. By (3), every M_i where $i \in I$ is projective. Now the result follows. □

Theorem 3.19. *Let R be a ring such that every δ-cosingular R-module is semisimple. Then the following assertions are equivalent.*

(1) *Every δ-cosingular R-module is projective;*
(2) *Every simple δ-cosingular R-module is projective;*
(3) *R is a right GV-ring;*
(4) *Every cosingular R-module is projective;*
(5) *For every R-module M, $\overline{Z}_\delta(M)$ is a direct summand of M;*
(6) *R has (P).*

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) Follows from Theorem 3.18.
(1) $\Longrightarrow$ (5) Let M be an R-module. By [10, Proposition 2.5], $M/\overline{Z}_\delta(M)$ is δ-cosingular. Now by (1), $M/\overline{Z}_\delta(M)$ is projective. It follows that $\overline{Z}_\delta(M)$ is a direct summand of M.
(5) $\Longrightarrow$ (1) Let M be a δ-cosingular R-module. By assumption, there exists a decomposition $M = \bigoplus_{i\in I} S_i$, such that each S_i is simple. By a similar argument to the first part of the proof of Theorem 3.11, each S_i is projective. Therefore, M is projective.
(4) $\Longleftrightarrow$ (6) It follows from Corollary 3.12. □

Let M be an R-module. Recall from [12] that a module M has C^* *property* provided that every submodule N of M contains a direct summand K of M such that N/K is cosingular.

A ring R is called *right* C^* if every R-module has C^* property. It is shown that R is right C^* if and only if every R-module is a direct sum of a cosingular module and an injective module (see [12, Theorem 2.9]).

Remark 3.20. Let R be a ring. Consider the following statements.

(1) R is right C^*;
(2) R has (P).

If R is right hereditary, then (1) $\Longrightarrow$ (2) and if every noncosingular R-module is injective, then (2) $\Longrightarrow$ (1).

Proof. (1) $\Longrightarrow$ (2) Let M be an R-module. By (1), there exists a decomposition $M = C \oplus E$, where C is cosingular and E is injective. Since R is right hereditary, E is noncosingular. So $\overline{Z}(M) = \overline{Z}(E) = E$.
(2) $\Longrightarrow$ (1) Since R has (P), we conclude that $M = \overline{Z}(M) \oplus C$. Then C is cosingular. Clearly $\overline{Z}(M)$ is noncosingular and by assumption is injective. So the result follows from [12, Theorem 2.9]. □

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