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Research Article

Generalized Lucas numbers of the form $11x^2 \mp 1$

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Abstract

Let $P \geq 3$ be an integer and (V_n) denote generalized Lucas sequence defined by $V_0 = 2, V_1 = P$, and $V_{n+1} = PV_n - V_{n-1}$ for $n \geq 1$. In this study, we solve the equation $V_n = 11x^2 \mp 1$. We show that the equation $V_n = 11x^2 + 1$ has a solution only when $n = 1$ and $P \equiv 1 (\bmod 11)$. Moreover, we show that if the equation $V_n = 11x^2 - 1$ has a solution, then $P \equiv 2 (\bmod 8)$ and $P \equiv -1 (\bmod 11)$.

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1. Introduction

Let P and Q be nonzero integers. Generalized Fibonacci sequence (U_n) and Lucas sequence (V_n) are defined by $U_0(P,Q) = 0, U_1(P,Q) = 1$; $V_0(P,Q) = 2, V_1(P,Q) = P$, and $U_{n+1}(P,Q) = PU_n(P,Q) + QU_{n-1}(P,Q)$, $V_{n+1}(P,Q) = PV_n(P,Q) + QV_{n-1}(P,Q)$ for $n \geq 1$. $U_n(P,Q)$ and $V_n(P,Q)$ are called n-th generalized Fibonacci number and n-th generalized Lucas number, respectively. Generalized Fibonacci and Lucas numbers for negative subscripts are defined as $U_{-n}(P,Q) = -(-Q)^{-n}U_n(P,Q)$ and $V_{-n}(P,Q) = (-Q)^{-n}V_n(P,Q)$, respectively.

Since

$$U_n(-P,Q) = (-1)^{n-1}U_n(P,Q) \text{ and } V_n(-P,Q) = (-1)^n V_n(P,Q)$$

it will be assumed that $P \geq 1$. Moreover, to exclude the degenerate case we also assume that $P^2 + 4Q > 0$. For $P = Q = 1$, we have classical Fibonacci and Lucas sequences (F_n) and (L_n). For $P = 2$ and $Q = 1$, we have Pell and Pell-Lucas sequences (P_n) and (Q_n). For more information about generalized Fibonacci and Lucas sequences one can consult [25].

Investigation of the square terms of the second-order linear recurrence sequences has been of interest to many mathematicians and generated an extensive literature. In 1963, Moser and Carlitz [21] and Rollett [30] proposed the problem of finding all square Fibonacci numbers. This problem was solved by Cohn [7] and Wyler [39], independently. Later, Alfred [2] and Cohn [8] determined the square Lucas numbers. Pethő [24] and Cohn [10] independently determined the perfect powers in Pell sequence. In 2006, Bugeaud, Mignotte and Siksek [4] showed that the perfect powers in Fibonacci and Lucas sequences are exactly $F_0 = 0, F_1 = F_2 = 1, F_6 = 8, F_{12} = 144$ and $L_1 = 1, L_3 = 4$, respectively. The problem

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of determining the terms of the linear recurrence sequences which can be represented by quadratic or cubic polynomials has been also of interest to many mathematicians. Pethő [23] determined the full cubes in Fibonacci sequence. Finkelstein [11] found Fibonaci numbers of the form x^2+1. The same author [12] determined Lucas numbers of the form x^2+1. Luca [17] determined Fibonacci numbers of the form x^2+x+2. Robbins [28] solved the equations $F_n = x^2 - 1$ and $F_n = x^3 \mp 1$. The same author [29] solved the equations $F_n = px^2 \mp 1$ and $F_n = px^3 \mp 1$. Recently, Bugeaud, Mignotte and Siksek [5] showed that the only Fibonacci numbers which are at distance 1 from a perfect power are $1, 2, 3, 5$, and 8. Moreover, combinatorial numbers in the second-order linear recurrence sequences have been investigated by many authors. Ming [19] proved that the only triangular numbers in Fibonacci sequence are $F_0 = 0, F_1 = F_2 = 1, F_4 = 3, F_8 = 21$, and $F_{10} = 55$. The same author [20] showed that the triangular numbers in Lucas sequence are $L_1 = 1, L_2 = 3$ and $L_{18} = 5778$. McDaniel [18] proved that the only triangular number in Pell sequence is 1. Tengely [37] determined the g-gonal numbers in Fibonacci, Lucas, Pell and Pell-Lucas sequences for $g \leq 20$. The same author [38] proved that $(n, x) = (1, 5)$ is the only positive solution to the Diophantine equation $L_n = \binom{x}{5}$. For an extensive bibliography concerning combinatorial numbers in second-order linear recurrence sequences, see [16].

For relatively prime integers P and Q, the square terms of generalized Fibonacci sequence (U_n) and Lucas sequence (V_n) have been investigated in the literature. In [29], Ribenboim and McDaniel showed that if U_n or V_n is x^2 or $2x^2$, then $n \leq 12$. In [3], Bremmer and Tzanakis discussed the more general problem of finding all integers P and Q for which $U_m = kx^2$ for fixed integers k and m. It is shown that for fixed nonzero integers k and m with $m \geq 8$, the equation $U_m = kx^2$ can hold for finitely many integers P and Q. In [33] and [34], Şiar and Keskin determined the terms of the sequences (U_n) and (V_n) of the form kx^2 or $2kx^2$ for odd P and Q when $k|P$ with $k > 1$.

In [1], the authors showed that when $a \neq 0$ and $b \neq \mp 2$ are integers, the equation $V_n(P, \mp 1) = ax^2 + b$ has only a finite number of solutions n. Moreover, the same authors showed that when $a \neq 0$ and b are integers, the equation $U_n(P, \pm 1) = ax^2 + b$ has only a finite number of solutions n. Keskin [14] solved the equation $V_n(P, -1) = wx^2 \mp 1$ for $w = 1, 2, 3, 6$ when P is odd. Keskin and Öğüt [15] solved the equations $U_n(P, -1) = wx^2 + 1$ for $w = 1, 2, 3, 5, 7, 10$ when P is odd. Öğüt and Keskin [22] showed that only $U_1(P, -1)$ and $U_2(P, -1)$ may be of the form $11x^2 + 1$ if P is odd. When P is odd, Karaatlı and Keskin [13] solved the equations $V_n(P, -1) = 5x^2 \mp 1$ and $V_n(P, -1) = 7x^2 \mp 1$.

In the present study, we solve the equation $V_n(P, -1) = 11x^2 \mp 1$. We show that the equation $V_n(P, -1) = 11x^2 + 1$ has a solution only when $n = 1$ and $P \equiv 1 (\bmod 11)$. We show that the equation $V_n(P, -1) = 11x^2 - 1$ has only the solutions $n = 1, 2$ when P is odd. In case P is even, we prove that if the equation $V_n(P, -1) = 11x^2 - 1$ has a solution, then $P \equiv 2 (\bmod 8)$ and $P \equiv -1 (\bmod 11)$. Our main results are Theorems 3.1 and 3.2. We will use the Jacobi symbol throughout this study. Our method of proof is similar to that presented by Cohn, Ribenboim and McDaniel in [9] and [26, 27], respectively.

2. Preliminaries

From now on, instead of $V_n(P, -1)$ and $U_n(P, -1)$ we write V_n and U_n, respectively. Moreover, we will assume that $P \geq 3$.

The following lemma can be proved by induction.

Lemma 2.1. *If n is a positive integer, then $V_{2n} \equiv \mp 2 (\bmod P^2)$ and $V_{2n+1} \equiv (-1)^n (2n+1)P (\bmod P^2)$.*

The following theorem is given in [32].

Theorem 2.2. *Let* $n \in \mathbb{N} \cup \{0\}$ *and* $m,\ r \in \mathbb{Z}$. *Then*

$$V_{2mn+r} \equiv (-1)^n V_r \,(\operatorname{mod} V_m)\,, \tag{2.1}$$

and if $m \neq 0$, *then*

$$V_{2mn+r} \equiv V_r \,(\operatorname{mod} U_m)\,. \tag{2.2}$$

If $n = 2 \cdot 2^k a + r$ with a odd and $k \geq 1$, then we get

$$V_n = V_{2 \cdot 2^k a + r} \equiv -V_r (\operatorname{mod} V_{2^k}) \tag{2.3}$$

by (2.1).

When P is odd, an induction method shows that

$$V_{2^k} \equiv 7 (\operatorname{mod} 8)$$

and thus

$$\left(\frac{-1}{V_{2^k}}\right) = -1 \tag{2.4}$$

for all $k \geq 1$.

When P is odd, we have

$$\left(\frac{P-1}{V_{2^k}}\right) = \left(\frac{P+1}{V_{2^k}}\right) = 1 \tag{2.5}$$

for all $k \geq 1$.

If P is odd and $k \geq 2$, then $V_{2^k} \equiv -1 \left(\operatorname{mod} P^2 - 3\right)$ and thus

$$\left(\frac{P^2-3}{V_{2^k}}\right) = 1. \tag{2.6}$$

When P is odd and $P^2 \equiv 1 (\operatorname{mod} 11)$, we have

$$\left(\frac{11}{V_{2^k}}\right) = 1 \tag{2.7}$$

for all $k \geq 1$.

If P is odd and $P^2 \equiv 3 (\operatorname{mod} 11)$, then we get

$$\left(\frac{11}{V_{2^k}}\right) = 1 \tag{2.8}$$

for all $k \geq 2$.

Now we give some identities concerning generalized Fibonacci and Lucas numbers:

$$V_{-n} = V_n,$$
$$V_{2n} = V_n^2 - 2. \tag{2.9}$$

If $P^2 \equiv 5, 9 (\operatorname{mod} 11)$, then we have $11 | U_5$. Since $V_{10q+r} \equiv V_r (\operatorname{mod} U_5)$ by (2.2), it follows that

$$V_n \equiv 2, 3, 4, 7, 8, 9 (\operatorname{mod} 11). \tag{2.10}$$

Similarly, if $P^2 \equiv 4 (\operatorname{mod} 11)$, then we get

$$V_n \equiv \begin{cases} \mp 2 (\operatorname{mod} 11) \text{ if } P \equiv -2 (\operatorname{mod} 11), \\ 2 (\operatorname{mod} 11) \quad \text{if} \quad P \equiv 2 (\operatorname{mod} 11). \end{cases} \tag{2.11}$$

and if $P^2 \equiv 1 (\operatorname{mod} 11)$, then

$$V_n \equiv \begin{cases} -1, 2 (\operatorname{mod} 11) \text{ if } P \equiv -1 (\operatorname{mod} 11), \\ \mp 1, 2 (\operatorname{mod} 11) \text{ if } P \equiv 1 (\operatorname{mod} 11). \end{cases} \tag{2.12}$$

$$\text{If } P \equiv 2 (\operatorname{mod} 8), \text{ then } V_n \equiv 2 (\operatorname{mod} 8) \text{ for every natural number } n. \tag{2.13}$$

3. Main theorems

From now on, we will assume that n is a positive integer.

Theorem 3.1. *If* $P \equiv -1 \pmod{11}$ *or* $P^2 \equiv 0,3,4,5,9 \pmod{11}$*, then the equation* $V_n = 11x^2+1$ *has no solution. The equation* $V_n = 11x^2+1$ *has only the solution* $n=1$ *when* $P \equiv 1 \pmod{11}$*.*

Proof. Assume that $11|P$. Then by Lemma 2.1, $V_n \equiv \mp 2, \mp nP \pmod{P^2}$, i.e., $V_n \equiv \mp 2, 0 \pmod{11}$, which is impossible since $V_n \equiv 1 \pmod{11}$. Assume that $P^2 \equiv 5,9 \pmod{11}$. Then by (2.10), it follows that $V_n \equiv 2,3,4,7,8,9 \pmod{11}$, which contradicts the fact that $V_n \equiv 1 \pmod{11}$. Assume that $P^2 \equiv 4 \pmod{11}$. Then by (2.11), we get $V_n \equiv \mp 2 \pmod{11}$, which is impossible since $V_n \equiv 1 \pmod{11}$. Assume that $P^2 \equiv 3 \pmod{11}$. Let $n = 12q \mp r$ with $0 \leq r \leq 6$. Then

$$V_n = V_{12q \mp r} \equiv V_{\mp r} \pmod{U_6}$$

by (2.2). Since $11|U_6$ when $P^2 \equiv 3 \pmod{11}$, we get

$$V_n \equiv V_{\mp r} \pmod{11}.$$

An easy calculation shows that $r = 2$. Then it follows that n is even and $4 \nmid n$. Let $n > 2$. So, $n = 8t \mp 2$ for some positive integer t. Then $n = 8t \mp 2 = 2 \cdot 2^k a \mp 2$ with a odd and $k \geq 2$. Let P be odd. We get

$$11x^2 = -1 + V_n \equiv -1 - V_{\mp 2} \pmod{V_{2^k}}$$

by (2.3). This shows that

$$11x^2 \equiv -(P^2-1) \pmod{V_{2^k}},$$

which is impossible since $\left(\frac{11}{V_{2^k}}\right) = 1$, $\left(\frac{-1}{V_{2^k}}\right) = -1$, and $\left(\frac{P^2-1}{V_{2^k}}\right) = 1$ by (2.8), (2.4) and (2.5), respectively. Thus $n = 2$. Then it follows that $V_2 = P^2 - 2 = 11x^2 + 1$, which implies that $11x^2 \equiv -2 \pmod 8$ since $P^2 \equiv 1 \pmod 8$. Therefore $3x^2 \equiv -2 \pmod 8$, i.e., $x^2 \equiv 2 \pmod 8$, which is impossible. Let P be even. Then,

$$11x^2 = -1 + V_n \equiv -1 + V_{\mp 2} \pmod{V_2}$$

by (2.1). This shows that

$$11x^2 \equiv -1 \pmod{P^2 - 2},$$

and therefore

$$11x^2 \equiv -1 \left(\bmod \left(P^2-2\right)/2\right).$$

Then we get

$$\left(\frac{11}{(P^2-2)/2}\right) = \left(\frac{-1}{(P^2-2)/2}\right),$$

which implies that

$$\left(\frac{(P^2-2)/2}{11}\right) = 1.$$

Thus,

$$1 = \left(\frac{(P^2-2)/2}{11}\right) = \left(\frac{P^2-2}{11}\right)\left(\frac{2}{11}\right) = \left(\frac{1}{11}\right)\left(\frac{2}{11}\right) = -1,$$

a contradiction. Assume that $P^2 \equiv 1 \pmod{11}$. Then $P \equiv \mp 1 \pmod{11}$. If $P \equiv -1 \pmod{11}$, we get $V_n \equiv -1, 2 \pmod{11}$ by (2.12). This is impossible since $V_n \equiv 1 \pmod{11}$. So, $P \equiv 1 \pmod{11}$. Let $n = 6q + r$ with $0 \leq r \leq 5$. Then

$$V_n = V_{6q+r} \equiv V_r \pmod{U_3}$$

by (2.2). Since $11|U_3$, when $P^2 \equiv 1 \pmod{11}$, we get

$$V_n \equiv V_r \pmod{11}.$$

An easy calculation shows that $r = 1$ or $r = 5$. Then it follows that n is odd and $3 \nmid n$. Let P be odd and $n > 1$. Then $n = 4q \mp 1 = 2 \cdot 2^k a \mp 1$ with a odd and $k \geq 1$. Thus,

$$11x^2 = -1 + V_n \equiv -1 - V_{\mp 1} (\bmod V_{2^k})$$

by (2.3). This shows that

$$11x^2 \equiv -(P+1) (\bmod V_{2^k}),$$

which is impossible since $\left(\frac{11}{V_{2^k}}\right) = 1$, $\left(\frac{-1}{V_{2^k}}\right) = -1$, and $\left(\frac{P+1}{V_{2^k}}\right) = 1$ by (2.7), (2.4), and (2.5), respectively. Let P be even. Since n is odd and $3 \nmid n$, we can write $n = 6q \mp 1$ for some $q \geq 0$. Then

$$11x^2 = -1 + V_n \equiv -1 + V_{\mp 1} (\bmod U_3)$$

by (2.2). This shows that $11x^2 \equiv P - 1 (\bmod P^2 - 1)$ and therefore $11x^2 \equiv -2 (\bmod P + 1)$. Then we get

$$\left(\frac{11}{P+1}\right) = \left(\frac{-2}{P+1}\right),$$

which implies that

$$\left(\frac{P+1}{11}\right) = \left(\frac{2}{P+1}\right).$$

Since $P + 1 \equiv 2 (\bmod 11)$, we get

$$-1 = \left(\frac{2}{11}\right) = \left(\frac{P+1}{11}\right) = \left(\frac{2}{P+1}\right).$$

Therefore, $P + 1 \equiv 3, 5 (\bmod 8)$, i.e., $P \equiv 2, 4 (\bmod 8)$. Assume that $P \equiv 2 (\bmod 8)$. Then $V_n \equiv 2 (\bmod 8)$ by (2.13). This contradicts the fact that $V_n = 11x^2 + 1 \equiv 1, 4, 5 (\bmod 8)$. Now, assume that $P \equiv 4 (\bmod 8)$. Then it is seen that $V_{2^k} \equiv \mp 2 (\bmod 8)$ for $k \geq 1$. Let $n > 1$. So, $n = 4q \mp 1 = 2 \cdot 2^k a \mp 1$ with a odd and $k \geq 1$. Thus,

$$11x^2 = -1 + V_n \equiv -1 - V_{\mp 1} (\bmod V_{2^k})$$

by (2.3). This shows that $11x^2 \equiv -(P+1) (\bmod V_{2^k}/2)$.Then it follows that

$$\left(\frac{11}{V_{2^k}/2}\right) = \left(\frac{-1}{V_{2^k}/2}\right)\left(\frac{P+1}{V_{2^k}/2}\right)$$

and therefore

$$\left(\frac{V_{2^k}/2}{11}\right) = \left(\frac{P+1}{V_{2^k}/2}\right) = \left(\frac{V_{2^k}/2}{P+1}\right)$$

since $P+1 \equiv 5 (\bmod 8)$. By using the facts that $V_{2^k} \equiv -1 (\bmod P+1)$ and $V_{2^k} \equiv -1 (\bmod 11)$ for $k \geq 1$, we get

$$\begin{aligned} 1 = \left(\frac{-1}{11}\right)\left(\frac{2}{11}\right) &= \left(\frac{V_{2^k}}{11}\right)\left(\frac{2}{11}\right) = \left(\frac{V_{2^k}/2}{11}\right) = \left(\frac{V_{2^k}/2}{P+1}\right) \\ &= \left(\frac{V_{2^k}}{P+1}\right)\left(\frac{2}{P+1}\right) = \left(\frac{-1}{P+1}\right)\left(\frac{2}{P+1}\right) = -1, \end{aligned}$$

a contradiction. Thus $n = 1$. □

Theorem 3.2. *If $P^2 \equiv 0, 3, 4, 5, 9 (\bmod 11)$, then the equation $V_n = 11x^2 - 1$ has no solution. The equation $V_n = 11x^2 - 1$ has the solutions $n = 1$ when P is odd and $P \equiv -1 (\bmod 11)$ or $n = 2$ when P is odd and $P \equiv 1 (\bmod 11)$. Let P be even. If the equation $V_n = 11x^2 - 1$ has a solution, then $P \equiv 2 (\bmod 8)$ and $P \equiv -1 (\bmod 11)$.*

Proof. If $n = 4q$ for some $q > 0$, then we get

$$11x^2 - 1 = V_{4q} = V_{2q}^2 - 2 = \left(V_q^2 - 2\right)^2 - 2 = V_q^4 - 4V_q^2 + 2,$$

i.e., $11x^2 = V_q^4 - 4V_q^2 + 3$. Hence, we obtain elliptic curve $11X^2 = Y^4 - 4Y^2 + 3$, where $X = x$ and $Y = V_q$. A practical method for the explicit computation of all integral points on an elliptic curve for the quadratic case has been developed by Stroker and Tzanakis in [31] and [35]. Their methods were based on lower bounds for linear forms in elliptic logarithms (see [36] for the details about elliptic logarithms). Such a method was implemented in the computational package MAGMA [6]. Here, we use the subroutine "Integral Quartic Points" of MAGMA. So, we have only the solutions $(X, Y) = (0, \mp 1)$. This implies that $V_q = 1$, which is impossible. Therefore $n \neq 4q$. Clearly, the case $11|P$ is impossible. For, otherwise we would have by Lemma 2.1, $V_n \equiv \mp 2, \mp nP (\operatorname{mod} P^2)$, i.e., $V_n \equiv \mp 2, 0 (\operatorname{mod} 11)$, which contradicts $V_n \equiv -1 (\operatorname{mod} 11)$. Assume that $P^2 \equiv 4 (\operatorname{mod} 11)$. Then by (2.11), we get $V_n \equiv \mp 2 (\operatorname{mod} 11)$, which is impossible since $V_n \equiv -1 (\operatorname{mod} 11)$. Assume that $P^2 \equiv 5, 9 (\operatorname{mod} 11)$. Then $V_n \equiv 2, 3, 4, 7, 8, 9 (\operatorname{mod} 11)$ by (2.10). This contradicts the fact that $V_n \equiv -1 (\operatorname{mod} 11)$. Now, assume that $P^2 \equiv 3 (\operatorname{mod} 11)$. Let $n = 12q + r$ with $0 \leq r \leq 11$. Then

$$V_n = V_{12q+r} \equiv V_r (\operatorname{mod} U_6)$$

by (2.2), which implies that $V_n \equiv V_r (\operatorname{mod} 11)$ since $11|U_6$ when $P^2 \equiv 3 (\operatorname{mod} 11)$. An easy calculation shows that $r = 4$ or $r = 8$. Then it follows that n is even and $4|n$. Therefore $n = 4t$ for some $t > 0$. But it was shown at the beginning of the proof that $n \neq 4t$. Assume that P is odd and $P^2 \equiv 1 (\operatorname{mod} 11)$. Let n be even. Since $n \neq 4q$, we get $n = 8q \mp 2$ for some $q \geq 0$. Let $n > 2$. So, $n = 8q \mp 2 = 2 \cdot 2^k a \mp 2$ with a odd and $k \geq 2$. Thus,

$$11x^2 = 1 + V_n \equiv 1 - V_{\mp 2} (\operatorname{mod} V_{2^k})$$

by (2.3). This shows that $11x^2 \equiv -(P^2 - 3) (\operatorname{mod} V_{2^k})$, which is impossible since $\left(\frac{11}{V_{2^k}}\right) = 1$, $\left(\frac{-1}{V_{2^k}}\right) = -1$, and $\left(\frac{P^2 - 3}{V_{2^k}}\right) = 1$ by (2.7), (2.4) and (2.6), respectively. Thus $n = 2$. It can be easily seen that $n = 2$ is a solution only when $P \equiv 1 (\operatorname{mod} 11)$. Let $n > 1$ be odd. So, $n = 4q \mp 1 = 2 \cdot 2^k a \mp 1$ with a odd and $k \geq 1$. Thus,

$$11x^2 = 1 + V_n \equiv 1 - V_{\mp 1} (\operatorname{mod} V_{2^k})$$

by (2.3). This shows that $11x^2 \equiv -(P - 1) (\operatorname{mod} V_{2^k})$, which is impossible since $\left(\frac{11}{V_{2^k}}\right) = 1$, $\left(\frac{-1}{V_{2^k}}\right) = -1$, and $\left(\frac{P - 1}{V_{2^k}}\right) = 1$ by (2.7), (2.4) and (2.5), respectively. Thus $n = 1$. Observe that $n = 1$ is a solution only when $P \equiv -1 (\operatorname{mod} 11)$. Assume that P is even and $P \equiv 1 (\operatorname{mod} 11)$. Let $n = 6q + r$ with $0 \leq r \leq 5$. Then

$$V_n = V_{6q+r} \equiv V_r (\operatorname{mod} U_3)$$

by (2.2). Since $11|U_3$ when $P \equiv 1 (\operatorname{mod} 11)$, we get $V_n \equiv V_r (\operatorname{mod} 11)$. An easy calculation shows that $r = 2$ or $r = 4$. Therefore n is even. Then $n = 4q$ or $n = 8q \mp 2$ for some $q \geq 0$. The first of these is impossible since it was shown at the beginning of the proof. Then, let $n = 8q \mp 2$ for some $q \geq 0$. By (2.2), we have

$$11x^2 = 1 + V_n \equiv 1 + V_{\mp 2} (\operatorname{mod} U_2).$$

It immediately follows that $11x^2 \equiv -1 (\operatorname{mod} P)$. Let $P = 2^r a$ with a odd and $r \geq 1$. Then we get $11x^2 \equiv -1 (\operatorname{mod} a)$, which implies that

$$\left(\frac{11}{a}\right) = \left(\frac{-1}{a}\right),$$

i.e.,

$$\left(\frac{a}{11}\right) = 1.$$

Then we can write

$$1 = \left(\frac{1}{11}\right) = \left(\frac{P}{11}\right) = \left(\frac{2^r}{11}\right)\left(\frac{a}{11}\right) = \left(\frac{2}{11}\right)^r \left(\frac{a}{11}\right) = \left(\frac{2}{11}\right)^r = (-1)^r.$$

Thus, r is even. So, $4|P$. Since $n = 8q \mp 2$, we get $11x^2 = 1 + V_n \equiv 1 + V_{\mp 2} (\bmod V_2)$ by (2.1). This shows that $11x^2 \equiv 1(\bmod\left(P^2-2\right)/2)$. Since $\left(P^2-2\right)/2$ is odd, we get

$$\left(\frac{11}{\left(P^2-2\right)/2}\right) = 1,$$

which is impossible since

$$\begin{aligned} 1 &= \left(\frac{11}{\left(P^2-2\right)/2}\right) = -\left(\frac{\left(P^2-2\right)/2}{11}\right) = -\left(\frac{P^2-2}{11}\right)\left(\frac{2}{11}\right) \\ &= \left(\frac{P^2-2}{11}\right) = \left(\frac{-1}{11}\right) = -1. \end{aligned}$$

Assume that P is even and $P \equiv -1(\bmod 11)$. Let n be odd. Writing $n = 4q \mp 1$ for some $q \geq 0$ gives

$$11x^2 = 1 + V_n \equiv 1 + V_{\mp 1}(\bmod U_2)$$

by (2.2). This implies that

$$11x^2 \equiv 1(\bmod P). \tag{3.1}$$

If $4|P$, then $11x^2 \equiv 1(\bmod 4)$, i.e., $x^2 \equiv 3(\bmod 4)$, which is impossible. Thus, $P \equiv \mp 2(\bmod 8)$. If $P \equiv -2(\bmod 8)$, then $11x^2 \equiv 1(\bmod P/2)$ by (3.1), which implies that $\left(\frac{11}{P/2}\right) = 1$. This is also impossible since

$$1 = \left(\frac{11}{P/2}\right) = -\left(\frac{P/2}{11}\right) = -\left(\frac{P}{11}\right)\left(\frac{2}{11}\right) = -\left(\frac{-1}{11}\right)\left(\frac{2}{11}\right) = -1.$$

Thus, $P \equiv 2(\bmod 8)$. Let n be even. Since $n \neq 4q$, we have $n = 8q \mp 2$ for some $q \geq 0$. Then $11x^2 = 1 + V_n \equiv 1 + V_{\mp 2}(\bmod V_2)$ by (2.1). This shows that $11x^2 \equiv 1(\bmod P^2 - 2)$ and therefore

$$11x^2 \equiv 1(\bmod\left(P^2-2\right)/2),$$

since P is even. Thus, we have

$$\left(\frac{11}{\left(P^2-2\right)/2}\right) = 1.$$

So,

$$1 = (-1)^{\left(P^2-4\right)/4}\left(\frac{\left(P^2-2\right)/2}{11}\right) = (-1)^{\left(P^2-4\right)/4}\left(\frac{P^2-2}{11}\right)\left(\frac{2}{11}\right) = (-1)^{\left(P^2-4\right)/4},$$

which implies that $P \equiv \mp 2(\bmod 8)$. Let $P \equiv -2(\bmod 8)$. Then it follows that

$$V_n \equiv \begin{cases} -2(\bmod 8) \text{ if } n \text{ is odd,} \\ 2(\bmod 8) \text{ if } n \text{ is even.} \end{cases} \tag{3.2}$$

Moreover,

$$11x^2 - 1 = V_{8q\mp 2} = V_{2(4q\mp 1)} = V_{4q\mp 1}^2 - 2,$$

i.e.,

$$V_{4q\mp 1}^2 - 11x^2 = 1.$$

It can be seen that all positive integer solutions of the equation $u^2 - 11v^2 = 1$ are given by $(u, v) = (V_k(20, -1)/2, 3U_k(20, -1))$ with $k \geq 1$. It can be shown that

$$\frac{V_k(20,-1)}{2} \equiv \begin{cases} 1 (\bmod 8) \text{ if } 4|k, \\ -1 (\bmod 8) \text{ if } k = 2m, \ m \text{ is odd}, \\ 2 (\bmod 8) \text{ if } k \text{ is odd}. \end{cases} \tag{3.3}$$

Since $V_{4q\mp1}^2 - 11x^2 = 1$, we get $V_{4q\mp1} = V_{k_0}(20, -1)/2$ for some natural number k_0. This is impossible by (3.2) and (3.3). Thus, $P \equiv 2(\bmod 8)$. □

Lemma 3.3. *Let $P \equiv 2(\bmod 8)$ and $m > 1$ be an odd integer. Suppose that the equation $V_m = 11x^2 - 1$ has no solution. Then, the equation $V_{2m} = 11x^2 - 1$ has no solution.*

Proof. Assume that $V_{2m} = 11x^2 - 1$ for some integer x. Since $V_{2m} = V_m^2 - 2$ by (2.9), it follows that $V_m^2 - 1 = 11x^2$. This implies that $(V_m - 1)(V_m + 1) = 11x^2$. It is seen that $(V_m - 1, V_m + 1) = 1$ since V_m is even when P is even. Then, either

$$V_m - 1 = a^2 \text{ and } V_m + 1 = 11b^2 \tag{3.4}$$

or

$$V_m - 1 = 11a^2 \text{ and } V_m + 1 = b^2 \tag{3.5}$$

for some integers a and b. The identity (3.4) is impossible by the hypothesis. Assume that (3.5) is satisfied. Then $11a^2 = V_m - 1 \equiv 1(\bmod 8)$ by (2.13). This is impossible since $a^2 \equiv 3(\bmod 8)$ in this case. So, the equation $V_{2m} = 11x^2 - 1$ has no solution. □

Lemma 3.4. *Let $P \equiv -1(\bmod 11)$. Then the equation $V_{3n} = 11x^2 - 1$ has no solution.*

Proof. Suppose that $V_{3n} = 11x^2 - 1$ for some integer x. We can write $3n = 12q \mp r$ with $r \in \{0, 3, 6\}$. Thus, $V_{3n} = V_{12q\mp r} \equiv V_r(\bmod U_3)$ by (2.2). Since $11|U_3$, we have $V_{3n} \equiv V_r$ $(\bmod 11)$. Since $P \equiv -1(\bmod 11)$, it can be seen that $V_r \equiv 2(\bmod 11)$ for all $r \in \{0, 3, 6\}$. So, we have $V_{3n} \equiv 2(\bmod 11)$. This is impossible since $V_{3n} \equiv -1(\bmod 11)$. □

From Lemma 3.4 and Theorem 3.2, we can give the following corollary.

Corollary 3.5. *The equation $V_{3n} = 11x^2 - 1$ has no solution.*

Lemma 3.6. *Let $P \equiv 2(\bmod 8)$ and $P \equiv -1(\bmod 11)$. If $m = 5, 7, 13, 19,$ or 23, then the equation $V_m = 11x^2 - 1$ has no solution.*

Proof. Suppose that $V_m = 11x^2 - 1$ for some integer x. Let $m = 5$. Then

$$11x^2 = V_5 + 1 = (P + 1)F(P), \tag{3.6}$$

where $F(P) = P^4 - P^3 - 4P^2 + 4P + 1$. Since $F(-1) = -5$, $F(P) \equiv -5(\bmod 11)$ and therefore $11 \nmid F(P)$. It can be seen that $(P + 1, F(P)) = 1$ or 5. Then from (3.6), we have

$$P + 1 = 11a^2 \text{ and } F(P) = b^2 \tag{3.7}$$

or

$$P + 1 = 55a^2 \text{ and } F(P) = 5b^2 \tag{3.8}$$

for some integers a and b. Assume that (3.7) is satisfied. Then $b^2 = F(P) \equiv -5(\bmod 11)$, which is impossible since $\left(\frac{-5}{11}\right) = -1$. The identity (3.8) is impossible because $5b^2 \equiv -5(\bmod 11)$ in this case. Let $m = 7$. Then

$$11x^2 = V_7 + 1 = (P + 1)F(P), \tag{3.9}$$

where $F(P) = P^6 - P^5 - 6P^4 + 6P^3 + 8P^2 - 8P + 1$. Since $F(-1) = 7$, $F(P) \equiv 7(\bmod 11)$ and therefore $11 \nmid F(P)$. Moreover, we have $(P + 1, F(P)) = 1$ or 7. Then from (3.9), we have

$$P + 1 = 11a^2 \text{ and } F(P) = b^2, \tag{3.10}$$

or

$$P + 1 = 77a^2 \text{ and } F(P) = 7b^2 \tag{3.11}$$

for some integers a and b. Assume that (3.10) is satisfied. Then $b^2 = F(P) \equiv 7 (\bmod 11)$, which is impossible since $\left(\frac{7}{11}\right) = -1$. The identity (3.11) is impossible since $P + 1 \equiv 3(\bmod 8)$. The proof is similar for $m = 13, 19$, or 23 and we omit the details. □

Since the equation $V_m = 11x^2 - 1$ has no solution for $m = 5, 7, 13, 19$, or 23 by Lemma 3.6 and Theorem 3.2, the equation $V_n = 11x^2 - 1$ has no solution for $n = 10, 14, 26, 38$, or 46 by Lemma 3.3 and Theorem 3.2. Therefore we can give the following corollary.

Corollary 3.7. *Let* $n \in \{5, 7, 9, 10, 13, 14, 23, 26, 38, 46\}$. *Then the equation* $V_n = 11x^2 - 1$ *has no solution.*

Let n be odd and $3 \nmid n$. Then $n = 6q \mp 1$ for some $q > 0$ and so $V_n + 1 \equiv V_{\mp 1} + 1 (\bmod U_3)$ by (2.2). This implies that $P + 1 | V_n + 1$. Therefore from the observations above, we can give the following lemma.

Lemma 3.8. *Let* p *be a prime number such that* $(p, 33) = 1$ *and* $V_p + 1 = (P + 1) F(P)$. *Assume that*

$$F(-1) = \begin{cases} p \text{ if } p \equiv 3(\bmod 4), \\ -p \text{ if } p \equiv 1(\bmod 4). \end{cases}$$

Also, assume that $\left(\frac{p}{11}\right) = -1$ *if* $p \equiv 3(\bmod 4)$ *and* $\left(\frac{p}{11}\right) = 1$ *if* $p \equiv 1(\bmod 4)$. *If* $P \equiv 2(\bmod 8)$ *and* $P \equiv -1(\bmod 11)$, *then the equation* $V_n = 11x^2 - 1$ *has no solution.*

4. Conclusion

We prove that the equation $V_n = 11x^2 + 1$ has only the solution $n = 1$ when $P \equiv 1(\bmod 11)$.

It can be seen that

$$\frac{V_k(20, -1)}{2} \equiv \begin{cases} 1(\bmod 11) \text{ if } k \text{ is even}, \\ -1(\bmod 11) \text{ if } k \text{ is odd}. \end{cases}$$

If $P = \frac{V_k(20,-1)}{2}$ and $x = 3U_k(20, -1)$ for some natural number k, then $V_2(P, -1) = 11x^2 - 1$. Clearly, the equation $V_n(P, -1) = 11x^2 - 1$ has the solutions $n = 1, 2$. We show that the equation $V_n(P, -1) = 11x^2 - 1$ has no solution for $n > 2$ when P is odd. In case P is even, we prove that if the equation $V_n(P, -1) = 11x^2 - 1$ has a solution, then $P \equiv 2(\bmod 8)$ and $P \equiv -1(\bmod 11)$. At the top of the proof of Theorem 3.2, we show that if $4|n$, then equation $V_n = 11x^2 - 1$ has no solution. Moreover, we prove that $V_n = 11x^2 - 1$ has no solution if $3|n$. If we show that the equation $V_m = 11x^2 - 1$ has no solution when $m > 1$ is odd and $3 \nmid m$, then it is seen that $V_{2m} = 11x^2 - 1$ has no solution by Lemma 3.3. Therefore we conclude that $V_n = 11x^2 - 1$ has no solution for $n > 2$. We think that the equation $V_m = 11x^2 - 1$ has no solution when $m > 1$ is odd and $3 \nmid m$. It is not known whether the equation $V_{11} = 11x^2 - 1$ has a solution in case $P \equiv -1(\bmod 11)$ and $P \equiv 2(\bmod 8)$.

References

[1] M.A. Alekseyev and S. Tengely, *On integral points on biquadratic curves and near-multiples of squares in Lucas sequences*, J. Integer Seq. **17** Article ID: 14.6.6, 1–15, 2014.

[2] B.U. Alfred, *On square Lucas numbers*, Fibonacci Quart. **2**, 11–12, 1964.

[3] A. Bremner and N. Tzanakis, *Lucas sequences whose nth term is a square or an almost square*, Acta Arith.**126**, 261–280, 2007.

[4] Y. Bugeaud, M. Mignotte, and S. Siksek, *Classical and modular approaches to exponential Diophantine equations. I. Fibonacci and Lucas perfect powers*, Ann. of Math. (2) **163** (3), 969–1018, 2006.

[5] Y. Bugeaud, M. Mignotte, F. Luca and S. Siksek, *Fibonacci numbers at most one away from a perfect square*, Elem. Math. **63**, 65–75, 2008.

[6] W. Bosma, J. Cannon and C. Playoust, *The MAGMA algebra system. I: The user language*, J. Symbolic Comput. **24**(3-4), 235–265, 1997.

[7] J.H.E. Cohn, *On square Fibonacci numbers*, J. London Math. Soc. **39**, 537–540, 1964.

[8] J.H.E. Cohn, *Lucas and Fibonacci numbers and some Diophantine equations*, Proc. Glasgow Math. Assoc. **7**, 24–28, 1965.

[9] J.H.E. Cohn, *Squares in some recurrent sequences*, Pacific J. Math., **41**, 631–646, 1972.

[10] J.H.E. Cohn, *Perfect Pell powers*, Glasgow Math. J. **38** (1), 19–20, 1996.

[11] R. Finkelstein, *On Fibonacci numbers which are one more than a square*, Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, J. Reine Angew. Math. **262/263**, 171–178, 1973.

[12] R. Finkelstein, *On Lucas numbers which are one more than a square*, Fibonacci Quart. **13** (4), 340–342, 1975.

[13] O. Karaatlı and R. Keskin, *Generalized Lucas numbers of the form $5kx^2$ and $7kx^2$*, Bull. Korean Math. Soc., **52** (5), 1467–1480, 2015.

[14] R. Keskin, *Generalized Fibonacci and Lucas Numbers of the form wx^2 and $wx^2 \mp 1$*, Bull. Korean Math. Soc. **51**, 1041–1054, 2014.

[15] R. Keskin and Ü. Öğüt, *Generalized Fibonacci Numbers of the form $wx^2 + 1$*, Period. Math. Hungar. **73** (2), 165–178, 2016.

[16] T. Kovács, *Combinatorial numbers in binary recurrences*, Period. Math. Hungar. **58** (1), 83–98, 2009.

[17] F. Luca, *Fibonacci numbers of the form $k^2 + k + 2$*, in: Applications of Fibonacci numbers, **8** (Rochester, NY, 1998), 241–249, Kluwer Acad. Publ. Dordrecht, 1999.

[18] W.L. McDaniel, *Triangular numbers in the Pell sequence*, Fibonacci Quart. **34** (2), 105–107, 1996.

[19] L. Ming, *On triangular Fibonacci numbers*, Fibonacci Quart., **27** (2), 98–108, 1989.

[20] L. Ming, *On triangular Lucas numbers*, in: Applications of Fibonacci numbers, **4**, (Winston-Salem, NC, 1990), 231–240, Kluwer Acad. Publ. Dordrecht, 1991.

[21] L. Moser and L. Carlitz, *Advanced problem H-2*, Fibonacci Quart. **1**, 46, 1963.

[22] Ü. Öğüt and R. Keskin, *Generalized Fibonacci and Lucas Numbers of the form $11x^2 + 1$*, Honam Math. J. **40** (1), 139–153, 2018.

[23] A. Pethő, *Full cubes in the Fibonacci sequence*, Publ. Math. Debrecen, **30** (1-2), 117–127, 1983.

[24] A. Pethő, *The Pell sequence contains only trivial perfect powers*, in: Sets, graphs and numbers (Budapest, 1991), 561–568, Colloq. Math. Soc. János Bolyai, **60**, North-Holland, Amsterdam, 1992.

[25] P. Ribenboim, *My Numbers, My Friends*, Springer-Verlag, New York, Inc. 2000.

[26] P. Ribenboim and W.L. McDaniel, *The square terms in Lucas sequences*, J. Number Theory, **58**, 104–123, 1996.

[27] P. Ribenboim and W.L. McDaniel, *On Lucas sequence terms of the form kx^2*, Number Theory: proceedings of the Turku symposium on Number Theory in memory of Kustaa Inkeri (Turku, 1999), Walter de Gruyter, Berlin, 293–303, 2001.

[28] N. Robbins, *Fibonacci and Lucas numbers of the forms $w^2 - 1$, $w^3 \pm 1$*, Fibonacci Quart. **19** (4), 369–373, 1981.

[29] N. Robbins, *Fibonacci numbers of the forms $px^2 \pm 1$, $px^3 \pm 1$, where p is prime*, in: Applications of Fibonacci numbers (San Jose, CA, 1986), 77–88, Kluwer Acad. Publ. Dordrecht, 1988.

[30] A.P. Rollett, *Advanced Problem No. 5080*, Amer. Math. Monthly, February, 1963.
[31] R.J. Stroeker and N. Tzanakis, *Computing all integer solutions of a genus 1 equation*, Math. Comp. **72**, 1917–1933, 2003.
[32] Z. Şiar and R. Keskin, *Some new identities concerning generalized Fibonacci and Lucas numbers*, Hacet. J. Math. Stat. **42** (3), 211–222, 2013.
[33] Z. Şiar and R. Keskin, *The Square terms in generalized Lucas sequence with parameters P and Q*, Math. Scand. **118** (1), 13–26, 2016.
[34] Z. Şiar and R. Keskin, On square classes in generalized Fibonacci sequences, Acta Arith. **174** (3),277–295, 2016.
[35] N. Tzanakis, *Solving elliptic Diophantine equations by estimating linear forms in elliptic logarithms. The case of quartic equations*, Acta Arith. **75**, 165–190, 1996.
[36] N. Tzanakis, *Elliptic Diophantine Equations (A concrete approach via the elliptic logarithm)*, Discrete Math. Appl. 195 pp, 2013.
[37] S. Tengely, *Finding g-gonal numbers in recurrence sequences*, Fibonacci Quart. **46/47** (3), 235–240, 2008/09.
[38] S. Tengely, *On the Diophantine equation* $L_n = \binom{x}{5}$, Publ. Math. Debrecen, **79** (3-4), 749–758, 2011.
[39] O. Wyler, *In the Fibonacci series* $F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1}$ *the first, second and twelfth terms are squares*, Amer. Math. Monthly, **71**, 221–222, 1964.