



On Posner's second theorem in additively inverse semirings

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Abstract

In this paper, we generalize Posner's second theorem in additively inverse semirings. This can be regarded as the generalization of Posner's theorem in semirings.

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Introduction

Let $(S, +, \cdot)$ be a semiring with commutative addition and an absorbing zero 0. A semiring S is called additively inverse semiring [6] if for every $a \in S$ there exists a unique element $\acute{a} \in S$ such that $a + \acute{a} + a = a$ and $\acute{a} + a + \acute{a} = \acute{a}$. Karvellas [6] proved that for all $a, b \in S$, $(a.\acute{b}) = \acute{a}.b = a.\acute{b}$ and $\acute{a}\acute{b} = ab$. Throughout this paper, S will represent an additively inverse semiring which satisfies the condition that for every $a \in S$, $a + \acute{a}$ is in the center $Z(S)$ of S . This class of semirings is known as MA semiring [5]. In the last couple of years, MA semirings investigated and studied in the connection with certain additive maps (see [4, 11–15]). By [5], a commutator $[\cdot, \cdot]$ in an additively inverse semiring defines as $[x, y] = xy + \acute{y}x = xy + y\acute{x}$. We will make use of the commutator identities $[xy, z] = x[y, z] + [x, z]y$, $[x, yz] = [x, y]z + y[x, z]$ and $[xy, x] = x[y, x]$, $[x, yx] = [x, y]x$ (see [5, 14], for their proof). S is prime if $aSb = (0)$ implies either $a = 0$ or $b = 0$. It is easy to see that if S is prime then the nonzero elements of $Z(S)$ are not zero divisors in S . An additive mapping $d : S \rightarrow S$ is called derivation if $d(ab) = d(a)b + ad(b)$, $\forall a, b \in S$. Following [4], a mapping $F : S \rightarrow S$ is said to be commuting on S if $[F(x), x] = 0$ for all $x \in S$, and F is said to be centralizing on S if $[[F(x), x], S] = 0$, holds for all $x \in S$.

A well-known theorem of E.C. Posner [10] says that a non-zero centralizing derivation of a prime ring R forces R to be commutative. This result has the virtue of being an important tool in ring theory, especially in the study of centralizing and commuting mappings and derivations. Posner's result has been generalized by many authors in several ways (see for example, [1, 7–9, 16]). Recently, M.Anjum and M.Asalam [4] studied the notion of commuting mappings in MA semirings. In this paper, we shall attempt to generalize Posner's second theorem in additively inverse semirings.

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We begin with the following useful lemma.

Lemma 1 ([14, Lemma 1.1]). *Let S be additively inverse semiring and $a, b \in S$. Then $a + b = 0$ implies $a = \acute{b}$.*

Lemma 2 ([4, Lemma 2.6]). *Let d be a derivation of prime additively inverse semiring S . If $ad(x) = 0, x \in S$ then either $a = 0$ or $d = 0$.*

Proposition 3 ([4, Theorem 2.7]). *Let S be prime and 2-torsion free, d be a non-zero derivation of S . If $a \in S$ and $[d(x), a] = 0, \forall x \in S$, then $a \in Z(S)$.*

We also proved the following lemma in [12].

Lemma 4. *Let S be prime and 2-torsion free. If $a \in S$ such that $[a, [a, s]] = 0, \forall s \in S$, then $[a, S] = 0$.*

Proof. Replacing s by $st, t \in S$ in $[a, [a, s]] = 0$, we get, $[a, s][a, t] = 0$. Replacing t by st in the last expression, we have, $0 = [a, s]s[a, t]$. But S is prime so $[a, s] = 0$. \square

Lemma 5. *Let S be 2-torsion free and prime. If d_1, d_2 are derivations of S such that*

$$d_1(a)d_2(b) + d_2(a)d_1(b) = 0, \forall a, b \in S, \tag{1}$$

then one of d_1, d_2 is zero.

Proof. Replacing a by $ad_1(c), c \in S$ in (1), we get

$$d_1(a)d_1(c)d_2(b) + d_2(a)d_1(c)d_1(b) + a(d_1^2(c)d_2(b) + d_2(d_1(c))d_1(b)) = 0.$$

Replacing a by $d_1(c)$ in (1) and using it in the last expression, we have,

$$d_1(a)d_1(c)d_2(b) + d_2(a)d_1(c)d_1(b) = 0, \forall a, b, c \in S. \tag{2}$$

Applying Lemma 1 in (1), we have $d_1(c)d_2(b) = d_2(c)d_1(\acute{b})$, from this and (2), we obtain

$$(d_1(a)d_2(\acute{c}) + d_2(a)d_1(c))d_1(b) = 0.$$

By Lemma 2, either $d_1(a)d_2(\acute{c}) + d_2(a)d_1(c) = 0$ or $d_1 = 0$. If $d_1(a)d_2(\acute{c}) + d_2(a)d_1(c) = 0$ then we have $2d_2(a)d_1(\acute{c}) = 0$ or $d_2(a)d_1(c) = 0, \forall a, c \in S$. Replacing a by $d_2(a)$, we have $d_2(d_2(a))d_1(c) = 0$. Again by Lemma 2, either $d_1 = 0$ or $d_2(a) = 0, \forall a \in S$. This completes the proof. \square

Theorem 6. *Let S be prime additively inverse semiring and d be a nonzero derivation of S such that*

$$[a, d(a)] = 0, \forall a \in S. \tag{3}$$

Then S is commutative.

Proof. By linearization of (3), we get

$$ad(b) + d(b)\acute{a} + bd(a) + d(a)\acute{b} = 0.$$

Thus we have,

$$d(ab) + d(b)\acute{a} + bd(a) + 2d(a)\acute{b} = 0. \tag{4}$$

From the given hypothesis, we have, $ad(a) + d(a)a + 2d(a)\acute{a} = 0$ thus $d(a^2) = 2d(a)a$, by using Lemma 1. Replacing b by $ax, x \in S$ in (4) and then using the last relation, we obtain

$0 = 2d(a)ax + a^2d(x) + d(a)x\acute{a} + ad(x)\acute{a} + axd(a) + 2d(a)a\acute{x} = a^2d(x) + d(a)x\acute{a} + ad(x)\acute{a} + axd(a) + 2d(a)a(x + \acute{x}) = a^2d(x) + d(a)x\acute{a} + ad(x)\acute{a} + axd(a) + 2a(x + \acute{x})d(a)$. Because $a(x + \acute{x})$ is an additive idempotent so

$$a^2d(x) + d(a)x\acute{a} + ad(x)\acute{a} + axd(a) = 0. \tag{5}$$

Replacing b by xa in (4), we arrive at

$$d(x)a^2 + d(a)xa + ad(x)\acute{a} + a\acute{x}d(x) = 0.$$

By Lemma 1, we have

$$d(a)x\acute{a} + a\acute{x}d(x) = d(x)a^2 + ad(x)\acute{a}. \tag{6}$$

Using (6) in (5), we get

$$a^2d(x) + 2ad(x)\acute{a} + d(x)a^2 = 0.$$

Thus we obtain, $a[a, d(x)] + [a, d(x)]\acute{a} = 0$. Replacing a by $a + d(x)$ in the last equation, we obtain

$$d(x)[a, d(x)] + [a, d(x)]d(x) + [d(x), d(x)](a + \acute{a}) = 0.$$

But $[d(x), d(x)](a + \acute{a}) = d(x)(d(x) + d(x'))(a + \acute{a}) = d(x)d(x)(a + \acute{a}) + d(x)d(x')(a + \acute{a}) = d(x)d(x')(a + \acute{a})$. Thus we have

$$d(x)[a, d(x)] + [a, d(x)]d(x) + d(x)d(x')(a + \acute{a}) = 0.$$

This implies that

$$d(x)[a, d(x)] + ad(x)d(x) + d(x)ad(x) + d(x)d(x')(a + \acute{a}) = 0.$$

As $(a + \acute{a}) \in Z(S)$ so we have,

$$d(x)[a, d(x)] + ad(x)d(x)' + d(x)ad(x) + (a + a')d(x)d(x)' = 0.$$

From this, we arrive at $[d(x), [a, d(x)]] = 0$. Thus $[d(x), S] = 0$, by Lemma 4 and therefore, by Proposition 3, we conclude that S is commutative. \square

Theorem 7. *Let S be prime and 2-torsion free d be a derivation of S such that*

$$[[a, d(a)], S] = 0, \forall a \in S \tag{7}$$

then if $d \neq 0$, S is commutative.

Proof. Linearization of (7) gives

$$[[a, d(b)] + [b, d(a)], S] = 0. \tag{8}$$

Replacing b by ax in the above equation, we have

$$\Rightarrow [[a, ad(x) + d(a)x] + [ax, d(a)], S] = 0.$$

Thus, we obtain

$$\begin{aligned} & [a[a, d(x)] + [a, d(a)]x + d(a)[a, x] + a[x, d(a)] + [a, d(a)]x, S] = 0 \\ \Rightarrow & [(ad(a) + d(a)a')x + d(a)xa' + axd(a), S] + [a^2d(x) + ad(x)a', S] = 0. \end{aligned} \tag{9}$$

Replacing b by xa in (8), we have, $[[a, d(xa)] + [xa, d(a)], S] = 0$. From this, we get, $[[a, d(x)a] + [a, xd(a)] + x[a, d(a)] + [x, d(a)]a, S] = 0$. Thus, we have

$$[x(ad(a) + d(a)a') + d(x)a'^2 + d(a)xa' + ad(x)a + axd(a), S] = 0.$$

From the given hypothesis, we have, $[a, d(a)] \in Z(S)$. Thus we obtain

$$\begin{aligned} & [(ad(a) + d(a)a')x + d(x)a'^2 + d(a)xa' + ad(x)a + axd(a), S] = 0 \\ \Rightarrow & [(ad(a) + d(a)a')x + d(a)xa' + axd(a), S] + [d(x)a'^2 + ad(x)a, S] = 0. \end{aligned}$$

By Lemma 1, we have

$$[(ad(a) + d(a)a')x + d(a)xa' + axd(a), S] = [d(x)a^2 + ad(x)a', S]. \tag{10}$$

Combining (9) and (10), we obtain

$$[a^2d(x) + d(x)a^2 + 2a'd(x)a, S] = 0.$$

In particular, for $a \in S$, we have

$$[a^2d(x) + d(x)a^2 + 2a'd(x)a, a] = 0$$

$$\begin{aligned} \Rightarrow a^2d(x)a + d(x)a^3 + 2ad(x)a^2 + a'^3d(x) + a'd(x)a^2 + 2a^2d(x)a &= 0 \\ \Rightarrow 3a^2d(x)a + d(x)a^3 + a'^3d(x) + 3a'd(x)a^2 &= 0. \end{aligned} \tag{11}$$

Replacing x by a in (11), we have

$$d(a)a^3 + a'^3d(a) + 3a(d(a)a' + ad(a))a = 0.$$

Thus we get

$$d(a)a^3 + a'^3d(a) = 3(d(a)a' + ad(a))a^2. \tag{12}$$

Replacing x by $ad(x)$ in (11), we get

$$3a^2d(ad(x))a + d(ad(x))a^3 + a'^3d(ad(x)) + 3a'd(ad(x))a^2 = 0.$$

So, we have,

$$\begin{aligned} 3a^2d(a)d(x)a + 3a^2ad^2(x)a + d(a)d(x)a^3 + ad^2(x)a^3 + a'^3d(a)d(x) + \\ a'^3ad^2(x) + 3a'd(a)d(x)a^2 + 3a'ad^2(x)a^2 = 0. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} a(3a'd^2(x)a^2 + a'^3d^2(x) + 3a^2d^2(x)a + d^2(x)a^3) + 3a^2d(a)d(x)a + \\ d(a)d(x)a^3 + a'^3d(a)d(x) + 3a'd(a)d(x)a^2 = 0. \end{aligned}$$

Replacing x by $d(x)$ in (11) and using it in the last relation, we have

$$3a^2d(a)d(x)a + d(a)d(x)a^3 + a'^3d(a)d(x) + 3a'd(a)d(x)a^2 = 0. \tag{13}$$

Pre-multiplying (11) by $d(a)$, we have

$$3d(a)a^2d(x)a + d(a)d(x)a^3 + d(a)a'^3d(x) + 3d(a)a'd(x)a^2 = 0.$$

Applying Lemma 1 in the last equation to find value of $d(a)d(x)a^3$ and then using it in (13), we get

$$\begin{aligned} 3a^2d(a)d(x)a + 3d(a)a^2d(x)a' + d(a)a^3d(x) + 3d(a)ad(x)a^2 + a'^3d(a)d(x) + 3a'd(a)d(x)a^2 = 0 \\ \Rightarrow 3[a^2, d(a)]d(x)a + [d(a), a^3]d(x) + 3[d(a), a]d(x)a^2 = 0. \end{aligned}$$

In the view of (12), we obtain

$$3[a^2, d(a)]d(x)a + 3[a, d(a)]a'^2d(x) + 3[d(a), a]d(x)a^2 = 0. \tag{14}$$

By using the fact that $[d(a), a] \in Z(S)$, we have

$$[a^2, d(a)] = a[a, d(a)] + [a, d(a)]a = 2a[a, d(a)].$$

Thus from (14), we have

$$3[a, d(a)](2ad(x)a + a'^2d(x) + d(x)a'^2) = 0.$$

But S is prime and $[a, d(a)] \in Z(S)$, so by using the fact that center of S is free of zero divisor, we have, either $[a, d(a)] = 0$ or $2ad(x)a + a'^2d(x) + d(x)a'^2 = 0$. If $[a, d(a)] \neq 0$, for some a then for that a , we have

$$2ad(x)a + a'^2d(x) + d(x)a'^2 = 0, x \in S. \tag{15}$$

Replacing x by ax , we obtain

$$2ad(a)xa + a'^2d(a)x + d(a)xa'^2 = 0. \tag{16}$$

Replacing x by a in (15) and then post-multiplying by x we get

$$2ad(a)ax + a'^2d(a)x + d(a)a'^2x = 0. \tag{17}$$

By Lemma 1, (16) and (17), we arrive at

$$2ad(a)[x, a] + d(a)[x, a'^2] = 0. \tag{18}$$

Replacing x by ax in (18), we get

$$2ad(a)a[x, a] + d(a)a[x, a'^2] = 0. \quad (19)$$

Pre-multiplying (18) by a , we get

$$2a^2d(a)[x, a] + ad(a)[x, a'^2] = 0. \quad (20)$$

Combining (19) and (20), we have

$$[d(a), a](2a[x, a] + [x, a'^2]) = 0.$$

But $[d(a), a] \neq 0$, thus we have $2a[x, a] + [x, a'^2] = 0$. From this, we have $2axa + 2a'^2x + x'a^2 + a^2x = 0$ or $0 = 2axa' + 2a^2x + xa^2 + a'^2x = 2axa' + xa^2 + (2a^2x + a'^2x) = 2axa' + xa^2 + a^2x = a[a, x] + [a, x]a' = [a, [a, x]]$. Thus $[a, [a, x]] = 0$ so by Lemma 4, $a \in Z(S)$. However, if $[d(a), a] = 0, \forall a \in S$ then the required result follows from Theorem 6. \square

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